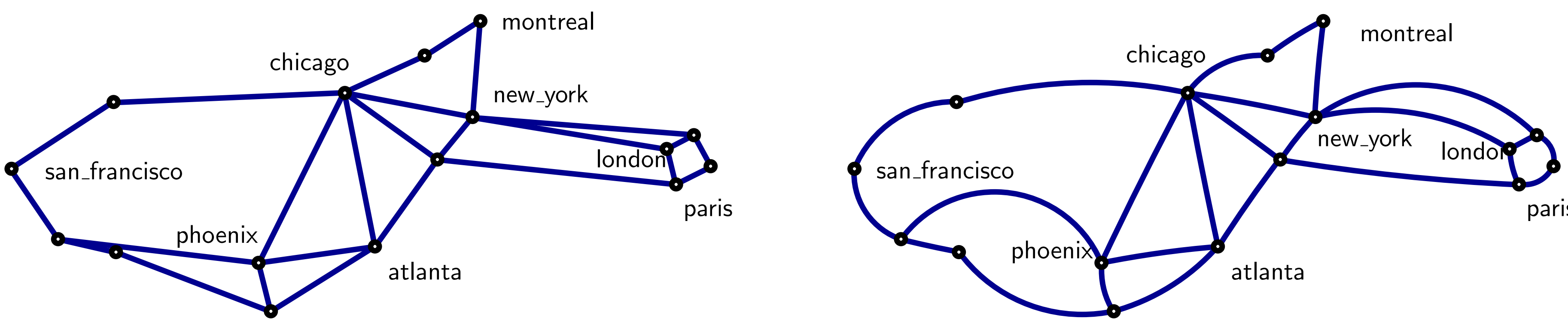
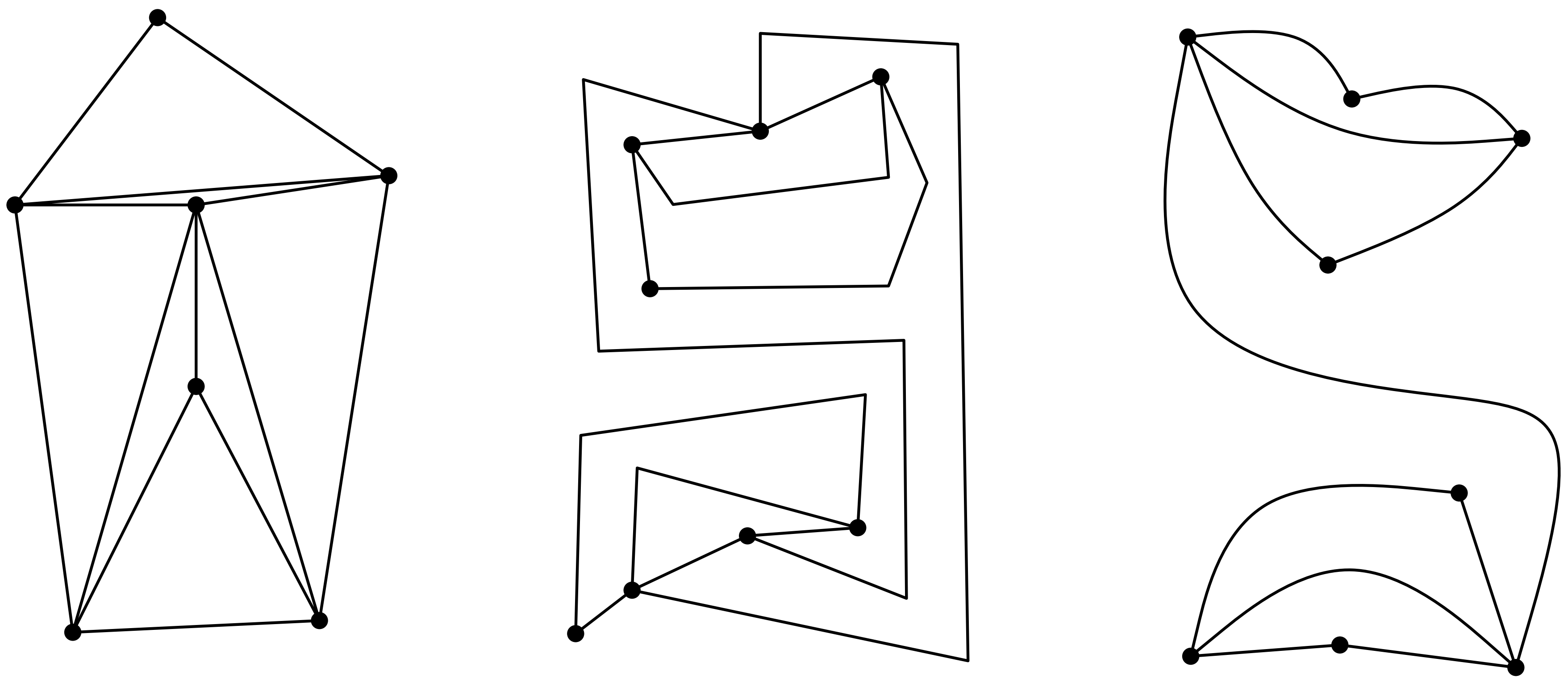
Triangulations with Circular Arcs

Oswin Aichholzer, Wolfgang Aigner,
Franz Aurenhammer, Kateřina Čech Dobiásova,
Bert Jüttler, Günter Rote



straight, with bends, curved



circular arcs!

Optimal Redrawings

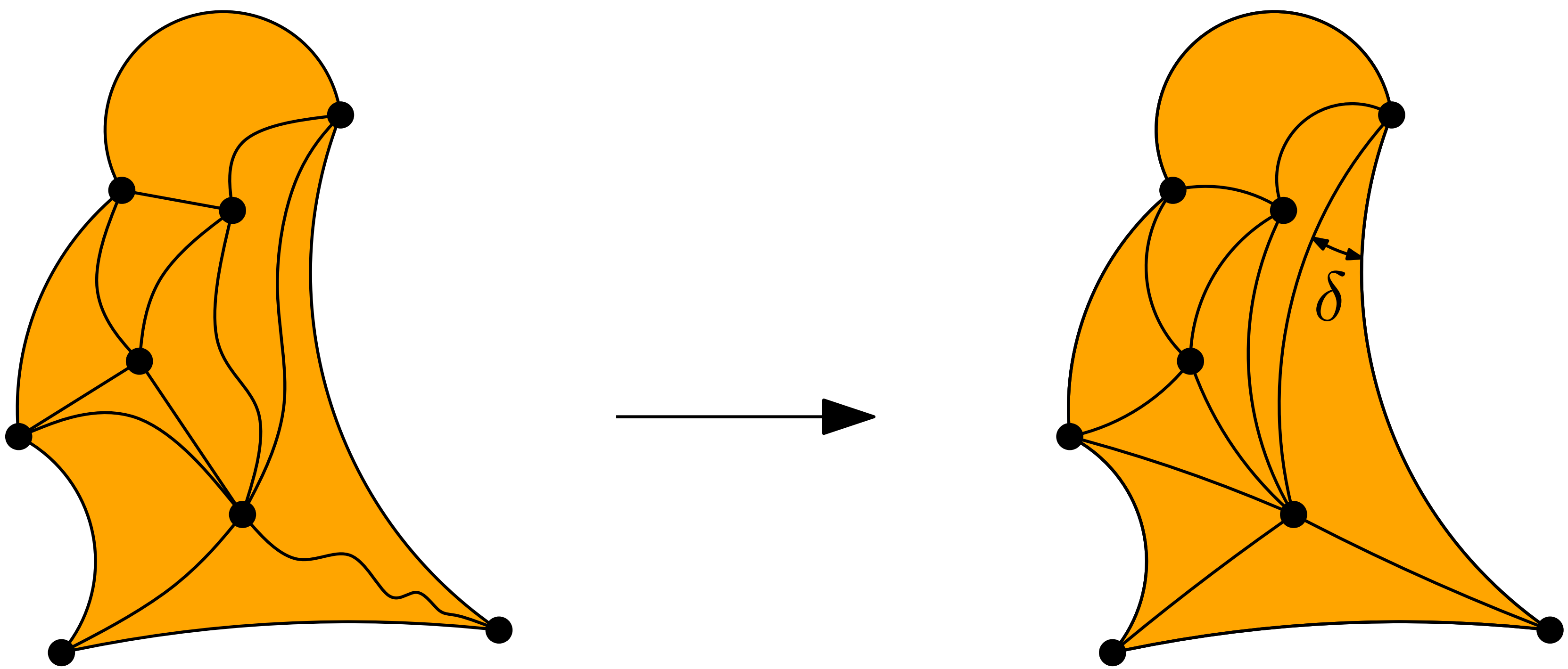
GIVEN:

A triangulation of a domain (with fixed boundary)

FIND:

A redrawing with circular arcs. (The vertices remain fixed.)

MAXIMIZE the smallest angle δ between adjacent edges.



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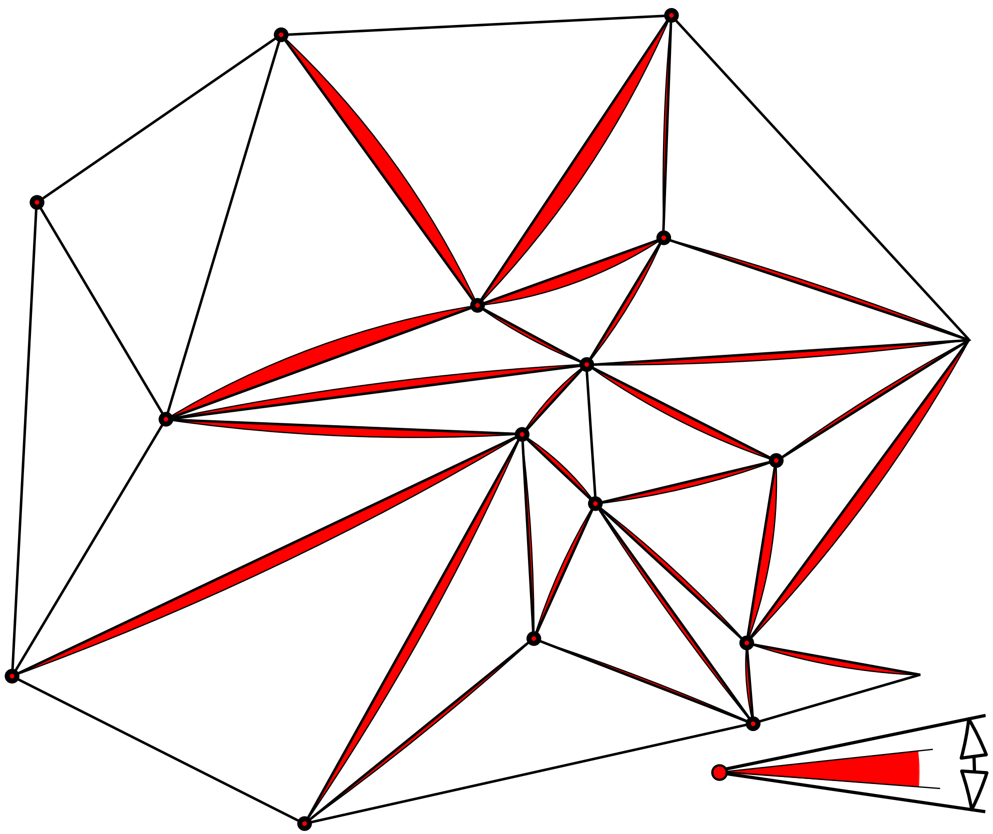

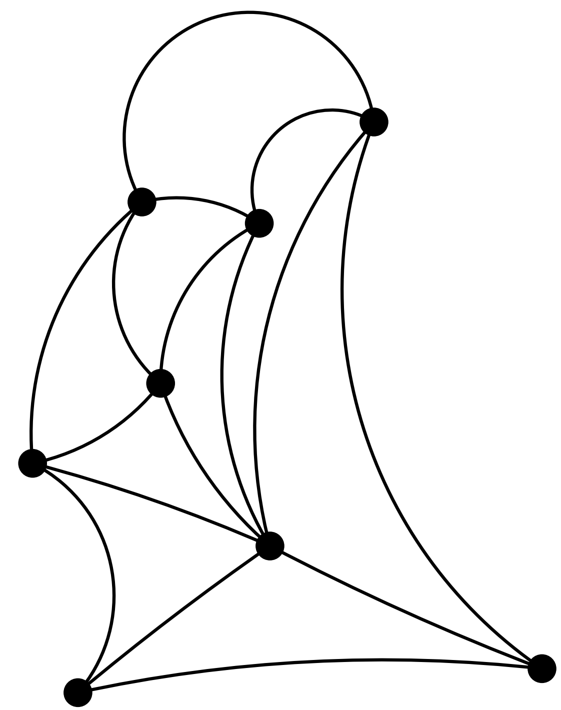

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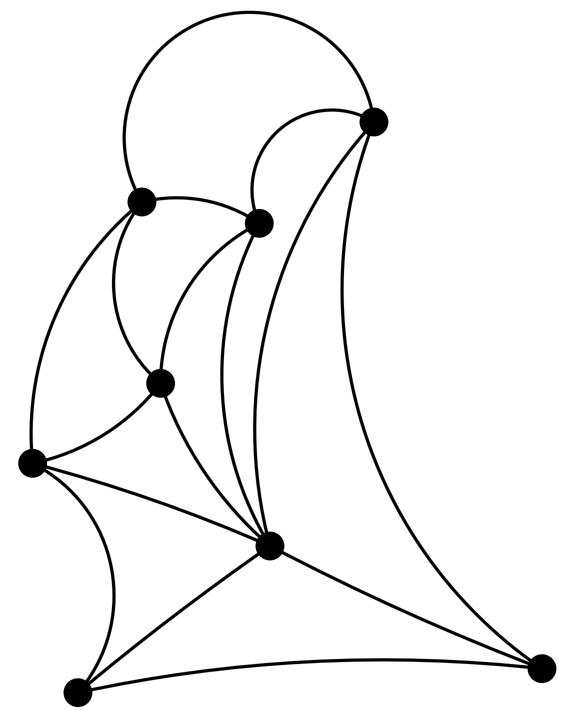
Applications:

- Graph Drawing: better visibility
- Meshing, Finite Element Methods:
  better quality of triangles
  ($\rightarrow$ better numerical properties)



Optimal Redrawings

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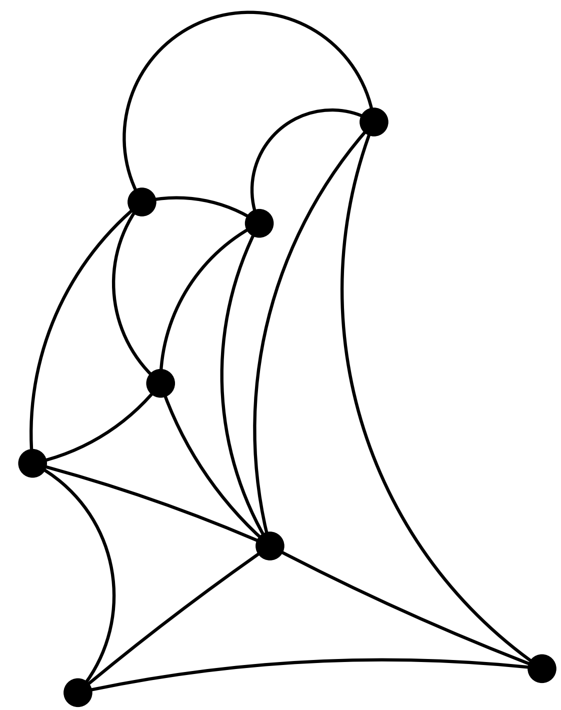
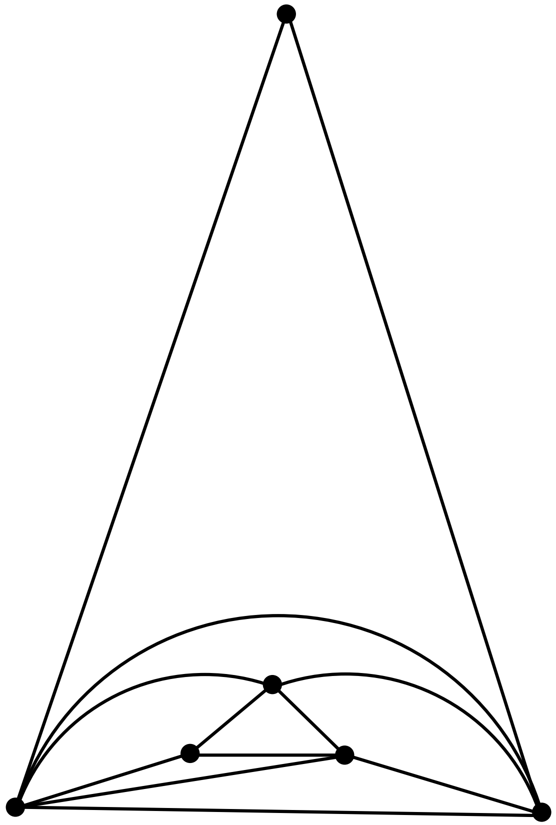
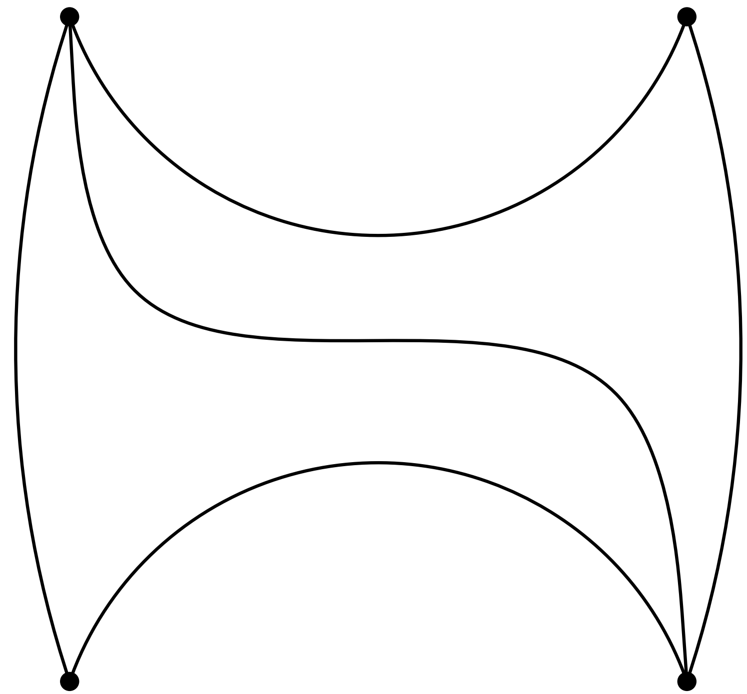
Results:

- A linear programming model
- An $O(n^2)$ algorithm



Remarks





Multiple edges are possible.

A solution need not exist.

Related Results

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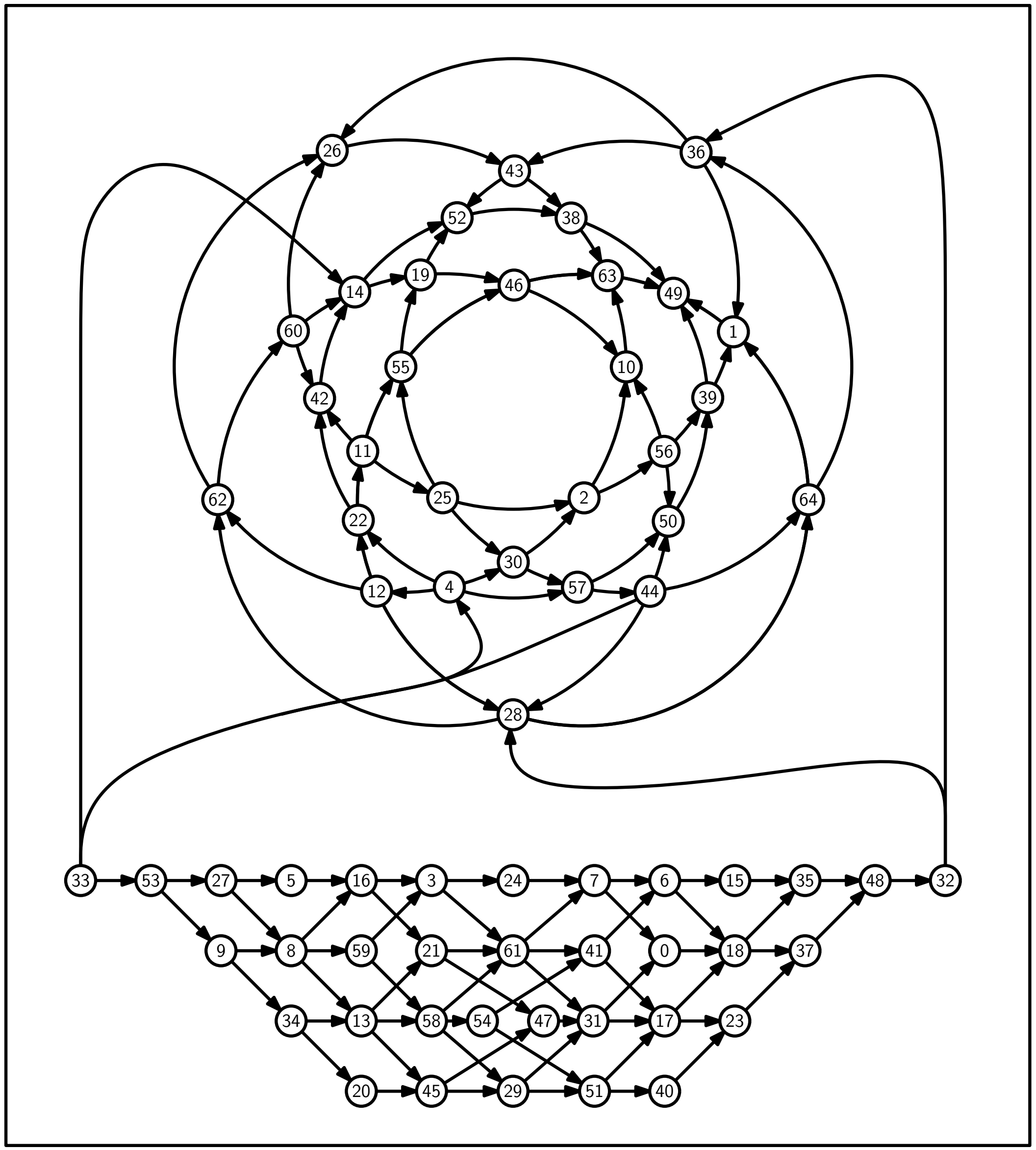
- *angle resolution* (Malitz and Papakostas, 1992)
- di Battista and Vismara (1996): angles in straight-line *triangulations* (vertices are not fixed)
- force-directed methods for curvilinear drawings (Finkel and Tamassia, GD 2004)
- *Lombardi* drawings (Duncan et al., GD 2010), 2 more papers in this session.

My entry for the GD 1996 contest

Graph C — The placement of the circles was optimized by computer. The rest was done by hand.



Günter Rote
Technische Universität Graz
Institut für Mathematik (501B)
Steyrergasse 30
A-8010 Graz, Austria
rote@opt.math.tu-graz.ac.at

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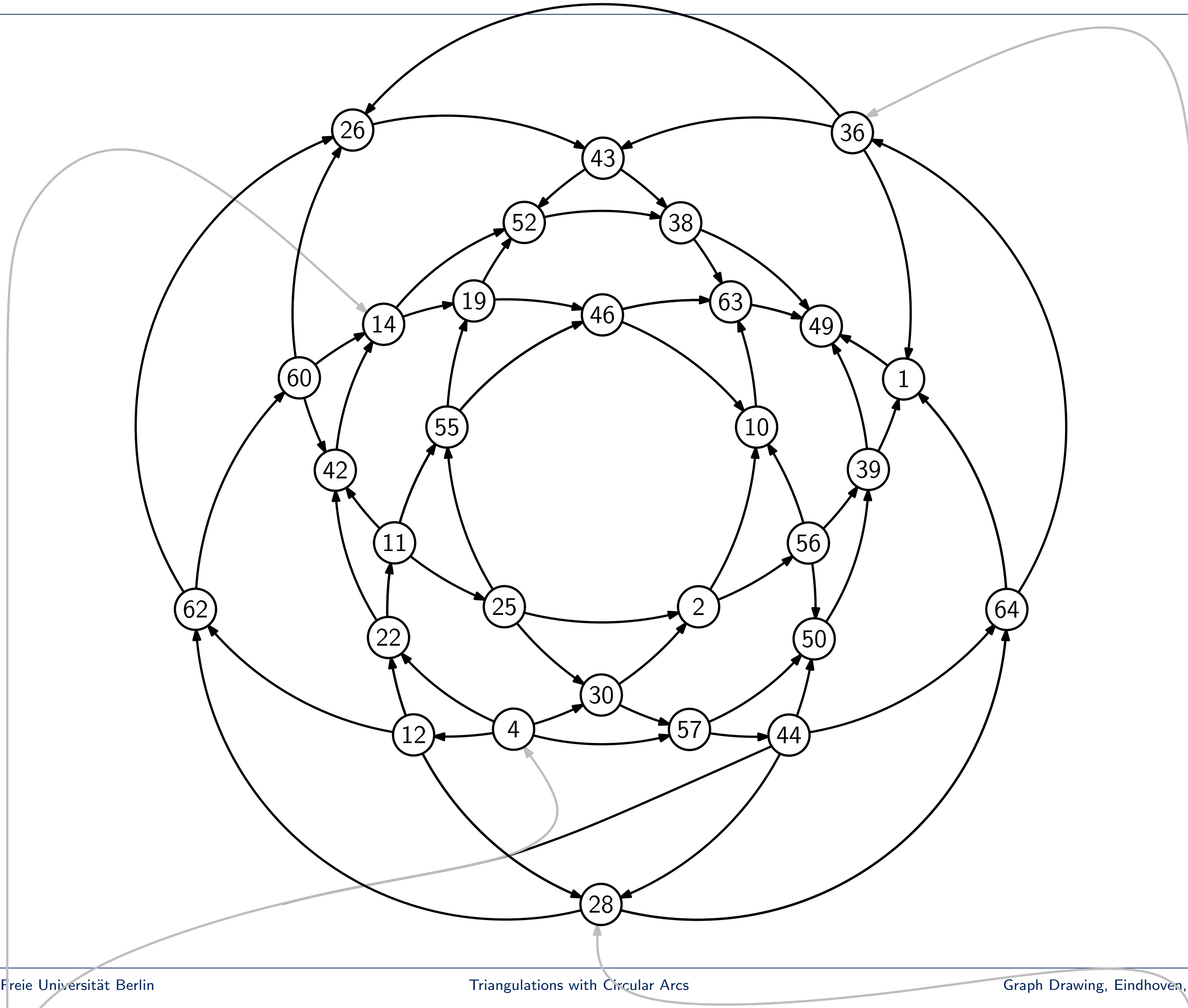
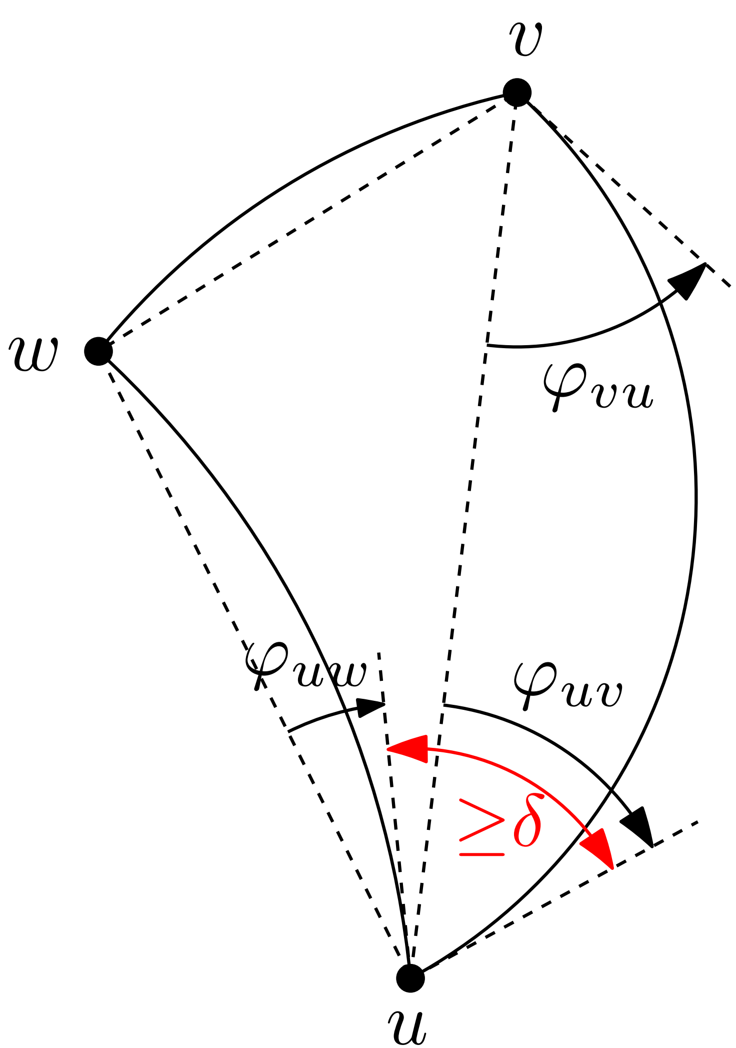


Modeling by Variables φ_{uv}



$\varphi_{uv} =$ the signed deviation from the straight edge uv (clockwise around u = positive, counterclockwise = negative.)

$$\varphi_{uv} = -\varphi_{vu}$$

$$\delta \leq \angle vuw + \varphi_{uv} - \varphi_{uw}$$

Maximize δ subject to these constraints.

Linear Programming Formulation

Maximize

$$\delta$$

subject to

$$\varphi_{uv} = -\varphi_{vu} \text{ for all edges } uv \tag{1}$$

$$\delta \leq \angle vuw + \varphi_{uv} - \varphi_{uw} \text{for all angles } vuw \tag{2}$$

Linear Programming Formulation

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For fixed δ, the constraints (2) are of the form

$$x_j \le x_i + c_{ij}$$

Checking feasibility of (2) for a given δ amounts to a *shortest path problem*.

Shortest Paths

A system of inequalities of the form

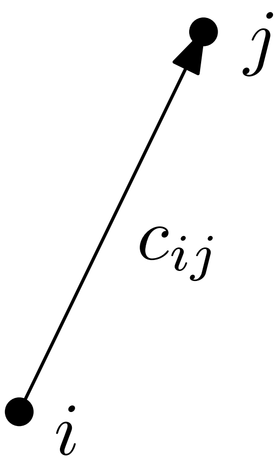
$$x_j \le x_i + c_{ij}$$

is feasible $\iff$ the directed graph with arc weights c_{ij} has no negative cycles.

---

add artificial source vertex S_0
$x_i :=$ shortest path from S_0 to i

Bellman-Ford algorithm: $O(mn) = O(n^2)$ time
($m = \#$arcs $= O(n)$.)



Shortest Paths

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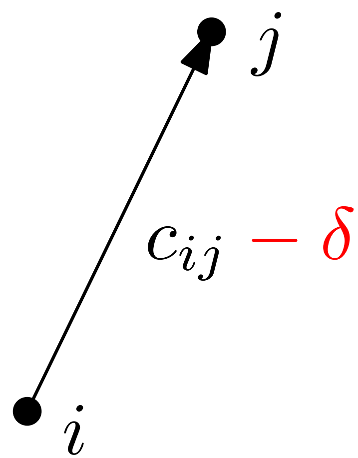
δ variable: Find the largest δ such that the graph with arc weights $c_{ij} - \delta$ has no negative cycles.
$\rightarrow$ the *minimum cycle mean problem* (Karp 1978):



In a cycle with k arcs, the weight changes like $C - k\delta$.
weight nonnegative $\implies$ $\delta \le C/k$.

$O(mn) = O(n^2)$ time, $O(n)$ space.

Getting rid of coupling equations

We have a system where variables come in pairs $x_i, \bar{x}_i$.

$$x_i = -\bar{x}_i \qquad (*)$$

$\bar{X}$ denotes the partner of X, $\bar{X} = -X$, $\bar{\bar{X}} = X$.

For each inequality of the form

$$X \le Y + c$$

add the (redundant) symmetric inequality

$$\bar{Y} \le \bar{X} + c$$

LEMMA: Then we can omit the equations $(*)$ without changing feasibility.

Getting rid of coupling equations

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$$x_i = -\bar{x}_i \qquad (*)$$

$$X \le Y + c$$

$$\overline{Y} \le \overline{X} + c$$

Proof (Shostak 1981):
Set

$$x_i^{\text{new}} := (x_i - \bar{x}_i)/2$$

$$\bar{x}_i^{\text{new}} := (\bar{x}_i - x_i)/2$$

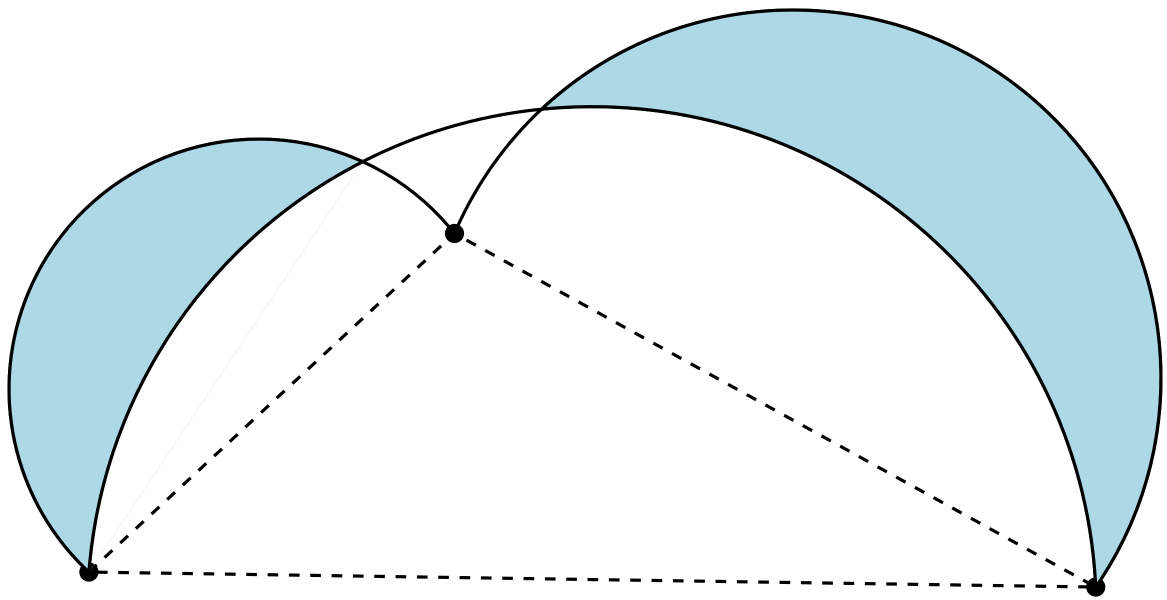
x_i^{new} and $\bar{x}_i^{\text{new}}$ will fulfill $(*)$.

Additional constraints

Upper and lower bounds on φ_{uv}:

$$\varphi_{uv}^{\min} \le \varphi_{uv} \le \varphi_{uv}^{\max}$$

In particular: fixed values for the boundary edges uv.
These constraints can be accommodated in the model.



avoid self-intersections

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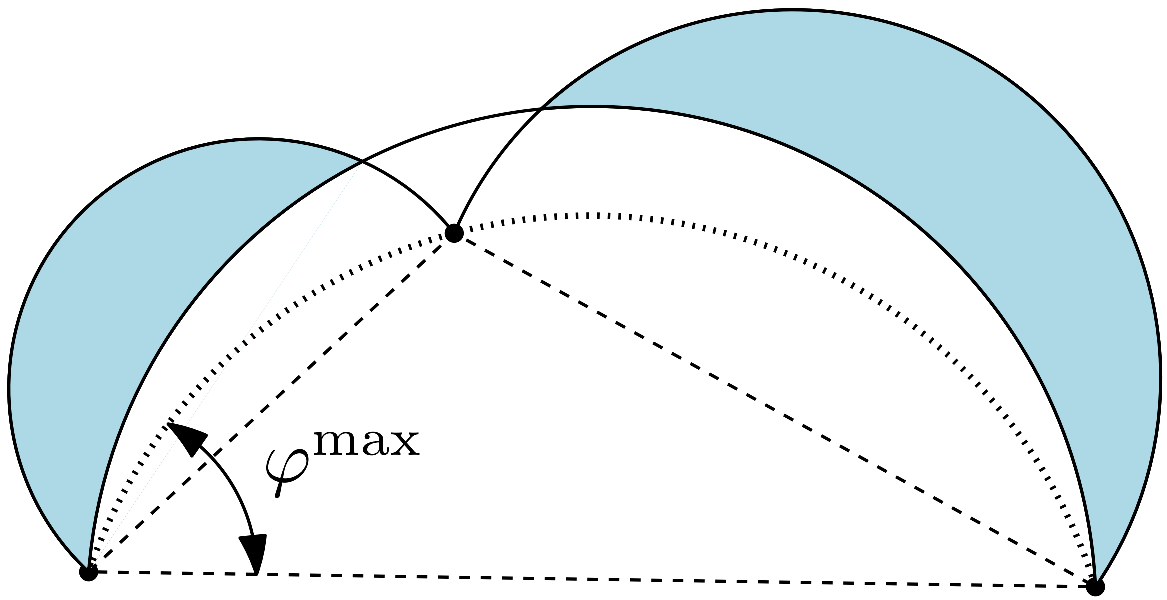

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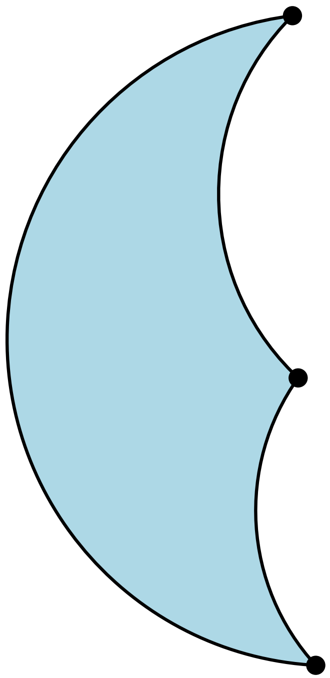
> THEOREM.
> The optimal redrawing of a triangulated domain with n vertices can be computed in $O(n^2)$ time.

Extension:
Maximize lexicographically the sorted sequence of angles, by solving a sequence of problems (Burkard and Rendl 1991).

π-triangulations



angle sum $= \pi$.

These are Möbius transforms (conformal images) of straight-line triangles.
$\rightarrow$ straightforward interpolation from vertex values into the interior

$$\varphi_{uv} + \varphi_{vw} + \varphi_{wu} = 0$$

Another linear equality.
Since it involves 3 variables, the reduction to a shortest path problem does not work. (General LP)

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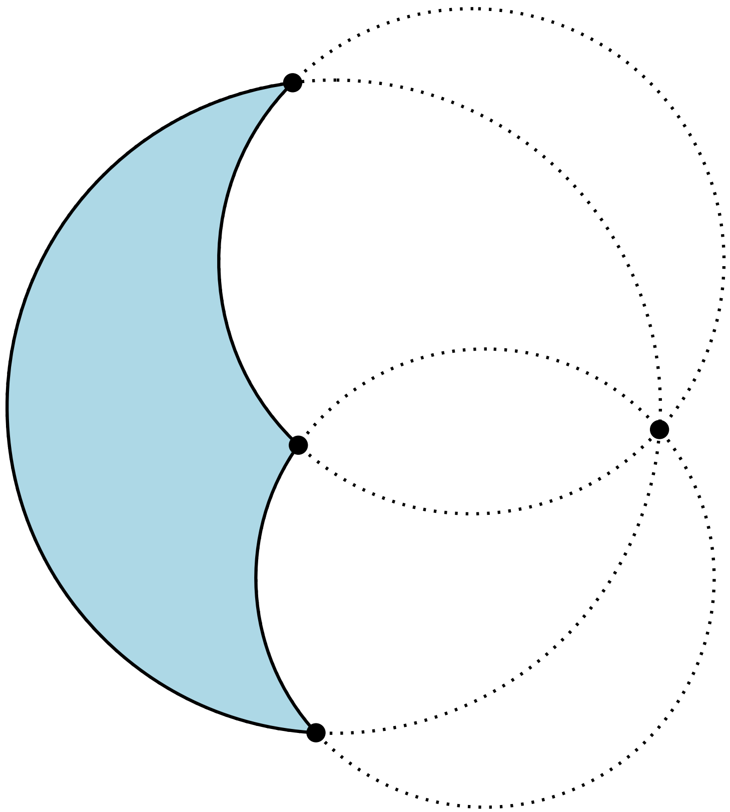

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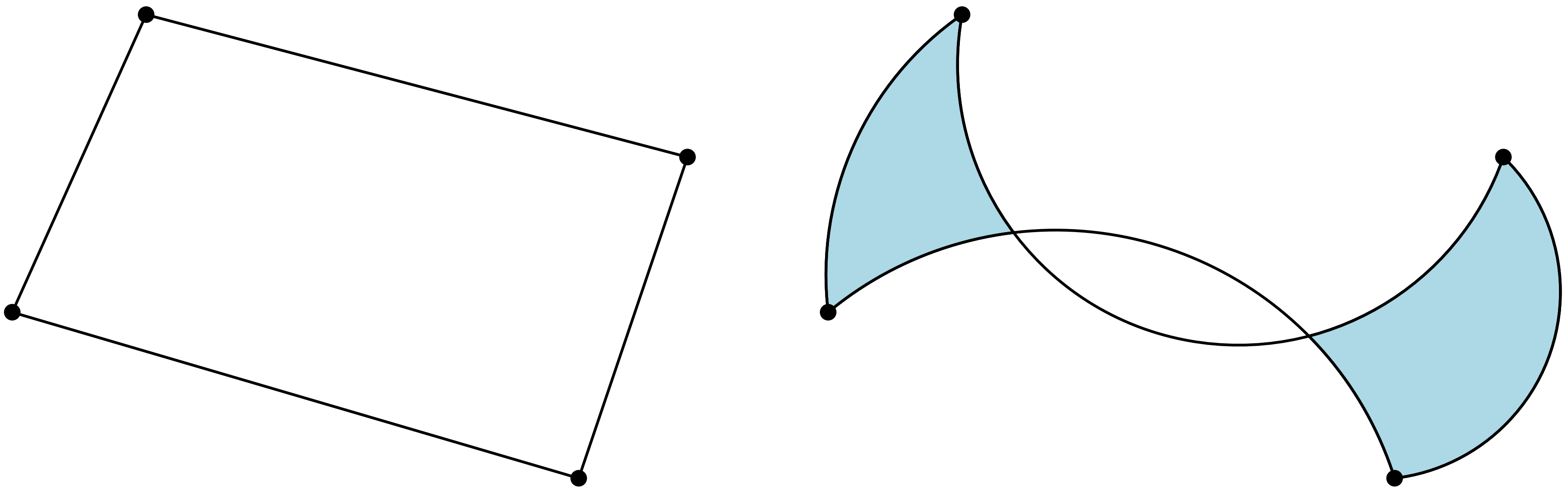
Graphs which are not triangulated



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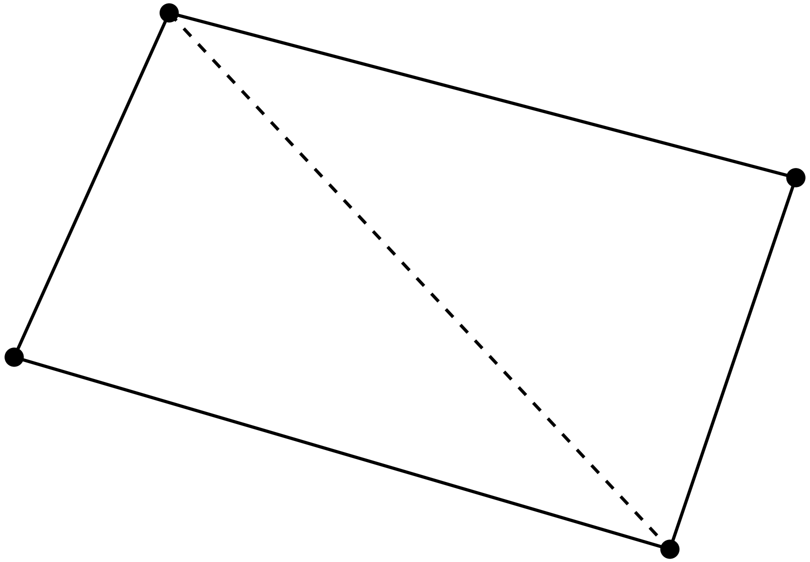

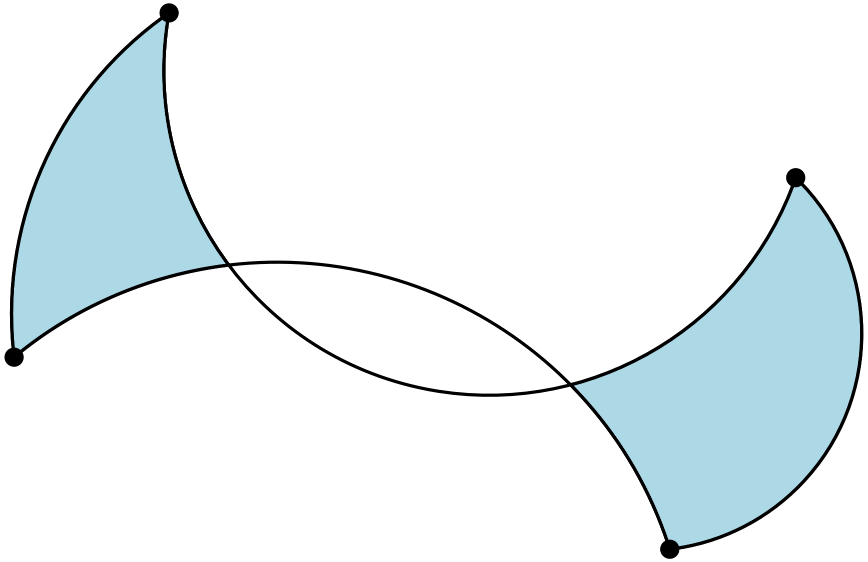

triangulate (arbitrarily)

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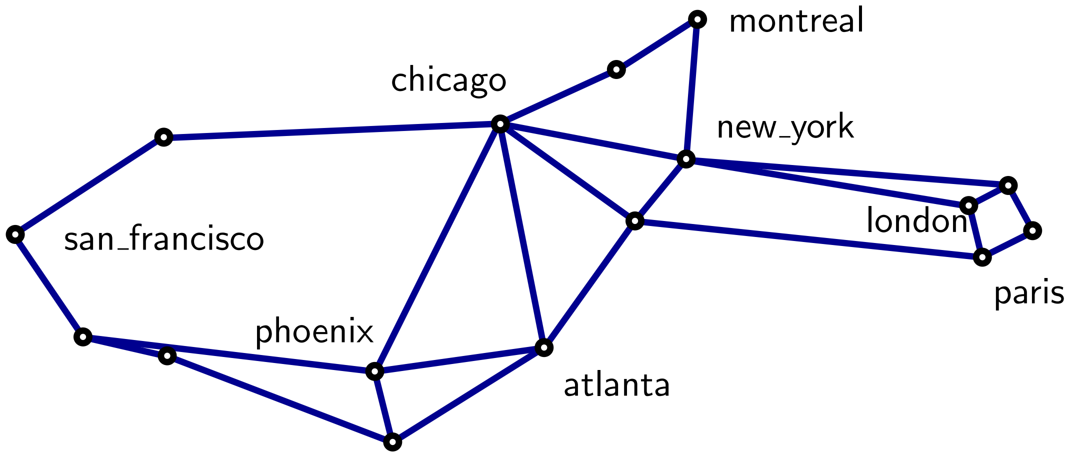
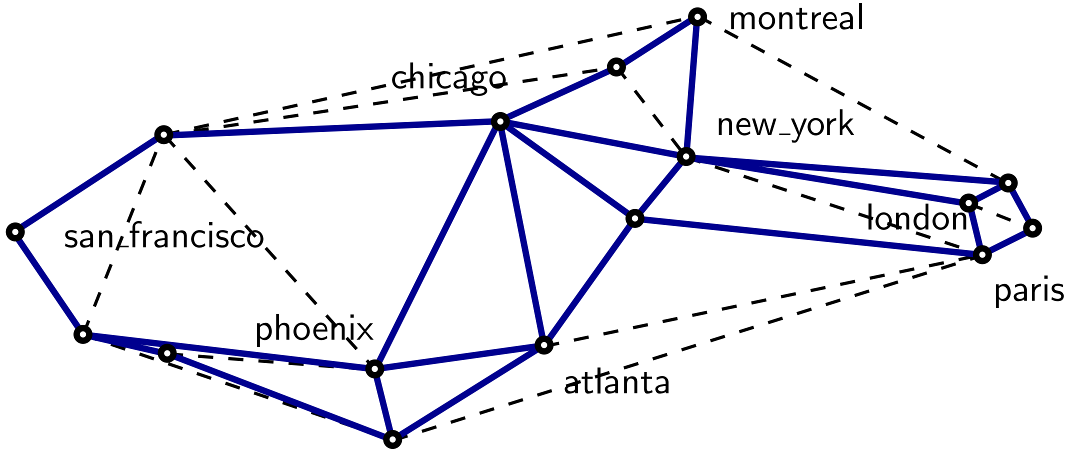


internet backbone network

Graphs which are not triangulated



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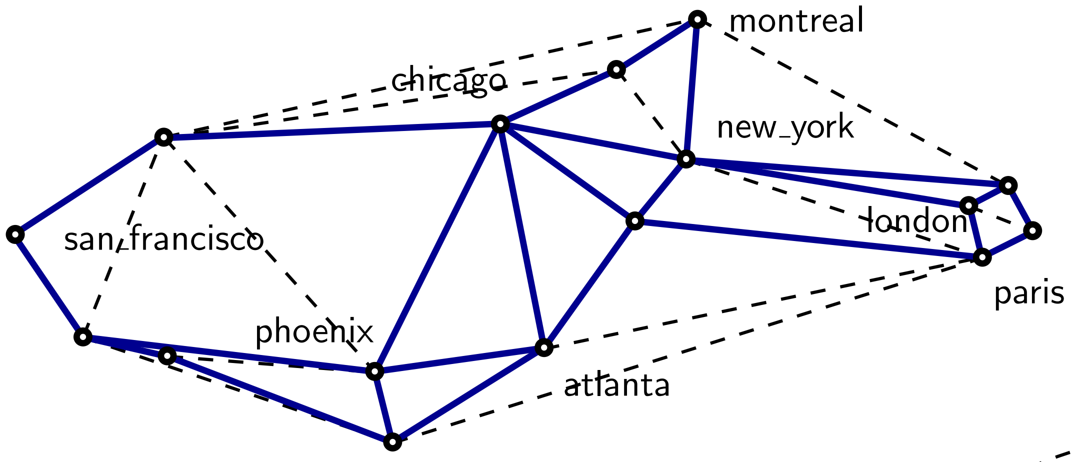

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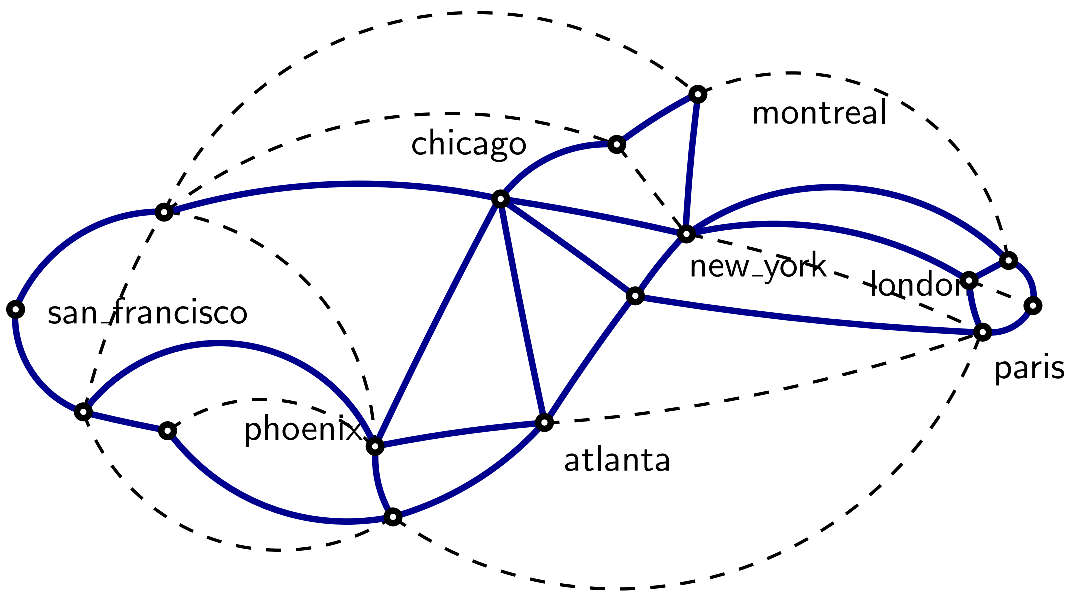

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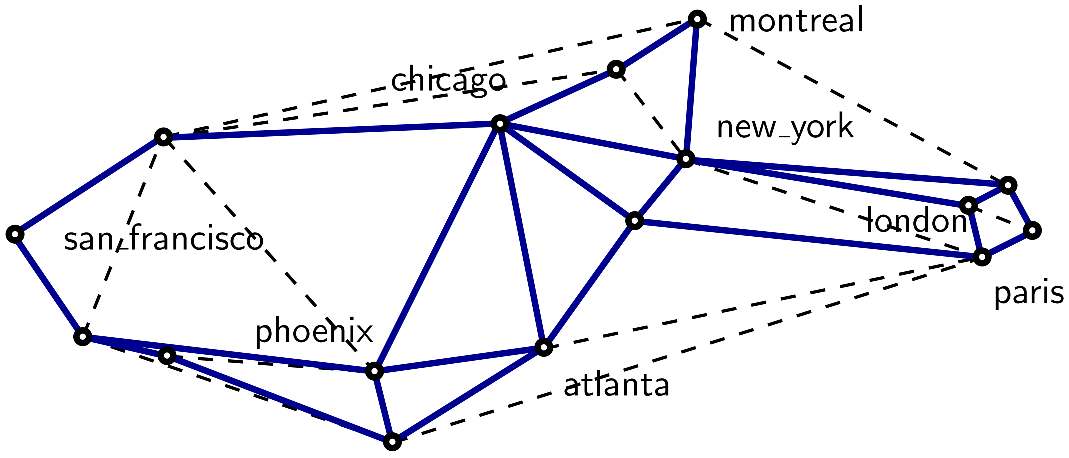

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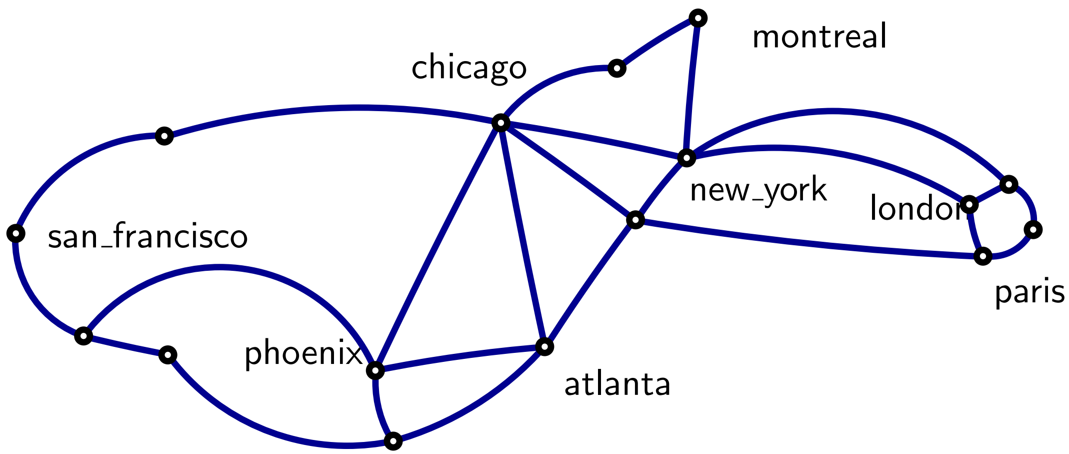
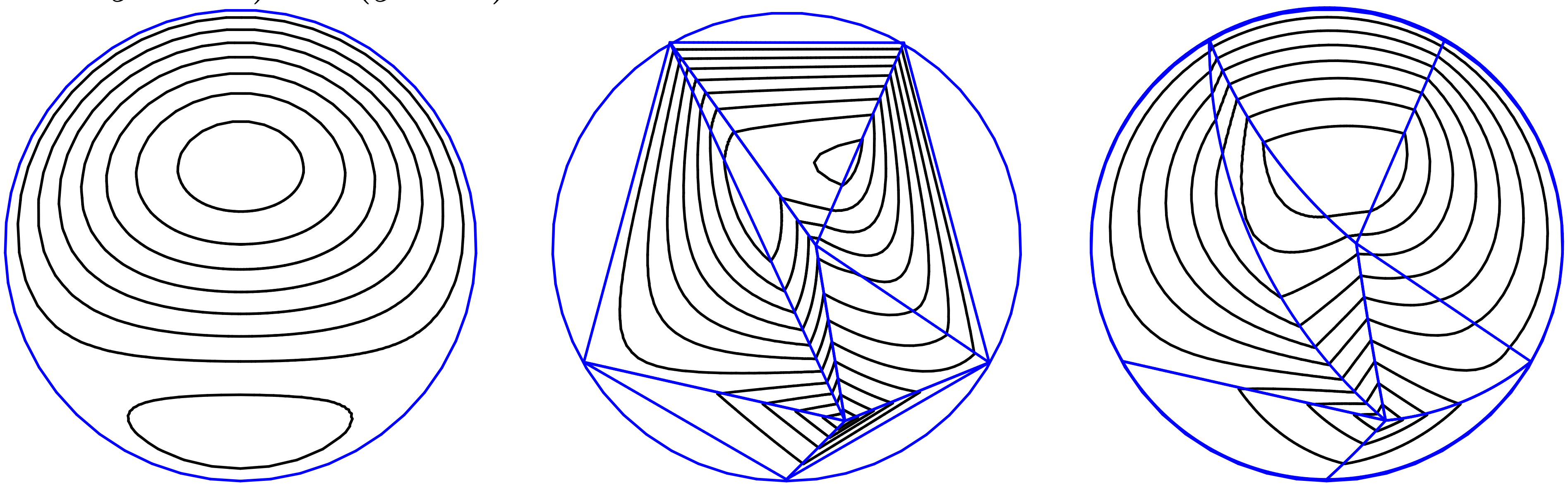


Interpolation for Finite Elements

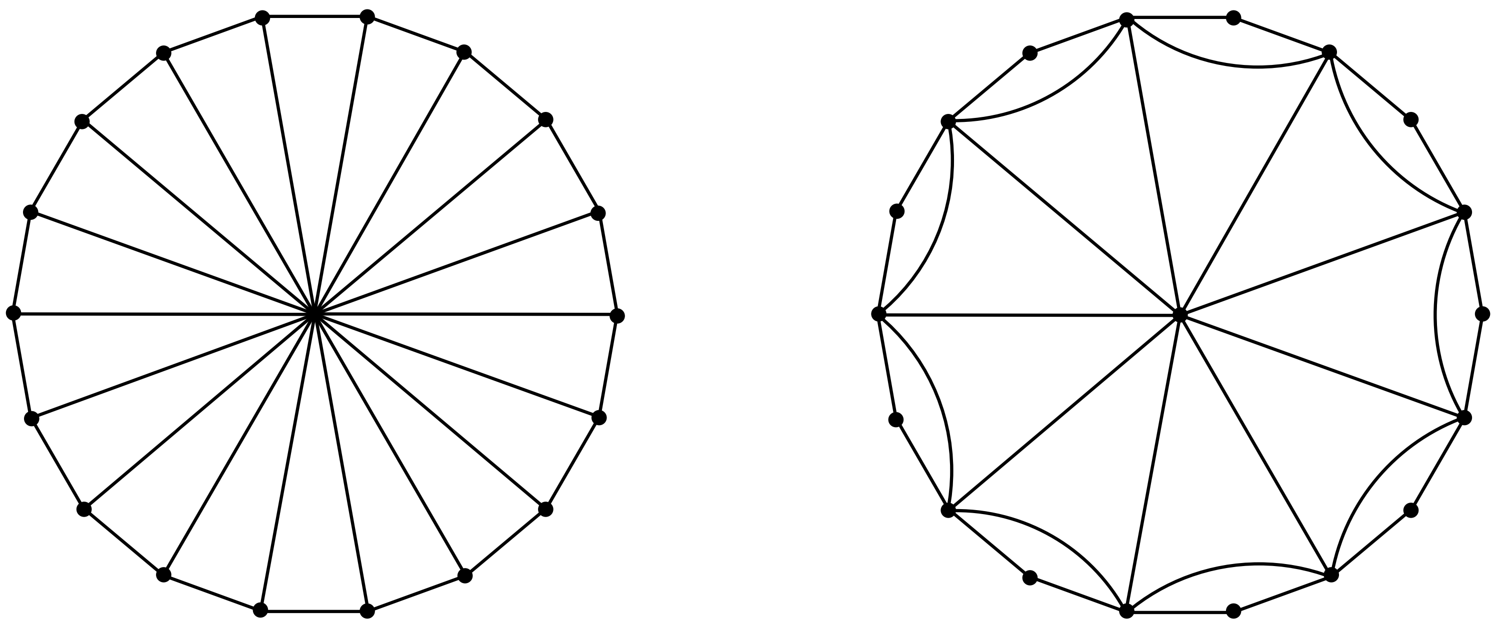
$(x^2 + y^2 - 1)\cos(y - 1)$



L^2-error: 0.18
max-error: 0.30

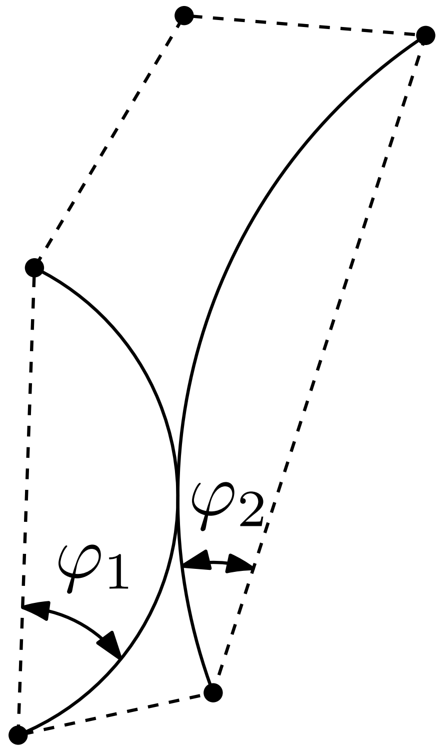
L^2-error: 0.07
max-error: 0.15

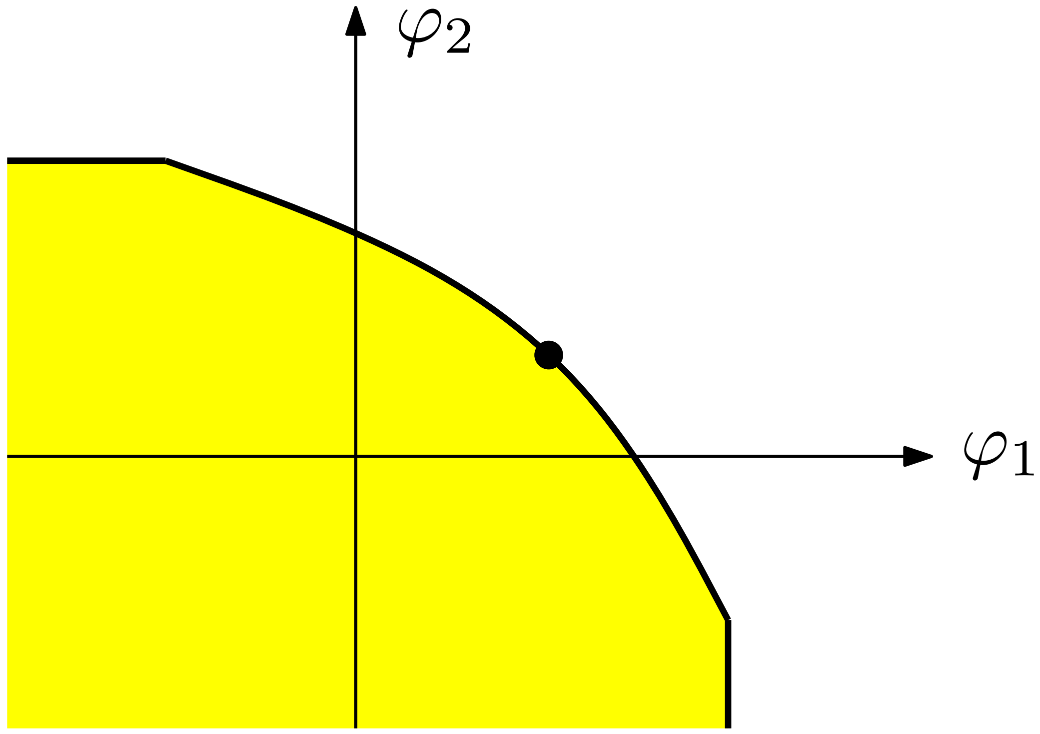
Improving the smallest angle by flipping



Open Question

Nontriangular faces.





Is the (φ_1, φ_2)-region of nonintersecting arcs *convex*?
(Perhaps with a different choice of parameters?)