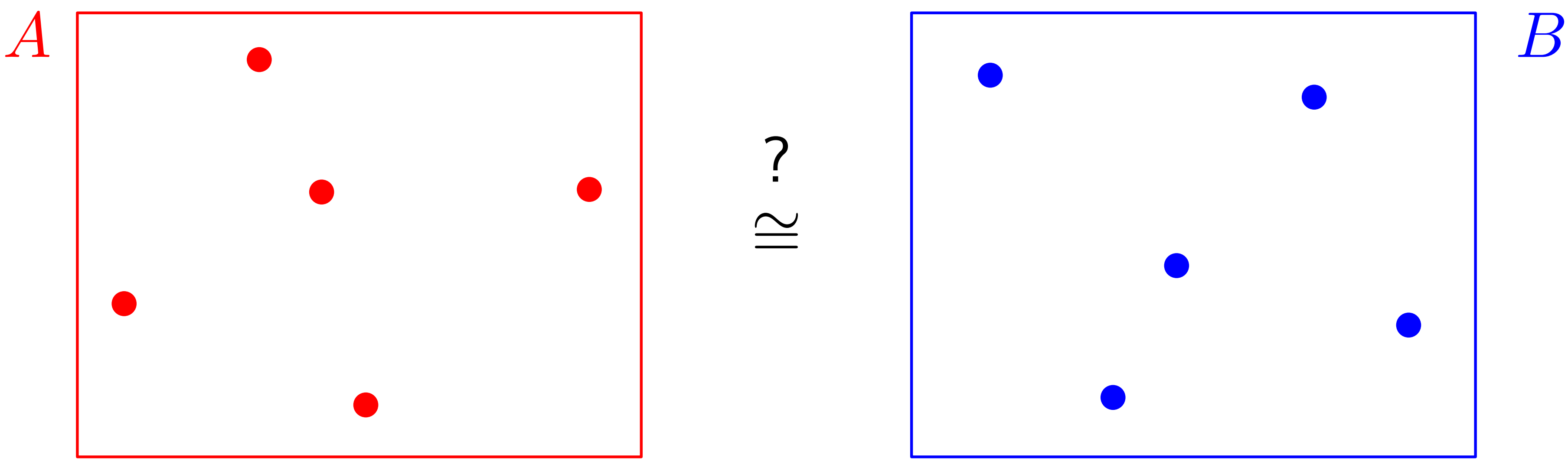
Testing Congruence of Point Sets

Günter Rote
Freie Universität Berlin



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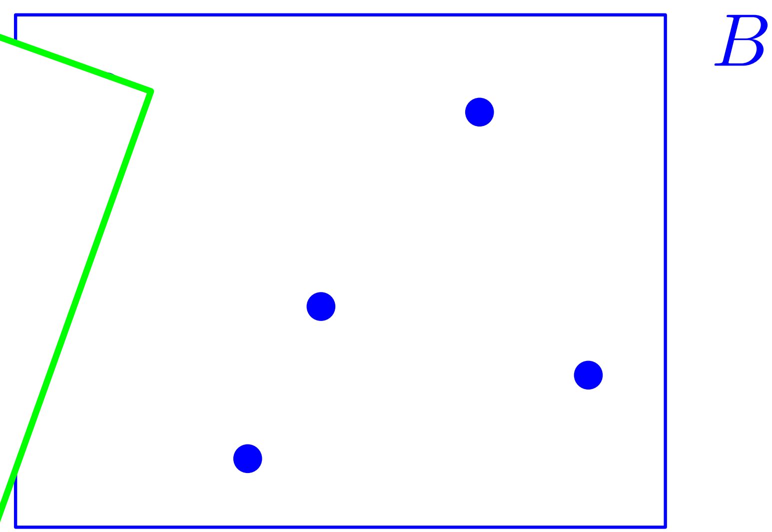
OPEN PROBLEM:

$t = 2, 3, \ldots$

Find a digraph such that

- every t-tuple of vertices has a common successor;
- with a (not necessarily proper) k-coloring of the arcs so that every directed cycle contains all colors

$(k \geq 2, k \to \max)$



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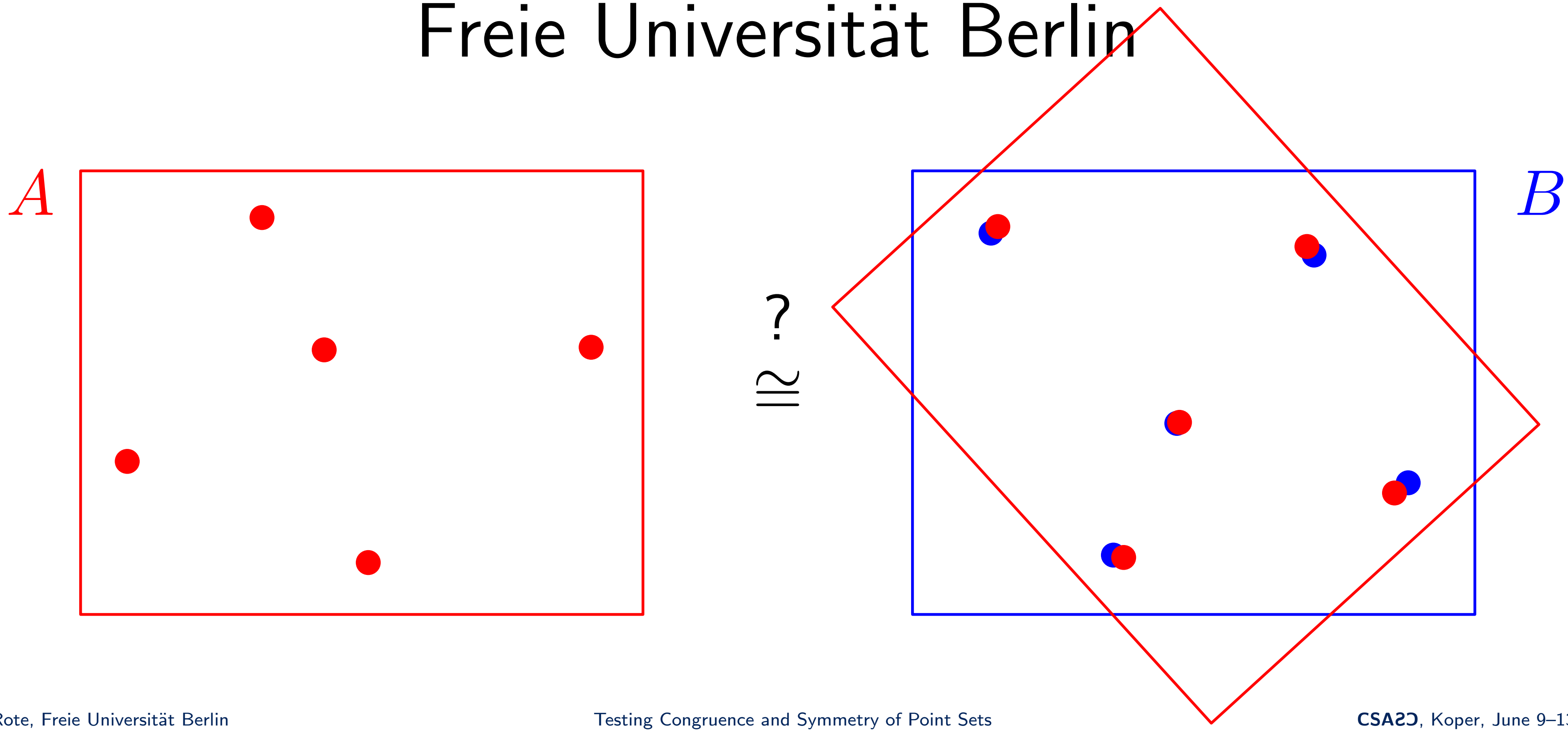
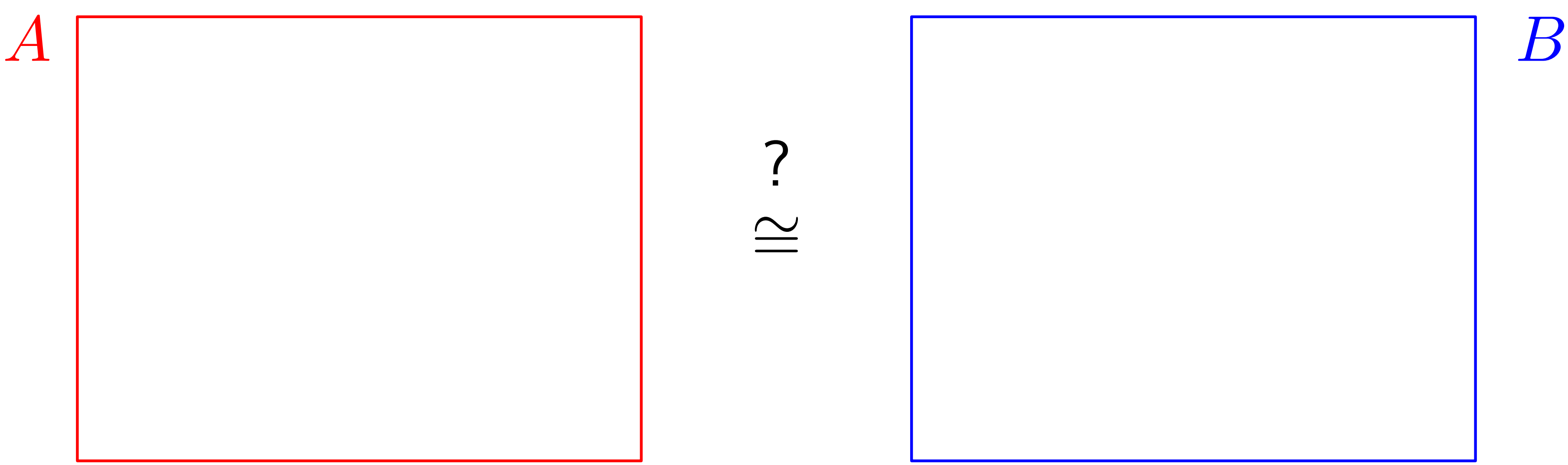


Testing Congruence of Point Sets and Symmetry

Günter Rote
Freie Universität Berlin



Testing Congruence and Symmetry of Point Sets

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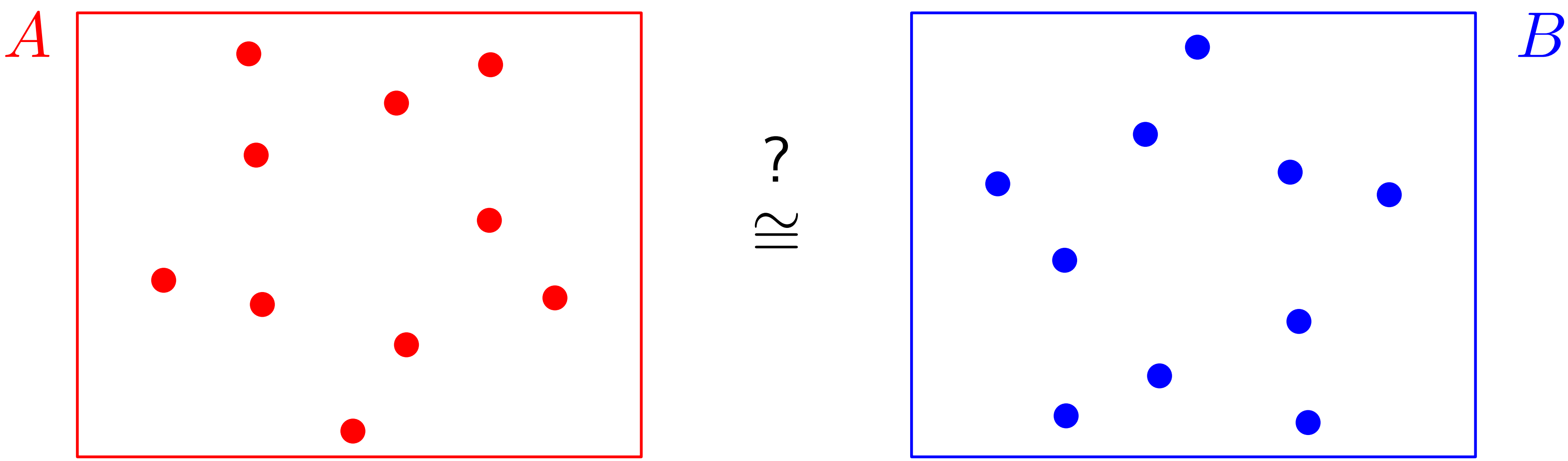
Freie Universität Berlin



Overview

- 1 dimension
- 2 dimensions
- 3 dimensions
- 4 dimensions
- d dimensions

Overview

- 1 dimension
- 2 dimensions
- 3 dimensions
- 4 dimensions ?

(1–4 dimensions:) $\Big\}\ O(n \log n)$ time

- d dimensions $O(n^{\lceil d/3 \rceil} \log n)$ time

Overview

- 1 dimension
- 2 dimensions
- 3 dimensions
- 4 dimensions ?

} $O(n \log n)$ time

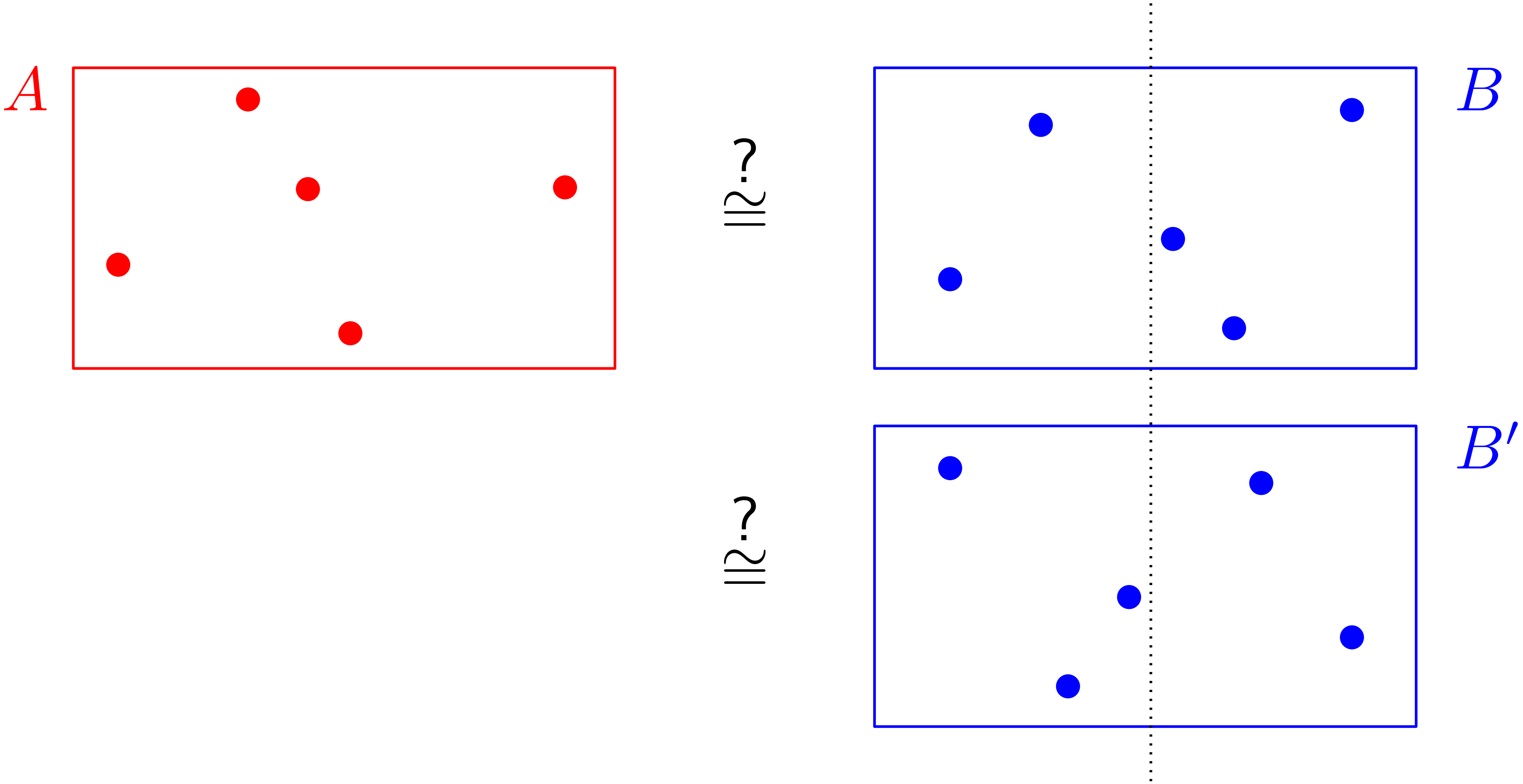
- d dimensions $O(n^{\lceil d/3 \rceil} \log n)$ time

- Problem statement and variations
- PRUNING and DIMENSION REDUCTION
- Point groups (discrete subgroups of the orthogonal group)

Rotation or Rotation+Reflection?

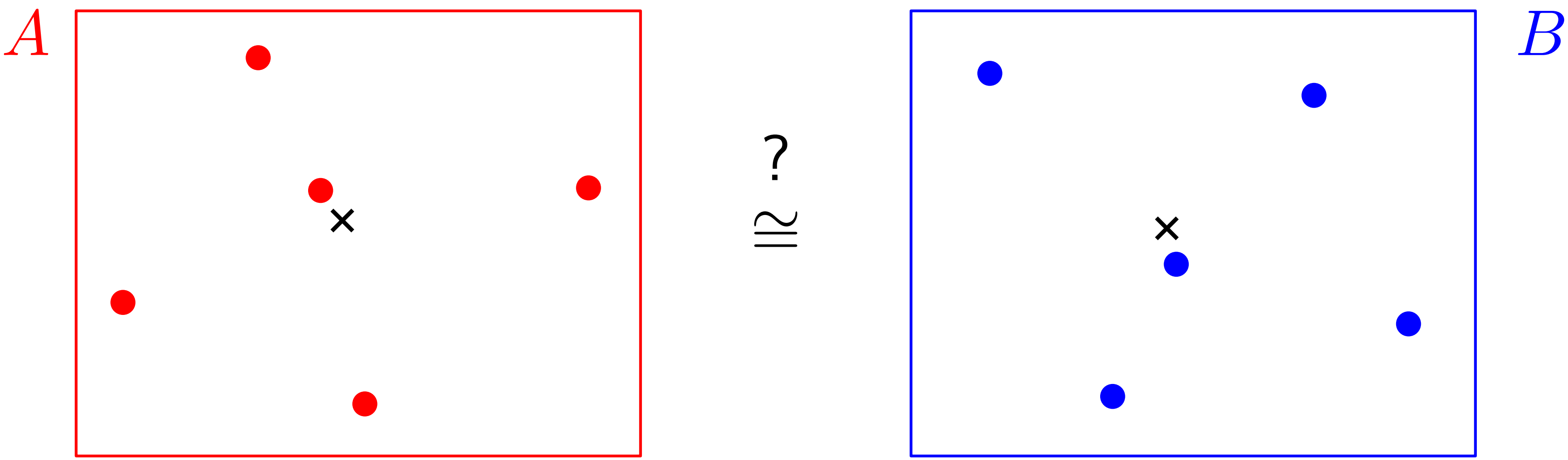
We only need to consider *proper* congruence (orientation-preserving congruence, of determinant $+1$).

If mirror-congruence is also desired, repeat the test twice, for B and its mirror image B'.



Congruence = Rotation + Translation

Translation is easy to determine:
The centroid of A must coincide with the centroid of B.



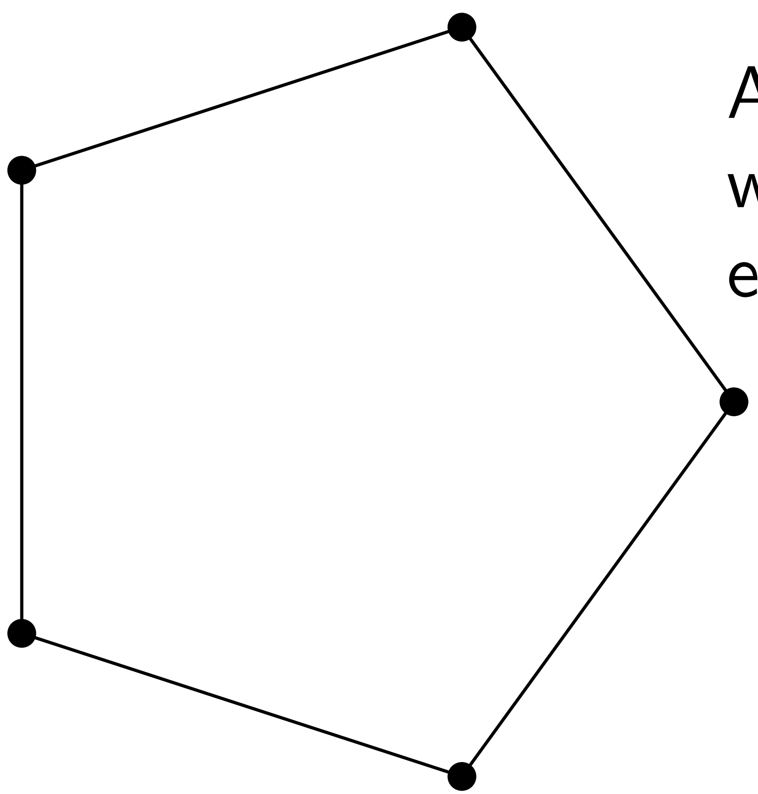
$\rightarrow$ from now on: All point sets are centered at the origin O:

$$\sum_{a \in A} a = \sum_{b \in B} b = 0$$

We need to find a rotation around the origin (orthogonal matrix T with determinant $+1$) which maps A to B.

Exact Arithmetic

The proper setting for this (mathematical) problem requires real numbers as inputs and exact arithmetic.
$\rightarrow$ the *Real RAM* model (RAM $=$ random access machine):
One elementary operation with real numbers ($+$, $\div$, $\sqrt{}$, $\sin$) is counted as one step.



A regular 5-gon, 7-gon, 8-gon, ...
with rational coordinates does not exist in any dimension.

Applications

Congruence testing is the basic problem for many pattern matching tasks

- computer vision
- star matching
- brain matching
- . . .

The proper setting for this applied problem requires tolerances, partial matchings, and other extensions.

Approximate matching

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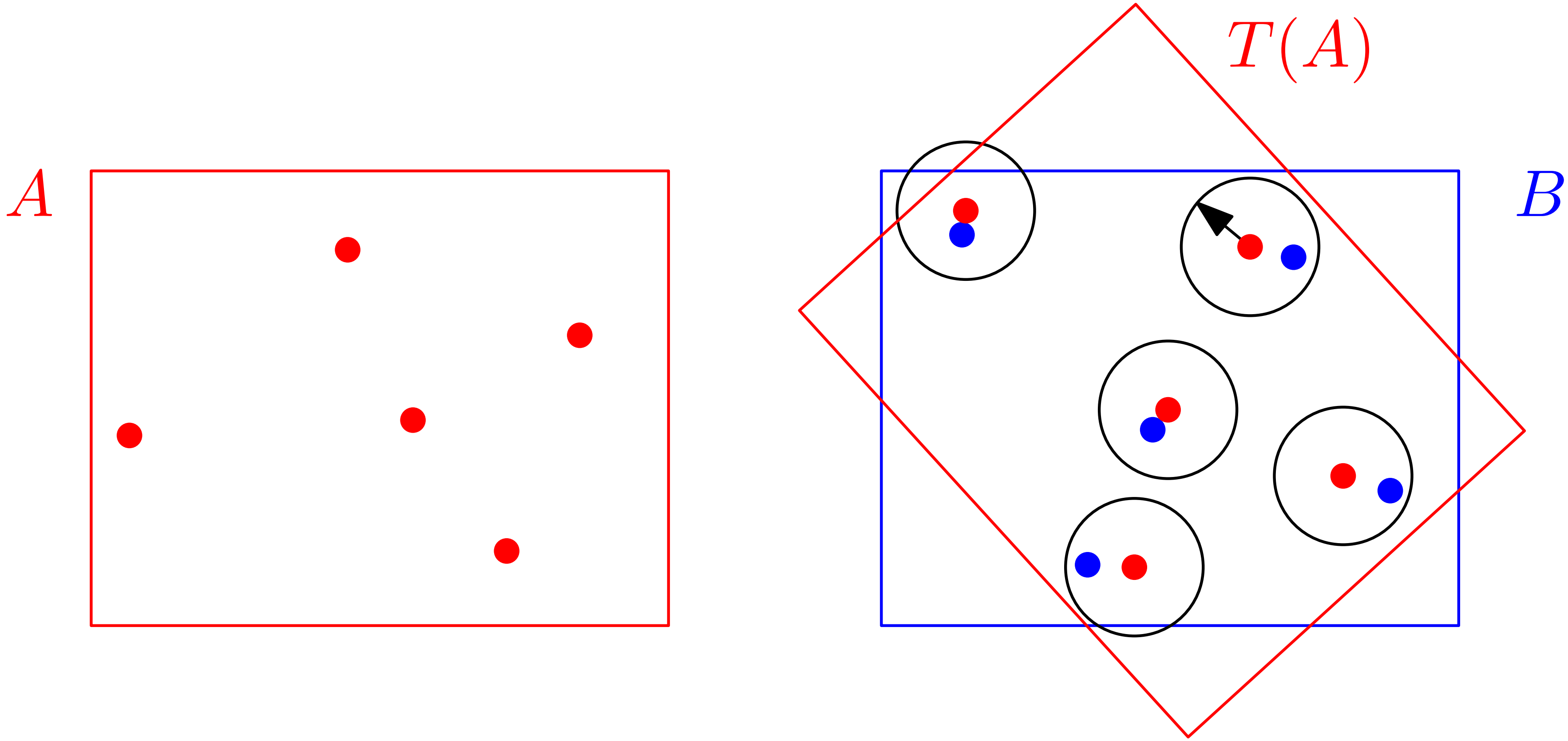
Given two sets A and B in the plane and an error tolerance ε, find a bijection $f\colon A \to B$ and a congruence T such that

$$\|T(a) - f(b)\| \leq \varepsilon, \text{ for all } a \in A.$$



This problem is NP-hard. [S.Iwanowski 1991, C.Dieckmann 2012]

Arbitrary Dimension

$A, B \subset \mathbb{R}^d$, $|A| = |B| = n$.
We consider the problem for fixed dimension d.

When d is unrestricted, the problem is equivalent to graph isomorphism:
$G = (V, E)$, $V = \{1, 2, \ldots, n\}$
$\mapsto A = \underbrace{\{e_1, \ldots, e_n\}}_{\text{regular simplex}} \cup \{ \frac{e_i + e_j}{2} \mid ij \in E \} \subset \mathbb{R}^n$

CONJECTURE:
Congruence can be tested in $O(n \log n)$ time for every fixed dimension d. ("fixed-parameter tractable")
Current best bound: $O(n^{\lceil d/3 \rceil} \log n)$ time

One dimension

Trivial.
(after shifting the centroid to the origin and getting rid of reflection):

Test if $A = B$. $O(n \log n)$ time.

Two dimensions

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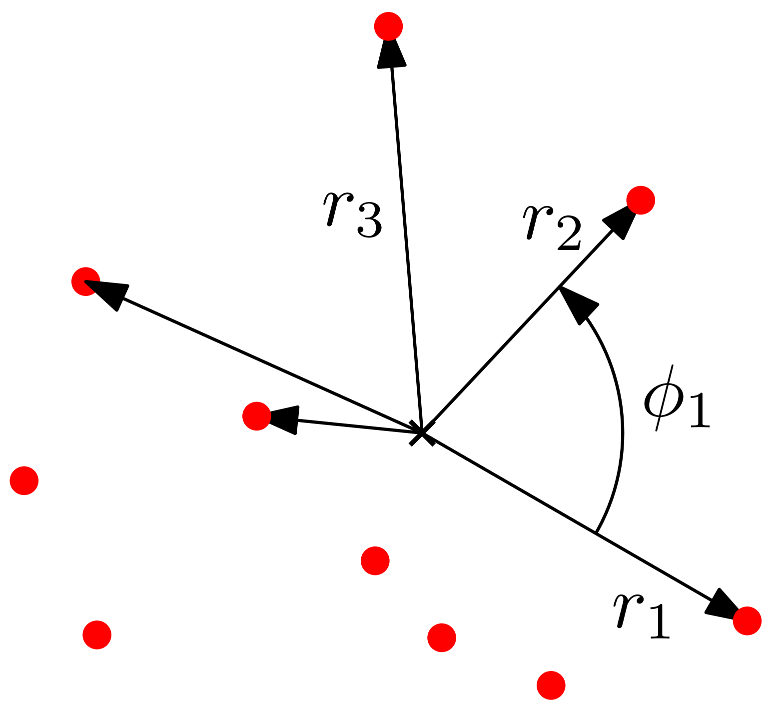
Can be done by string matching. [Manacher 1976]

Sort points around the origin.

Encode alternating sequence of distances r_i and angles ϕ_i.



$$(r_1, \phi_1, r_2, \phi_2, \ldots, r_n, \phi_n)$$

Check whether the corresponding sequence of B is a cyclic shift.

$\rightarrow O(n \log n) + O(n)$ time.

Three dimensions

Compute the convex hull $P(A)$ and $P(B)$, in $O(n \log n)$ time.

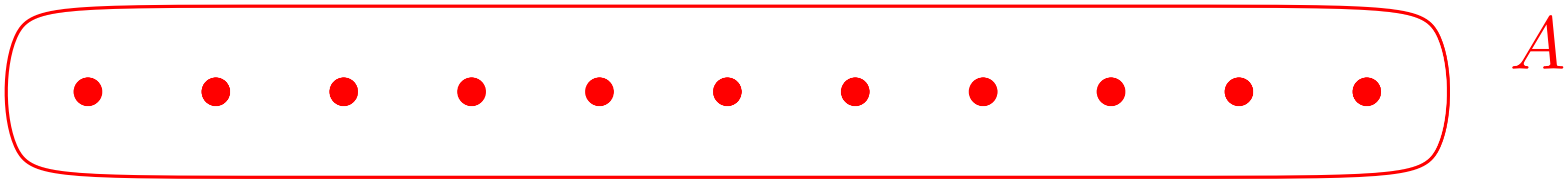
Check isomorphism between the corresponding planar graphs, in $O(n)$ time. [Hopcroft and Wong 1974]

The result is unique, up to

- the symmetries of a Platonic solid (at most 60 choices), or
- a rotation around an axis.
  $\rightarrow$ project down to a 2-dimensional problem.

[Sugihara 1984; Alt, Mehlhorn, Wagener, Welzl 1988]

Pruning



Pruning

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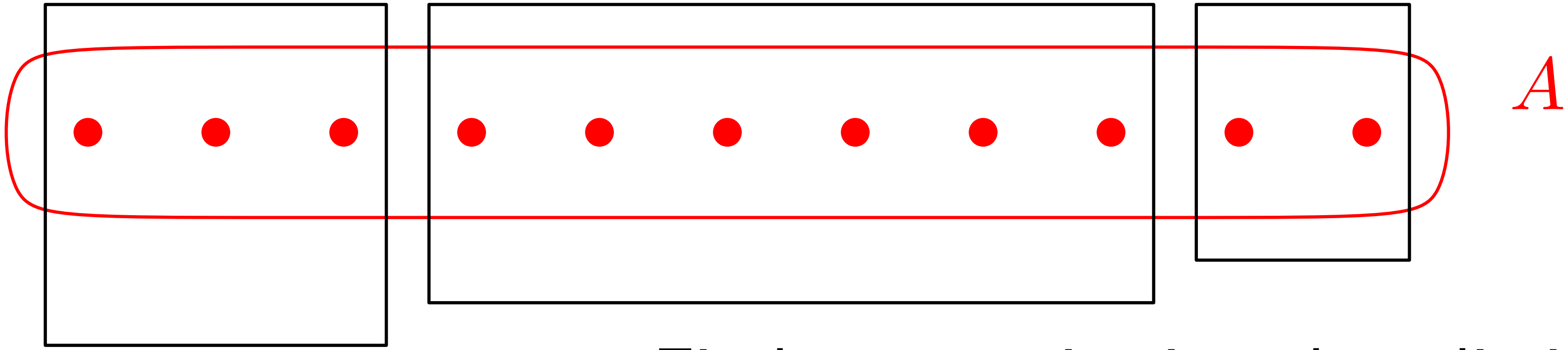
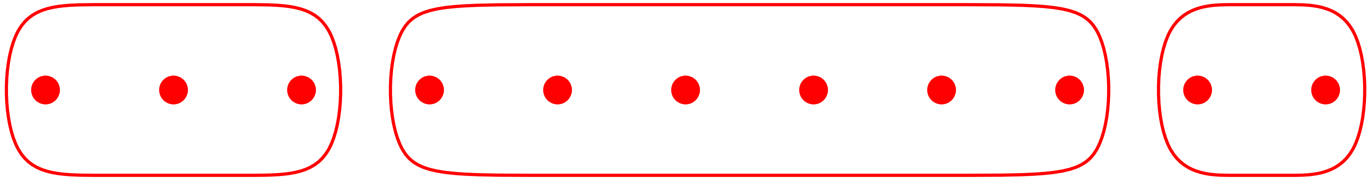
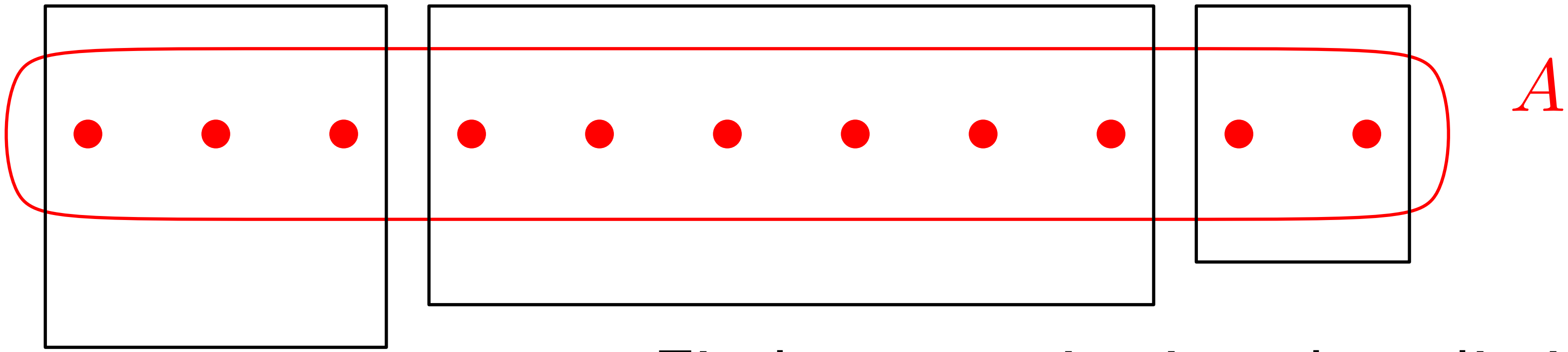


Find *some* criterion that distinguishes points (distance from the center, number of closest neighbors, . . .)



Pruning



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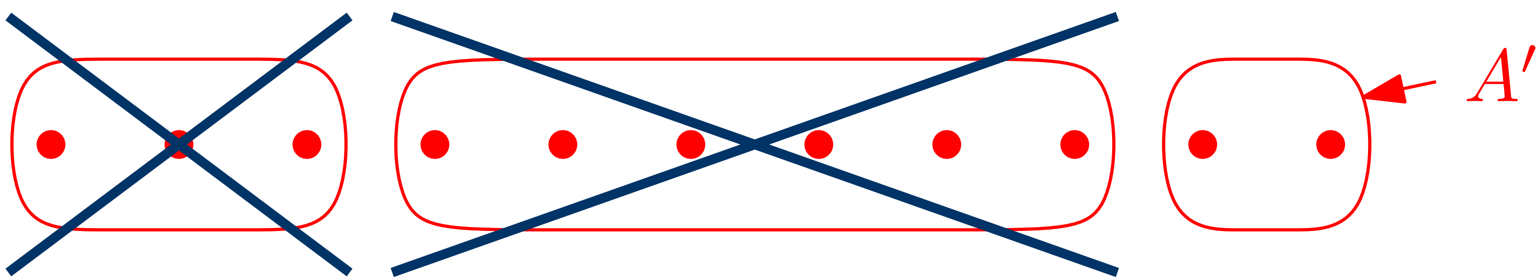

Throw away all but the smallest resulting class,
and repeat.

Simultaneously, apply the same pruning procedure to B.

Dimension Reduction

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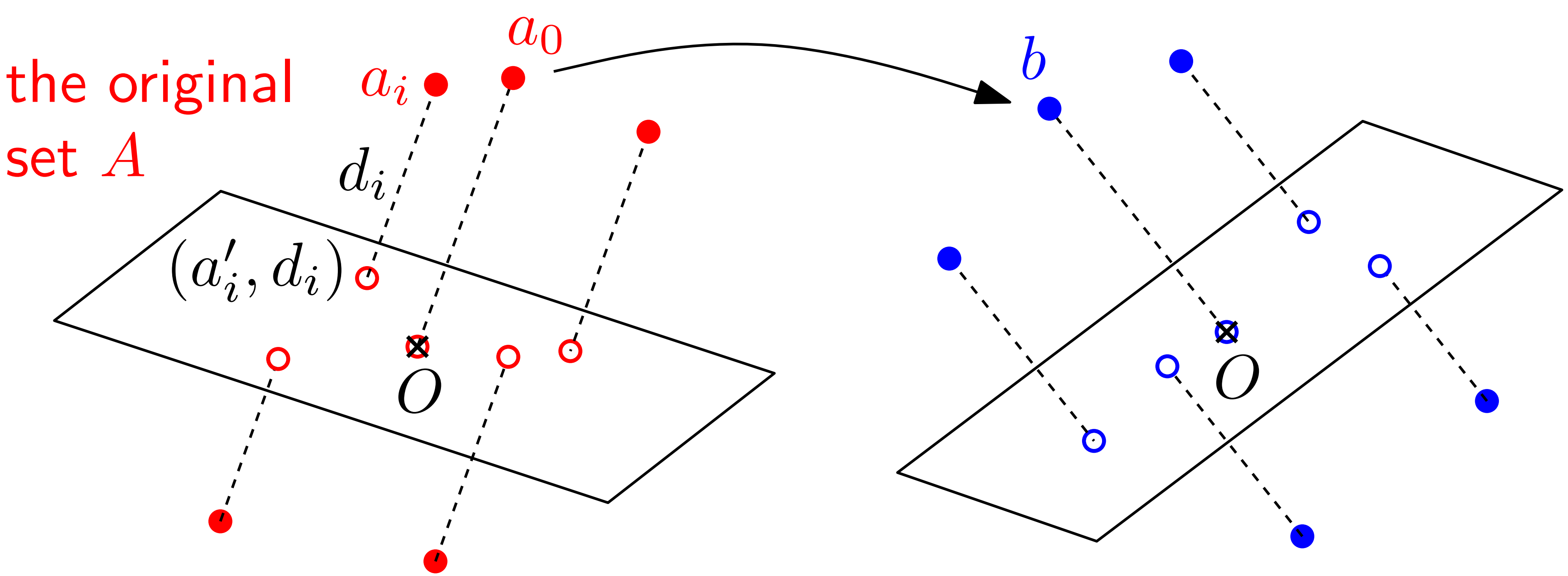
As soon as $|A'| = |B'| = k$ is small:
Choose a point $a_0 \in A'$ and try all k possibilities of mapping it to a point $b \in B'$.

Fixing $a_0 \mapsto b$ reduces the dimension by one.



Project perpendicular to Oa_0 and label projected points a_i' with the signed projection distance d_i as (a_i', d_i).

Pruning + Dimension Reduction

PRUNING:

Find some (geometric, combinatorial) characteristic that distinguishes points from each other.

Keep only the smallest equivalence class.

DIMENSION REDUCTION:

Reduce one d-dimensional problem to k problems of dimension $d-1$.

[M. D. Atkinson, J. Algorithms 1987, for $d = 3$]

Three Dimensions

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If the points lie in a plane or on a line
$\rightarrow$ DIMENSION REDUCTION.

Compute the convex hull $P(A)$.
If there are vertices of different degrees $\rightarrow$ PRUNE

The number n of vertices is reduced to $\leq n/2$. RESTART.

All n vertices have now degree 3, 4, or 5.
There are $f = \frac{n}{2} + 2$ or $f = n + 2$ or $f = \frac{3n}{2} + 2$ faces.

If the face degrees are not all equal
$\rightarrow$ switch to the centroids of the faces and PRUNE them.

n is reduced to $\leq \frac{3n}{4} + 1$. RESTART.

Now $P(A)$ must have the graph of a Platonic solid. $\rightarrow n \leq 20$.
$\rightarrow$ DIMENSION REDUCTION.

Three Dimensions

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If the points lie in a plane or on a line
$\rightarrow$ DIMENSION REDUCTION.

Compute the convex hull $P(A)$. ← $O(|A| \log |A|)$ time

If there are vertices of different degrees $\rightarrow$ PRUNE

The number n of vertices is reduced to $\leq n/2$. RESTART.

All n vertices have now degree 3, 4, or 5.
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If the face degrees are not all equal
$\rightarrow$ switch to the centroids of the faces and PRUNE them.

n is reduced to $\leq \frac{3n}{4} + 1$. RESTART.

TIME =
$O(n \log n) + O(\frac{3}{4} n \log \frac{3}{4} n) + O((\frac{3}{4})^2 n \log((\frac{3}{4})^2 n)) + \dots$
$= O(n \log n)$

Symmetry groups

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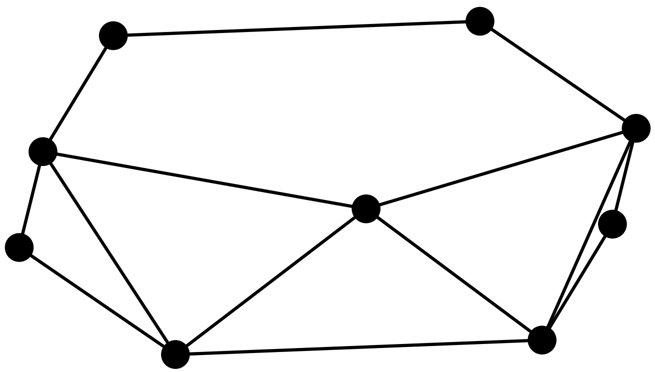
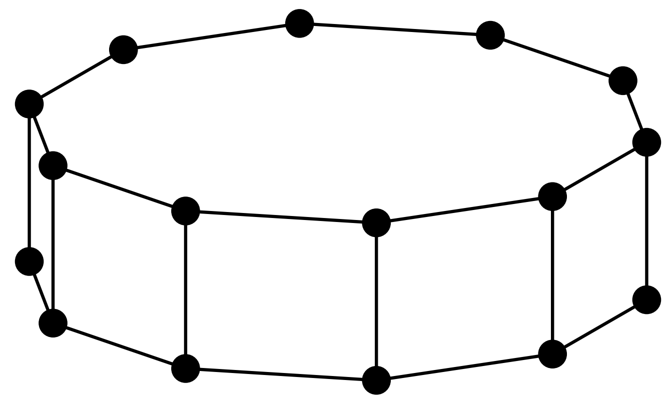
COROLLARY. The symmetry group of a finite full-dimensional point set in 3-space (= a discrete subgroup of $O(3)$) is

- the symmetry group of a Platonic solid,
- the symmetry group of a regular prism,
- or a subgroup of such a group.





The *point groups* (discrete subgroups of $O(3)$) are classified (Hessel's Theorem).

[F. Hessel 1830, M. L. Frankenheim 1826]

Point groups in higher dimensions

Is this true in higher dimensions?

¿The symmetry group of a finite full-dimensional point set in d-space (= a discrete subgroup of $O(d)$) is

- the symmetry group of a regular d-dimensional polytope:
  - a regular simplex
  - * a hypercube (or its dual, the crosspolytope)
  - a regular n-gon in two dimensions
  - a dodecahedron (or its dual, the icosahedron) in 3 d.
  - a 24-cell, or a 120-cell (or its dual, the 600-cell) in 4 d.
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

The symmetry groups of the root systems E_6, E_7, E_8 in 6, 7, and 8 dimensions might be counterexamples.

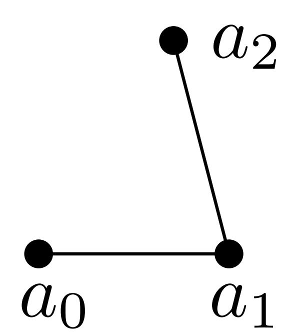
Dimension d

Dimension reduction without pruning:
Pick $a_0 \in A$. Try $a_0 \mapsto b$ for all $b \in B$ (n possibilities).
$\rightarrow O(n^{d-2} \log n)$ time

Improvement [Matoušek $\approx$ 1998]:
Consider all *closest pairs* of A and B. Each point belongs to $\leq C_d$ closest pairs. (packing argument, the kissing number).
$\implies O(n)$ closest pairs.

Pick a closest pair $a_0 a_1 \in A$. Try $(a_0, a_1) \mapsto (b, b')$ for all closest pairs $(b, b') \in B$.
$O(n)$ possibilities, reducing the dimension by *two*.
$\rightarrow O(n^{\lfloor d/2 \rfloor} \log n)$ time

Further improvement: Find a "closest triplet" . . .



Life in four dimensions

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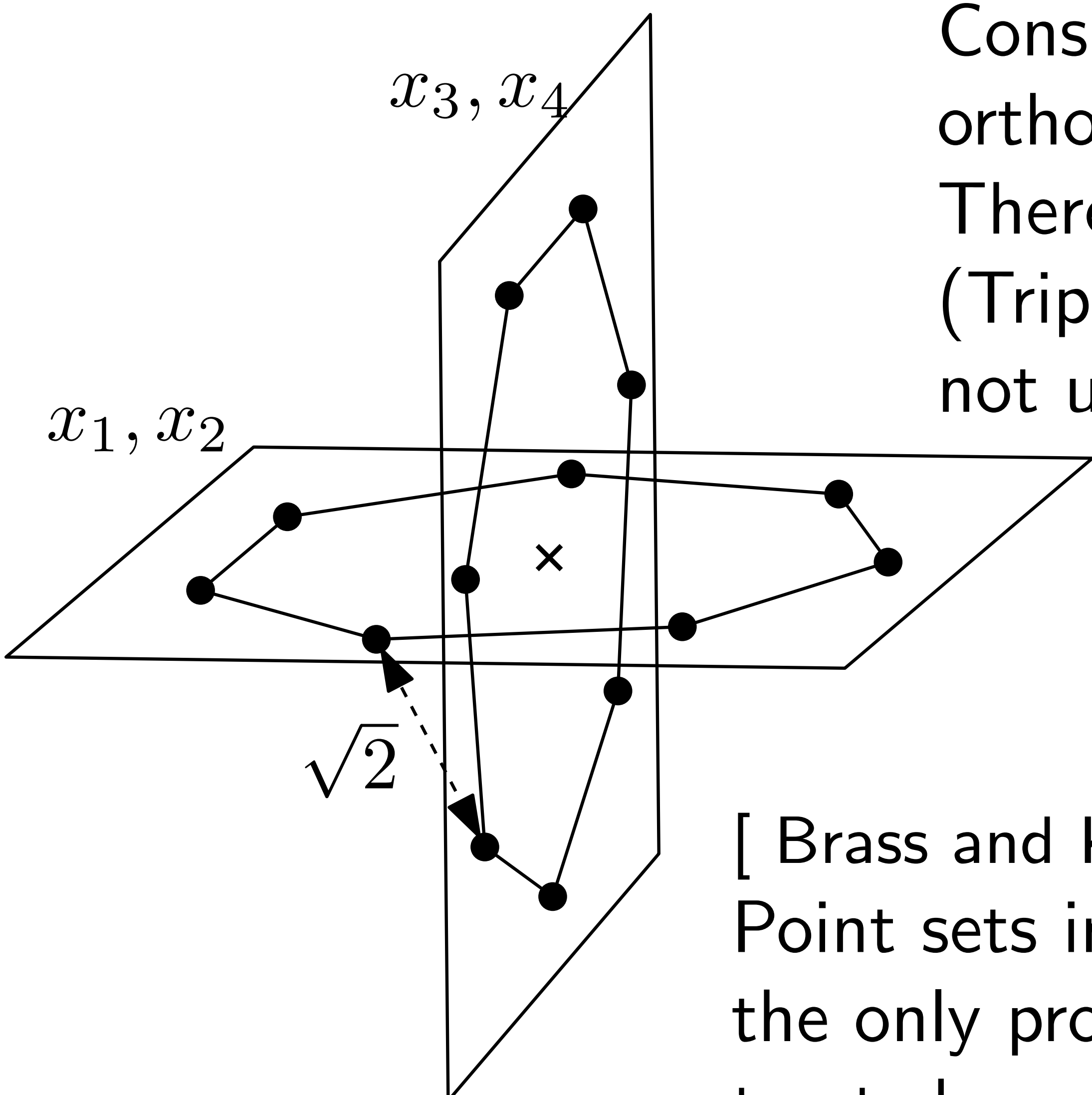


Consider two regular n-gons in orthogonal planes.
There are $O(n^2)$ "closest triplets".
(Triplets on the same n-gon are not useful.)

The convex hull has $\Theta(n^2)$ edges and facets.

[Brass and Knauer 2002]
Point sets in orthogonal subspaces are the only problematic case; they can be treated specially.
$\to O(n^{\lceil d/3 \rceil} \log n)$ time,
by looking at closest pairs

Life in four dimensions

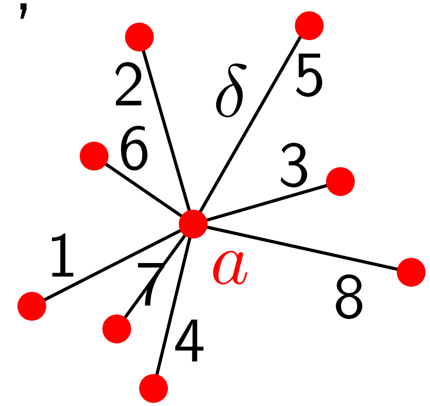
Compute the closest pair graph

$$G(A) = (A, \{\, aa' : \|a - a'\| = \delta \,\})$$

where δ is the distance of the closest pair.

By the PRUNING principle, we can assume that all points look locally the same:

- same distance from the origin. (A lies on the 3-sphere $\mathbb{S}^3$.)
- All points have congruent neighborhoods in $G(A)$.
  (The neighbors of a lie on a 2-sphere in $\mathbb{S}^3$;
  There are at most 12 neighbors.)

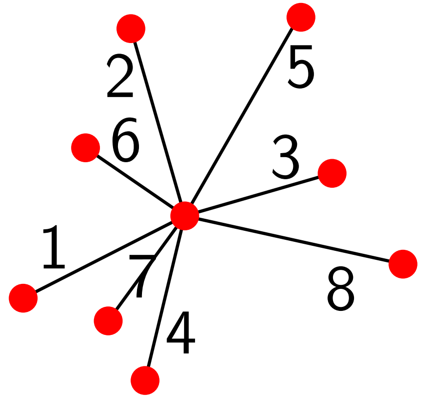


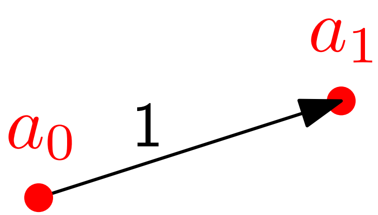
CASE 1.
The vertex figure has no symmetries.

Life in four dimensions

CASE 1.
The vertex figure has no symmetries.

Follow the sequence of "1"-neighbors





Life in four dimensions

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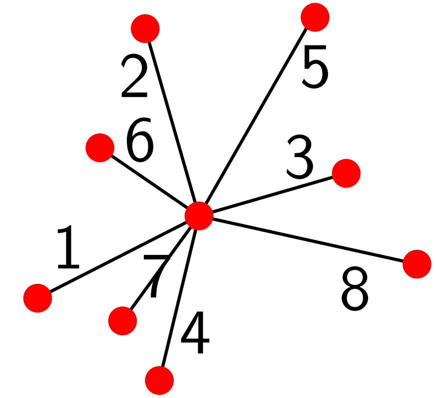

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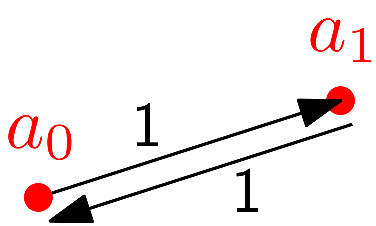

- If the "1"-edges form a matching:
  Take the midpoints of these edges. $n \to n/2$.

Life in four dimensions

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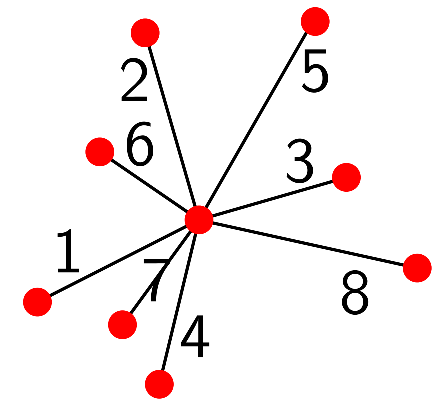
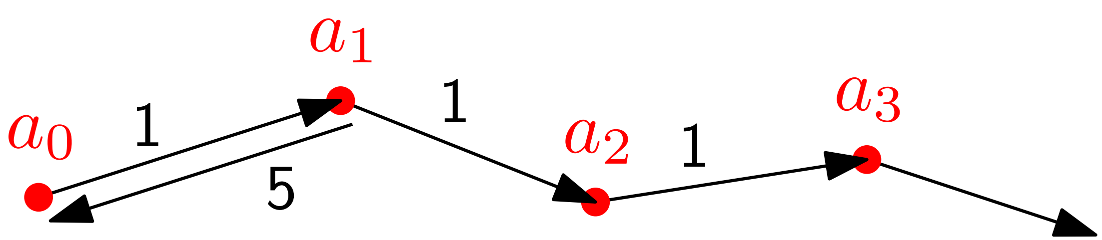
CASE 1.

The vertex figure has no symmetries.



Follow the sequence of "1"-neighbors



$(a_0, a_1, a_2) \cong (a_1, a_2, a_3) \cong (a_2, a_3, a_4) \cong \cdots$

A is partitioned into cycles of the same length.

- Cycles are short $\rightarrow$ their centroid is nonzero; replace them by their centroids.
- a_0, a_1, a_2 lie on a circle through the origin special situation: all these circles are parallel; or there is a bounded number of circles.
- a_0, a_1, a_2 span a hyperplane $\rightarrow$ take their normals

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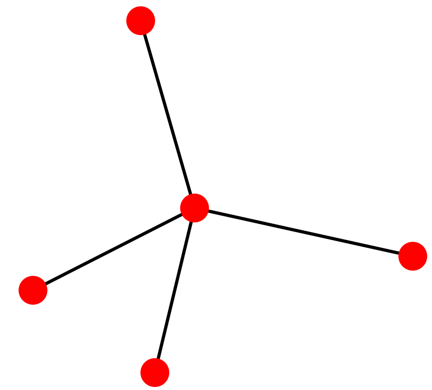
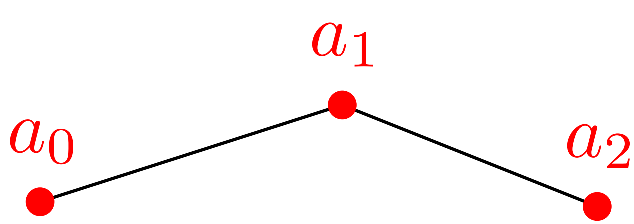


CASE 2.
The vertex figure has, say, tetrahedral symmetry.

Start with a path of length 3,
by extending a_0a_1 "as straight as possible"
(Finitely many starting patterns).
a_0, a_1, a_2 span a hyperplane.



Life in four dimensions

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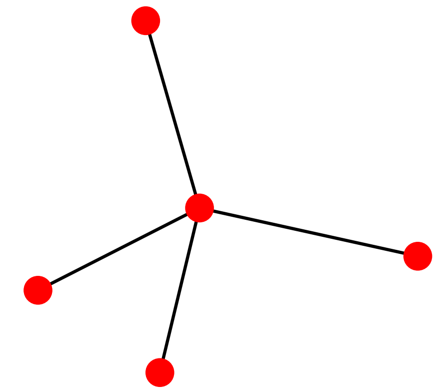

Start with a path of length 3,
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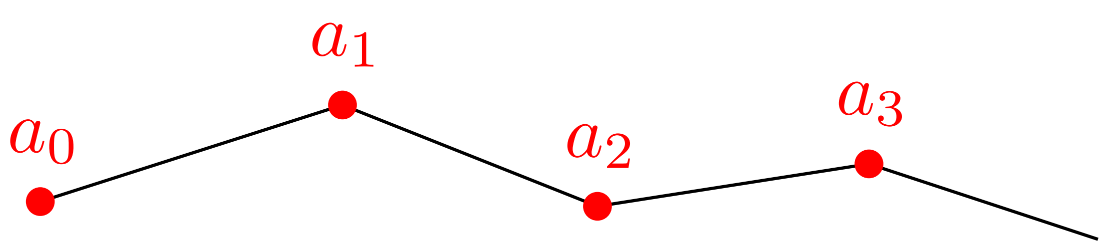


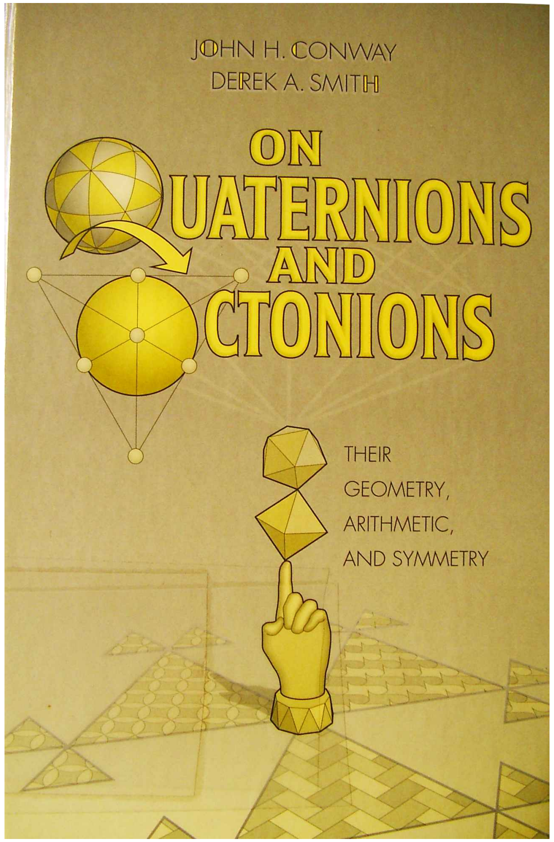
Extend the path by

$$(a_0, a_1, a_2) \cong (a_1, a_2, a_3) \cong (a_2, a_3, a_4) \cong \cdots$$

$\rightarrow A$ is partitioned into cycles of the same length.

The four-dimensional point groups

- [W. Threlfall and H. Seifert, Math. Annalen, 1931, 1933] enumerated discrete subgroups of $SO(4)$ (determinant $+1$)
- [J. Conway and D. Smith 2003] complete enumeration of point groups



4d-rotation $T \leftrightarrow$ pair (R, S) of 3d-rotations.
(for example, via quaternions)

Goursat's Lemma: [É. Goursat 1890]
Pairs of 3d point groups
$+$ additional information
$\rightarrow$ 4d point groups

- The groups generated by reflections (Coxeter groups) have been enumerated up to 8 dimensions.
  [Norman Johnson, unpublished book manuscript]

The four-dimensional point groups

umerate the groups. (see section 3

Group	Generators
$\pm[I\times O]$	$[i_I,1],[\omega,1],[1,i_O],[1,\omega];$
$\pm[I\times T]$	$[i_I,1],[\omega,1],[1,i],[1,\omega];$
$\pm[I\times D_{2n}]$	$[i_I,1],[\omega,1],[1,e_n],[1,j];$
$\pm[I\times C_n]$	$[i_I,1],[\omega,1],[1,e_n];$
$\pm[O\times T]$	$[i_O,1],[\omega,1],[1,i],[1,\omega];$
$\pm[O\times D_{2n}]$	$[i_O,1],[\omega,1],[1,e_n],[1,j];$
$\pm\frac{1}{2}[O\times D_{2n}]$	$[i,1],[\omega,1],[1,e_n];[i_O,j]$
$\pm\frac{1}{2}[O\times \overline{D}_{4n}]$	$[i,1],[\omega,1],[1,e_n],[1,j];[i_O,e_{2n}]$
$\pm\frac{1}{6}[O\times D_{6n}]$	$[i,1],[j,1],[1,e_n];[i_O,j],[\omega,e_{3n}]$
$\pm[O\times C_n]$	$[i_O,1],[\omega,1],[1,e_n];$
$\pm\frac{1}{2}[O\times C_{2n}]$	$[i,1],[\omega,1],[1,e_n];[i_O,e_{2n}]$
$\pm[T\times D_{2n}]$	$[i,1],[\omega,1],[1,e_n],[1,j];$
$\pm[T\times C_n]$	$[i,1],[\omega,1],[1,e_n];$
$\pm\frac{1}{3}[T\times C_{3n}]$	$[i,1],[1,e_n];[\omega,e_{3n}]$
$\pm\frac{1}{2}[D_{2m}\times \overline{D}_{4n}]$	$[e_m,1],[1,e_n],[1,j];[j,e_{2n}]$
$\pm[D_{2m}\times C_n]$	$[e_m,1],[j,1],[1,e_n];$
$\pm\frac{1}{2}[D_{2m}\times C_{2n}]$	$[e_m,1],[1,e_n];[j,e_{2n}]$
$+\frac{1}{2}[D_{2m}\times C_{2n}]$	$-\ ,\ -\ ;\ +$
$\pm\frac{1}{2}[\overline{D}_{4m}\times C_{2n}]$	$[e_m,1],[j,1],[1,e_n];[e_{2m},e_{2n}]$

Table 4.1. Chiral groups, I. These are most of the "metachiral" groups—s 4.6—some others appear in the last few lines of Table 4.2.

Table 4.1. The *chiral* groups (groups of orientation-preserving orthogonal transformations)

[Conway and Smith 2003]

← both m and n must be odd.

The four-dimensional point groups

Table 4.2.
The *chiral* groups
(continued)

Group	Generators	Coxeter Name
$\pm[I\times I]$	$[i_I,1],[\omega,1],[1,i_I],[1,\omega];$	$[3,3,5]^+$
$\pm\frac{1}{60}[I\times I]$	$;[\omega,\omega],[i_I,i_I]$	$2.[3,5]^+$
$+\frac{1}{60}[I\times I]$	$;\ +\ ,\ +$	$[3,5]^+$
$\pm\frac{1}{60}[I\times\overline{I}]$	$;[\omega,\omega],[i_I,i'_I]$	$2.[3,3,3]^+$
$+\frac{1}{60}[I\times\overline{I}]$	$;\ +\ ,\ +$	$[3,3,3]^+$
$\pm[O\times O]$	$[i_O,1],[\omega,1],[1,i_O],[1,\omega];$	$[3,4,3]^+:2$
$\pm\frac{1}{2}[O\times O]$	$[i,1],[\omega,1],[1,i],[1,\omega];[i_O,i_O]$	$[3,4,3]^+$
$\pm\frac{1}{6}[O\times O]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\omega],[i_O,i_O]$	$[3,3,4]^+$
$\pm\frac{1}{24}[O\times O]$	$;[\omega,\omega],[i_O,i_O]$	$2.[3,4]^+$
$+\frac{1}{24}[O\times O]$	$;\ +\ ,\ +$	$[3,4]^+$
$+\frac{1}{24}[O\times\overline{O}]$	$;\ +\ ,\ -$	$[2,3,3]^+$
$\pm[T\times T]$	$[i,1],[\omega,1],[1,i],[1,\omega];$	$[^+3,4,3^+]$
$\pm\frac{1}{3}[T\times T]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\omega]$	$[^+3,3,4^+]$
$\cong\pm\frac{1}{3}[T\times\overline{T}]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\overline{\omega}]$	”
$\pm\frac{1}{12}[T\times T]$	$;[\omega,\omega],[i,i]$	$2.[3,3]^+$
$\cong\pm\frac{1}{12}[T\times\overline{T}]$	$;[\omega,\overline{\omega}],[i,-i]$	”
$+\frac{1}{12}[T\times T]$	$;\ +\ ,\ +$	$[3,3]^+$
$\cong+\frac{1}{12}[T\times\overline{T}]$	$;\ +\ ,\ +$	”
$\pm[D_{2m}\times D_{2n}]$	$[e_m,1],[j,1],[1,e_n],[1,j];$	
$\pm\frac{1}{2}[\overline{D}_{4m}\times\overline{D}_{4n}]$	$[e_m,1],[j,1],[1,e_n],[1,j];[e_{2m},e_{2n}]$	
$\pm\frac{1}{4}[D_{4m}\times\overline{D}_{4n}]$	$[e_m,1],[1,e_n];[e_{2m},j],[j,e_{2n}]$	Conditions
$+\frac{1}{4}[D_{4m}\times\overline{D}_{4n}]$	$-\ ,\ -\ ;\ +\ ,\ +$	m,n odd
$\pm\frac{1}{2f}[D_{2mf}\times D^{(s)}_{2nf}]$	$[e_m,1],[1,e_n];[e_{mf},e^s_{nf}],[j,j]$	$(s,f)=1$
$+\frac{1}{2f}[D_{2mf}\times D^{(s)}_{2nf}]$	$-\ ,\ -\ ;\ +\ ,\ +$	m,n odd, $(s,2f)=1$
$\pm\frac{1}{f}[C_{mf}\times C^{(s)}_{nf}]$	$[e_m,1],[1,e_n];[e_{mf},e^s_{nf}]$	$(s,f)=1$
$+\frac{1}{f}[C_{mf}\times C^{(s)}_{nf}]$	$-\ ,\ -\ ;\ +$	m,n odd, $(s,2f)=1$

Table 4.2. Chiral groups, II. These groups are mostly "orthochiral," with a few "parachiral" …

The four-dimensional point groups

Freie Universität Berlin

Group	Extending element	Coxeter Name
$\pm[I \times I] \cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I \times I] \cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I \times I] \cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I \times \overline{I}] \cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{60}[I \times \overline{I}] \cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
$\pm[O \times O] \cdot 2$	$*$	$[3,4,3]:2$
$\pm\frac{1}{2}[O \times O] \cdot 2$ or $\overline{2}$	$*$ or $*[1, i_O]$	$[3,4,3]$ or $[3,4,3]^{+}\cdot 2$
$\pm\frac{1}{6}[O \times O] \cdot 2$	$*$	$[3,3,4]$
$\pm\frac{1}{24}[O \times O] \cdot 2$	$*$	$2.[3,4]$
$+\frac{1}{24}[O \times O] \cdot 2_3$ or 2_1	$*$ or $-*$	$[3,4]$ or $[3,4]^\circ$
$+\frac{1}{24}[O \times \overline{O}] \cdot 2_3$ or 2_1	$*$ or $-*$	$[2,3,3]^\circ$ or $[2,3,3]$
$\pm[T \times T] \cdot 2$	$*$	$[3,4,3^+]$
$\pm\frac{1}{3}[T \times T] \cdot 2$	$*$	$[^+3,3,4]$
$\pm\frac{1}{3}[T \times \overline{T}] \cdot 2$	$*$	$[3,3,4^+]$
$\pm\frac{1}{12}[T \times T] \cdot 2$	$*$	$2.[^+3,4]$
$\pm\frac{1}{12}[T \times \overline{T}] \cdot 2$	$*$	$2.[3,3]$
$+\frac{1}{12}[T \times T] \cdot 2_3$ or 2_1	$*$ or $-*$	$[^+3,4]$ or $[^+3,4]^\circ$
$+\frac{1}{12}[T \times \overline{T}] \cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3]^\circ$ or $[3,3]$
$\pm[D_{2n} \times D_{2n}] \cdot 2$	$*$	
$\pm\frac{1}{2}[\overline{D}_{4n} \times \overline{D}_{4n}] \cdot 2$ or $\overline{2}$	$*$ or $*[1, e_{2n}]$	
$\pm\frac{1}{4}[D_{4n} \times \overline{D}_{4n}] \cdot 2$	$*$	<u>Conditions</u>
$+\frac{1}{4}[D_{4n} \times \overline{D}_{4n}] \cdot 2_3$ or 2_1	$*$ or $-*$	n odd
$\pm\frac{1}{2f}[D_{2nf} \times D_{2nf}^{(s)}] \cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e_{2nf}^{\alpha}, e_{2nf}^{\alpha s+\beta f}]$ or $*[1,j]$	*See*
$+\frac{1}{2f}[D_{2nf} \times D_{2nf}^{(s)}] \cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e_{2nf}^{\alpha}, e_{2nf}^{\alpha s+\beta f}]$ or $*[1,j]$	*Text*
$\pm\frac{1}{f}[C_{nf} \times C_{nf}^{(s)}] \cdot 2^{(\gamma)}$	$*[1, e_{2nf}^{\gamma(f,s+1)}]$	*in*
$+\frac{1}{f}[C_{nf} \times C_{nf}^{(s)}] \cdot 2^{(\gamma)}$	$*[1, e_{2nf}^{\gamma(f,s+1)}]$	*Appendix*

Table 4.3. Achiral groups.

Table 4.3.
The *achiral* groups

The four-dimensional point groups

Group	Extending element	Coxeter Name
$\pm[I\times I]\cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I\times I]\cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I\times I]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I\times \overline{I}]\cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{60}[I\times \overline{I}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
$\pm[O\times O]\cdot 2$	$*$	$[3,4,3]:2$
$\pm\frac{1}{2}[O\times O]\cdot 2$ or $\overline{2}$	$*$ or $*[1,i_O]$	$[3,4,3]$ or $[3,4,3]^+\cdot 2$
$\pm\frac{1}{6}[O\times O]\cdot 2$	$*$	$[3,3,4]$
$\pm\frac{1}{24}[O\times O]\cdot 2$	$*$	$2.[3,4]$
$+\frac{1}{24}[O\times O]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,4]$ or $[3,4]^\circ$
$+\frac{1}{24}[O\times \overline{O}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[2,3,3]^\circ$ or $[2,3,3]$
$\pm[T\times T]\cdot 2$	$*$	$[3,4,3^+]$
$\pm\frac{1}{3}[T\times T]\cdot 2$	$*$	$[^+3,3,4]$
$\pm\frac{1}{3}[T\times \overline{T}]\cdot 2$	$*$	$[3,3,4^+]$
$\pm\frac{1}{12}[T\times T]\cdot 2$	$*$	$2.[^+3,4]$
$\pm\frac{1}{12}[T\times \overline{T}]\cdot 2$	$*$	$2.[3,3]$
$+\frac{1}{12}[T\times T]\cdot 2_3$ or 2_1	$*$ or $-*$	$[^+3,4]$ or $[^+3,4]^\circ$
$+\frac{1}{12}[T\times \overline{T}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3]^\circ$ or $[3,3]$
$\pm[D_{2n}\times D_{2n}]\cdot 2$	$*$	
$\pm\frac{1}{2}[\overline{D}_{4n}\times \overline{D}_{4n}]\cdot 2$ or $\overline{2}$	$*$ or $*[1,e_{2n}]$	
$\pm\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2$	$*$	Conditions
$+\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2_3$ or 2_1	$*$ or $-*$	n odd
$\pm\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*See*
$+\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*Text*
$\pm\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1, e^{\gamma(f,s+1)}_{2nf}]$	*in*
$+\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1, e^{\gamma(f,s+1)}_{2nf}]$	*Appendix*

Table 4.3. Achiral groups.

Table 4.3.
The *achiral* groups

- Visualize these groups:
Schlegel diagram of a 4-polytope which has these symmetries.

The four-dimensional point groups

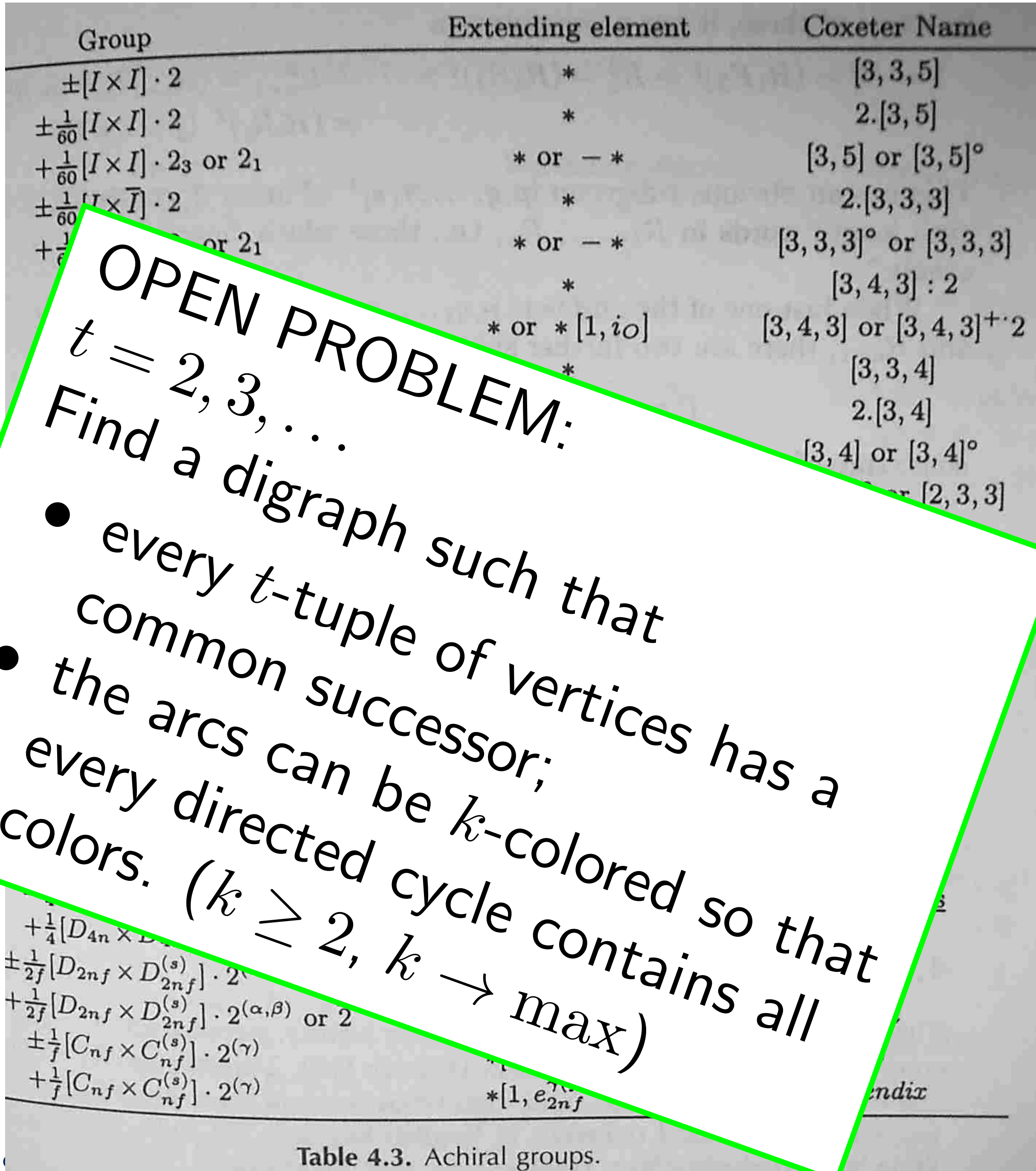
Group	Extending element	Coxeter Name
$\pm[I\times I]\cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I\times I]\cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I\times I]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I\times \overline{I}]\cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{60}[I\times \overline{I}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
$\pm[O\times O]\cdot 2$	$*$	$[3,4,3]:2$
$\pm\frac{1}{2}[O\times O]\cdot 2$ or $\overline{2}$	$*$ or $*[1,i_O]$	$[3,4,3]$ or $[3,4,3]^{+}\cdot 2$
$\pm\frac{1}{6}[O\times O]\cdot 2$	$*$	$[3,3,4]$
$\pm\frac{1}{24}[O\times O]\cdot 2$	$*$	$2.[3,4]$
$+\frac{1}{24}[O\times O]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,4]$ or $[3,4]^\circ$
$+\frac{1}{24}[O\times \overline{O}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[2,3,3]^\circ$ or $[2,3,3]$
$\pm[T\times T]\cdot 2$	$*$	$[3,4,3^+]$
$\pm\frac{1}{3}[T\times T]\cdot 2$	$*$	$[^+3,3,4]$
$\pm\frac{1}{3}[T\times \overline{T}]\cdot 2$	$*$	$[3,3,4^+]$
$\pm\frac{1}{12}[T\times T]\cdot 2$	$*$	$2.[^+3,4]$
$\pm\frac{1}{12}[T\times \overline{T}]\cdot 2$	$*$	$2.[3,3]$
$+\frac{1}{12}[T\times T]\cdot 2_3$ or 2_1	$*$ or $-*$	$[^+3,4]$ or $[^+3,4]^\circ$
$+\frac{1}{12}[T\times \overline{T}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3]^\circ$ or $[3,3]$
$\pm[D_{2n}\times D_{2n}]\cdot 2$	$*$	
$\pm\frac{1}{2}[\overline{D}_{4n}\times \overline{D}_{4n}]\cdot 2$ or $\overline{2}$	$*$ or $*[1,e_{2n}]$	
$\pm\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2$	$*$	Conditions
$+\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2_3$ or 2_1	$*$ or $-*$	n odd
$\pm\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*See*
$+\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*Text*
$\pm\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1,e^{\gamma(f,s+1)}_{2nf}]$	*in*
$+\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1,e^{\gamma(f,s+1)}_{2nf}]$	*Appendix*

Table 4.3. Achiral groups.

Table 4.3.
The *achiral* groups

- Visualize these groups:
  Schlegel diagram of a 4-polytope which has these symmetries.
- Then go to 5d and higher

The four-dimensional point groups



Group	Extending element	Coxeter Name
$\pm[I\times I]\cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I\times I]\cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I\times I]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I\times \bar{I}]\cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{6}$ … or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
	$*$	$[3,4,3]:2$
	$*$ or $*[1,io]$	$[3,4,3]$ or $[3,4,3]^{+}\cdot 2$
	$*$	$[3,3,4]$
		$2.[3,4]$
		$[3,4]$ or $[3,4]^\circ$
		… or $[2,3,3]$
$+\frac{1}{4}[D_{4n}\times$ …		
$\pm\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2$ …		
$+\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or 2		
$\pm\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$		
$+\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1,e_{2nf}$ …	…endix

Table 4.3. Achiral groups.

OPEN PROBLEM:
$t=2,3,\ldots$
Find a digraph such that
- every t-tuple of vertices has a common successor;
- the arcs can be k-colored so that every directed cycle contains all colors. ($k\geq 2$, $k\to \max$)

Table 4.3.
The *achiral* groups

- Visualize these groups: Schlegel diagram of a 4-polytope which has these symmetries.
- Then go to 5d and higher