Realization of Three-Dimensional Polytopes

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joint work with
Ares Ribó Mor and André Schulz

Graphs of polytopes

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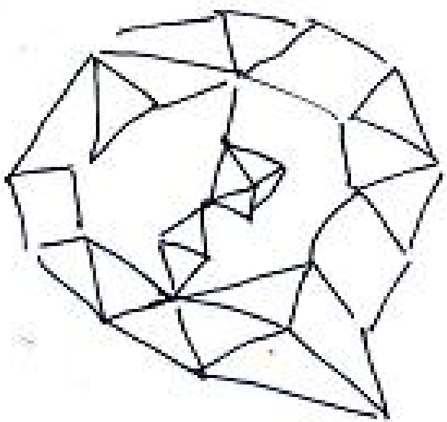
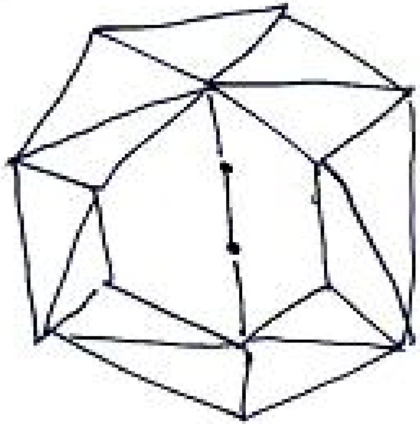
The graph of a 3-polytope is 3-connected.

(Removing 2 vertices does not disconnect the graph.)





The intersection of two faces is an edge, a vertex, or empty.

Graphs of polytopes

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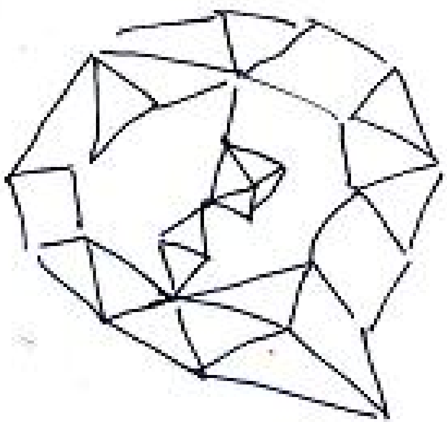
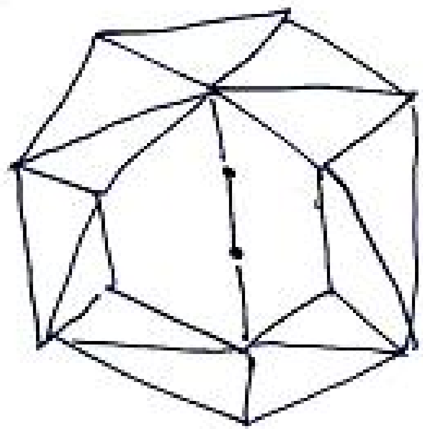




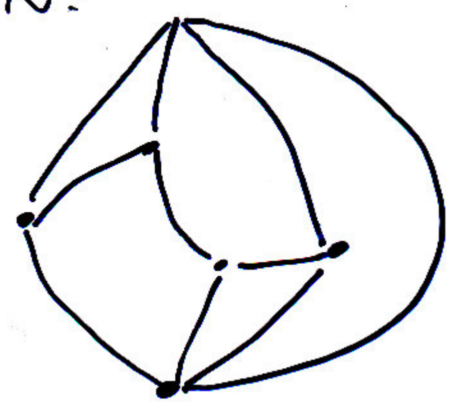
The intersection of two faces is an edge, a vertex, or empty.

Theorem (Steinitz)

Every 3-connected planar graph is the graph of a 3-polytope.

Polytope construction

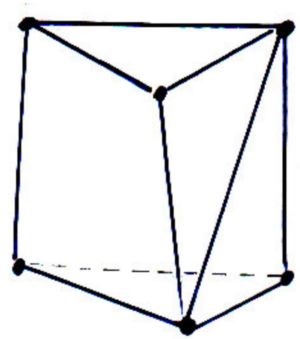
GIVEN:



a combinatorial type of 3-polytope (convex)
(= a 3-connected planar graph)

[+ additional data]

"CONSTRUCT:"



a geometric realization of the polytope.

Polytope construction

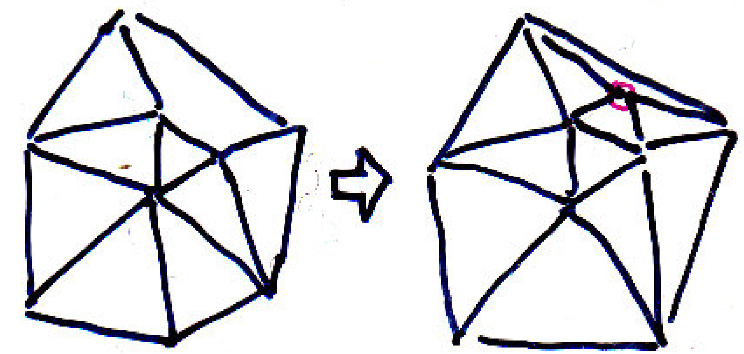
GIVEN a combinatorial type of convex 3-polytope
FIND a geometric realization with . . .
. . . certain properties

Polytope construction

GIVEN a combinatorial type of convex 3-polytope
FIND a geometric realization with . . .
. . . certain properties

Two approaches:

1. *inductive*: start with the simplest polytopes and make local modifications



2. *direct*: obtain the polytope as a result of

– a system of equations
– an optimization problem
– an existential proof

Polytope construction

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GIVEN a combinatorial type of convex 3-polytope
FIND a geometric realization with . . .
. . . small integer vertex coordinates

Two approaches:

1. *inductive*:
   - Steinitz (1922): coordinates $\le 2^{\exp(n)}$
   - Das & Goodrich (1997): coordinates $\le 2^{\mathrm{poly}(n)}$ for *triangulated* polytopes
2. *direct*: obtain the polytope as a result of
   - – a system of equations ← • Onn,Sturmfels('94): $\le n^{169n^3}$
   - – an optimization problem • Richter-Gebert('96): $\le 2^{20n^2}$
   - – an existential proof • Ribó, Rote, Schulz (2008): $\le 2^{8n}$

Polytope construction

GIVEN a combinatorial type of convex 3-polytope
FIND a geometric realization with . . .
. . . all vertices on the unit sphere (an *inscribed* polytope)

(cf. Delaunay triangulation)

Two approaches:

1. *inductive*:

2. *direct*: obtain the polytope as a result of

– a system of equations
– an optimization problem
– an existential proof

Rivin, Hodgson, Smith (1993): test inscribability in polynomial time

Polytope construction

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GIVEN a combinatorial type of convex 3-polytope
FIND a geometric realization with . . .
. . . all edges tangent to the unit sphere

(a *midscribed* polytope)

(cf. circle packings)

Two approaches:

1. *inductive*:

2. *direct*: obtain the polytope as a result of

– a system of equations ← Thurston's algorithm, Brightwell & Scheinerman

– an optimization problem ← Y. Colin de Verdière

– an existential proof ← Koebe-Andreyev-Thurston Theorem

Polytope construction

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face normals and face areas

GIVEN ~~a combinatorial type~~ of convex 3-polytope

FIND a geometric realization with . . .

. . . these face areas and face normals

Two approaches:

1. *inductive*:

2. *direct*: obtain the polytope as a result of

– a system of equations

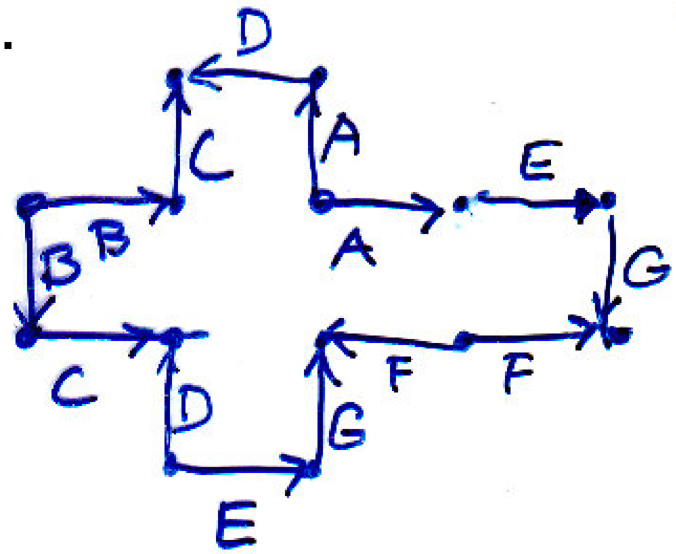
– an optimization problem Minkowski (~1897)

– an existential proof

Polytope construction

GIVEN ~~a combinatorial type~~ metric on the surface of convex 3-polytope (a *net*)

FIND a geometric realization with . . .



Two approaches:

1. *inductive*:

2. *direct*: obtain the polytope as a result of

– a system of equations Sabitov (1990)

– an optimization problem Bobenko & Izmestiev (2008)

– an existential proof Alexandrov (~1930)

Polytope construction

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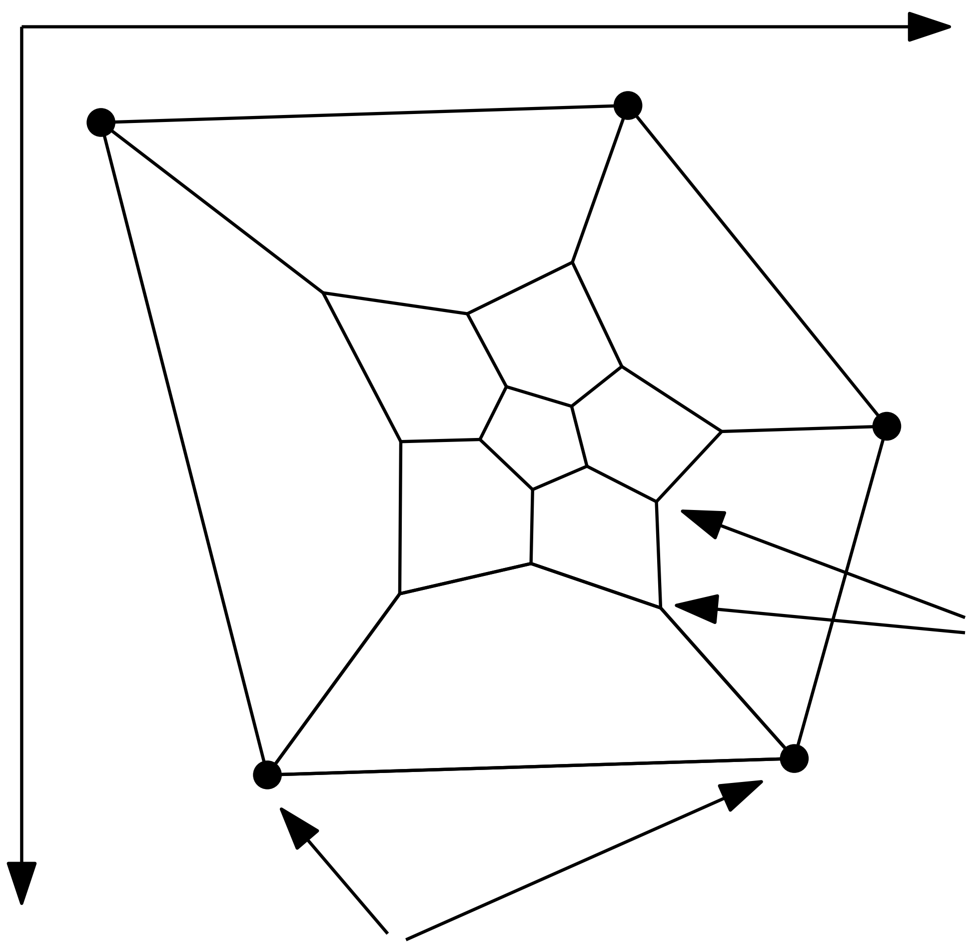
GIVEN a combinatorial type of convex 3-polytope
FIND a geometric realization with . . .
. . . all edge lengths rational

Two approaches:
1. *inductive*:

2. *direct*: obtain the polytope as a result of
– a system of equations
– an optimization problem
– an existential proof

OPEN

Tutte embedding



Tutte embedding (spiderweb embedding, equilibrium embedding)

All edges are springs with elasticity constant $\omega_{ij} = \omega_{ji} = 1$, obeying Hooke's law.

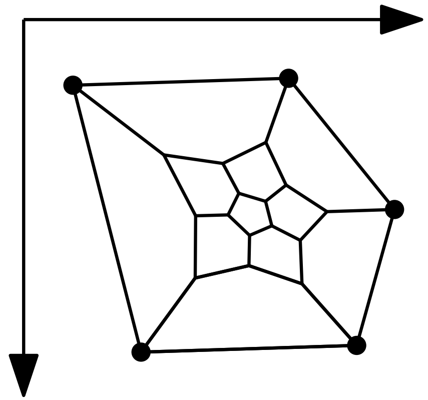
inner vertices are in *equilibrium*

fix *boundary* vertices $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$

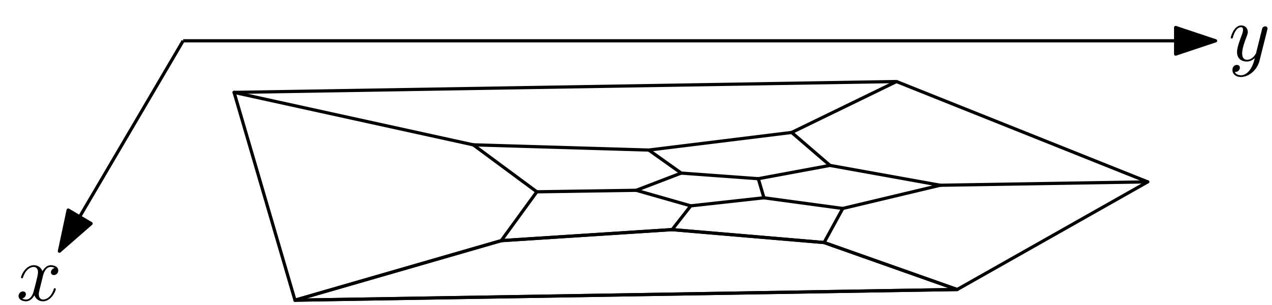
$\rightarrow$ drawing is planar.

Tutte (1961)

Tutte embedding



Lifting to 3-space (Maxwell–Cremona correspondence)



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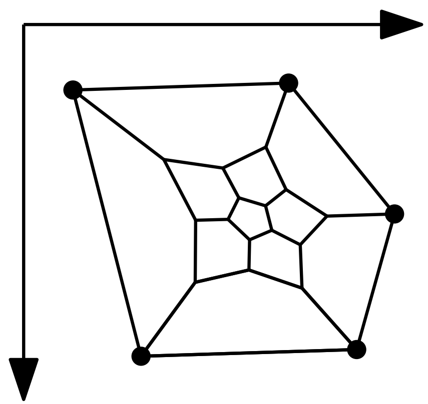

Lifting to 3-space
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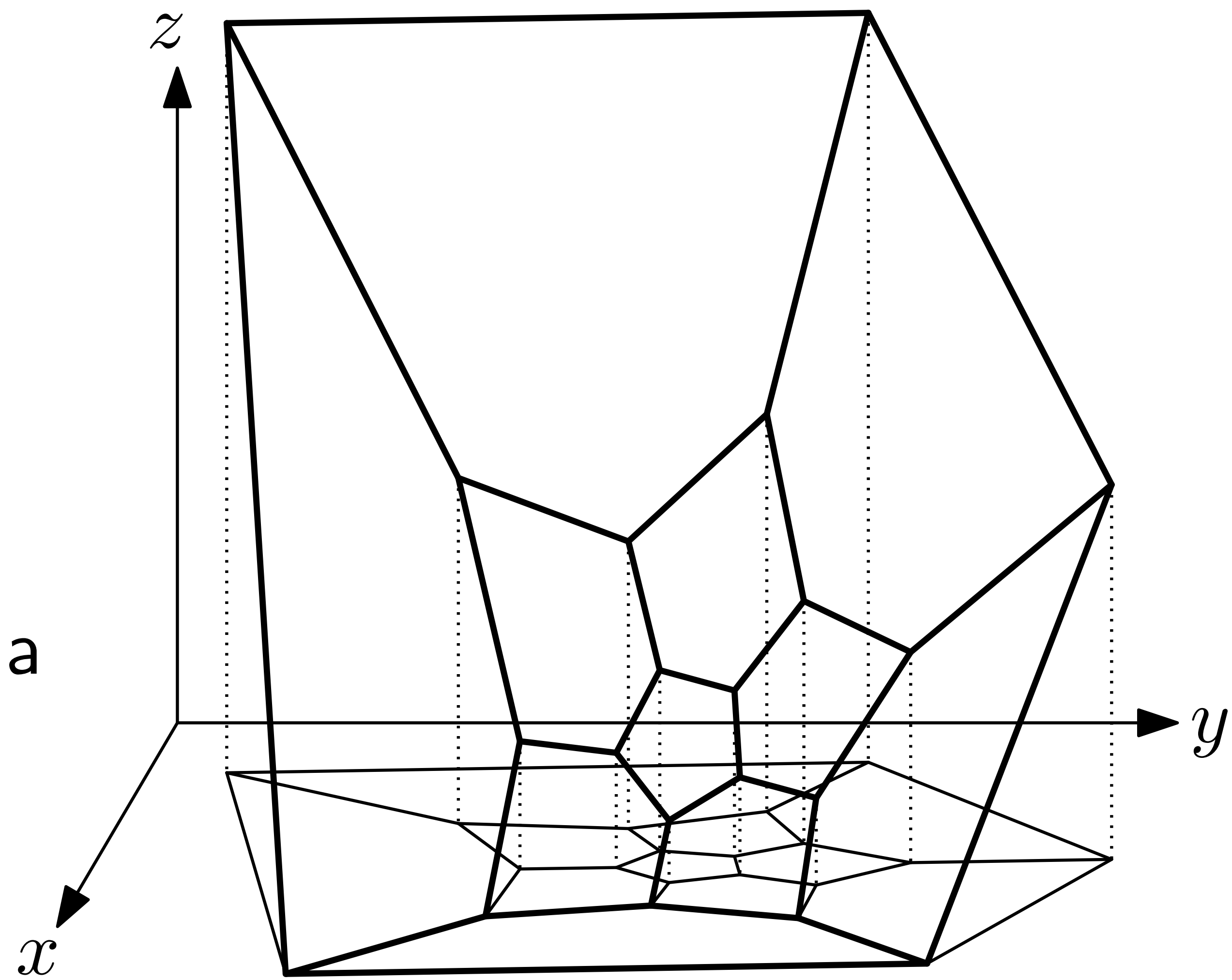
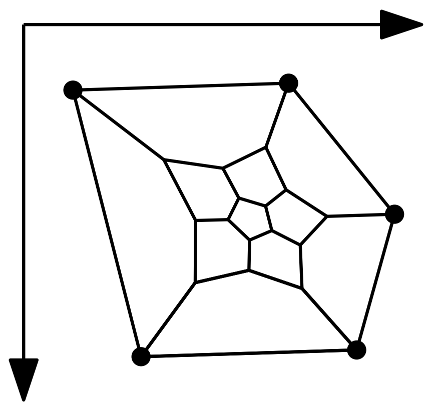


Tutte embedding



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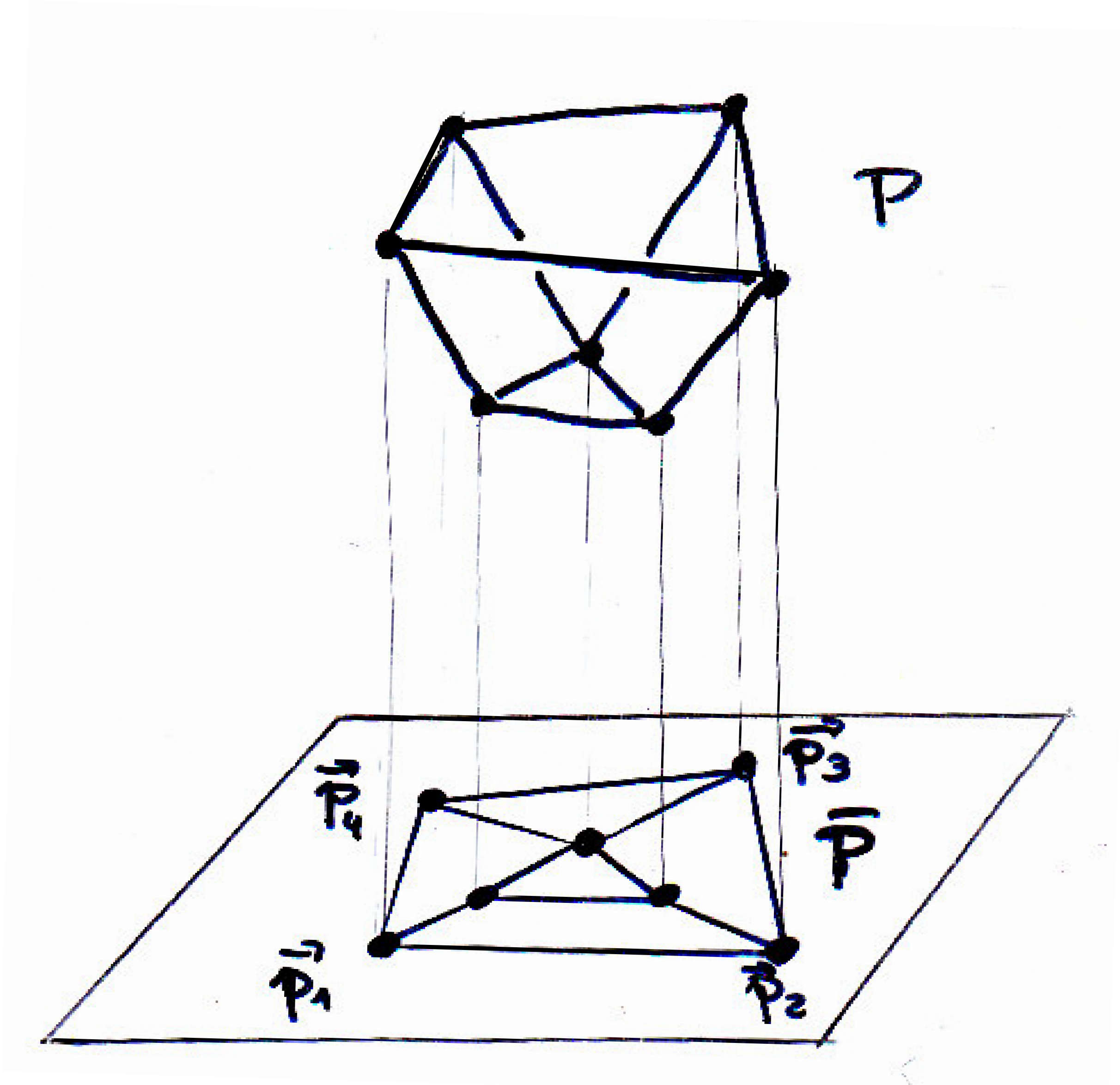


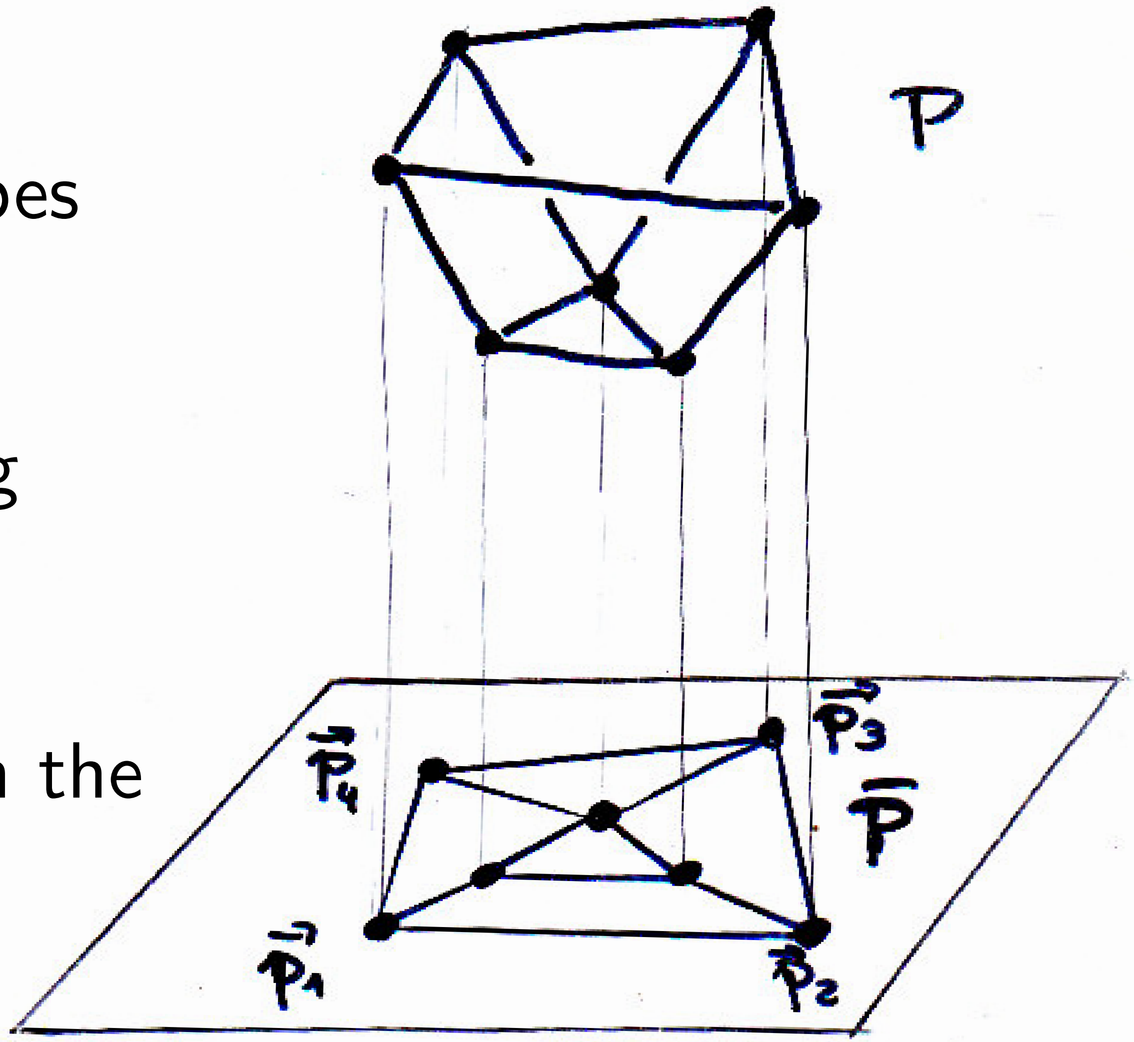
Maxwell-Cremona correspondence

between . . .

- 3-dimensional polytopes

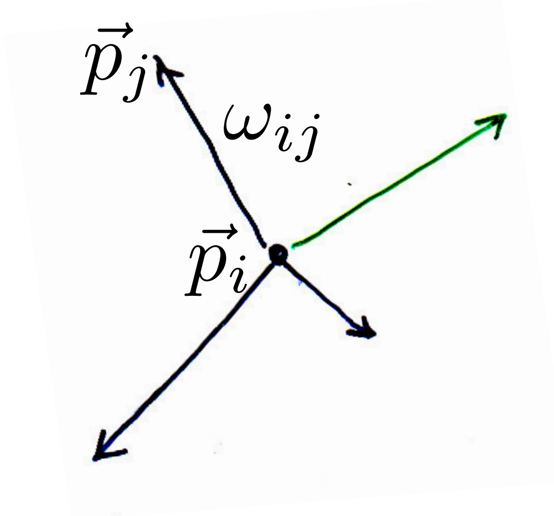
projection ↓ ↑ lifting

- equilibrium stresses in the plane projection



Maxwell (1864), Whiteley (1982)

Equilibrium



For every i: $\sum_{j \sim i} w_{ij} (\vec{p}_j - \vec{p}_i) = 0$ $\quad \left(\sum_{j \sim i} w_{ij}\right) \vec{p}_i = \sum_{j \sim i} (w_{ij} \cdot \vec{p}_j)$

$\vec{p}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ .... two separate systems for x_i and for y_i

(weighted) Laplacian matrix $L = \begin{pmatrix} \ddots & 0 & -w_{ij} \\ 0 & \ddots & 0 \\ -w_{ij} & 0 & \ddots \end{pmatrix}$

$\ell_{ii} = \sum_{j \sim i} w_{ij}$

remove rows (equations) and columns (variables) corresponding to boundary vertices $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_k$

$$\bar{L} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$$

$$\bar{L} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$$

Equilibrium

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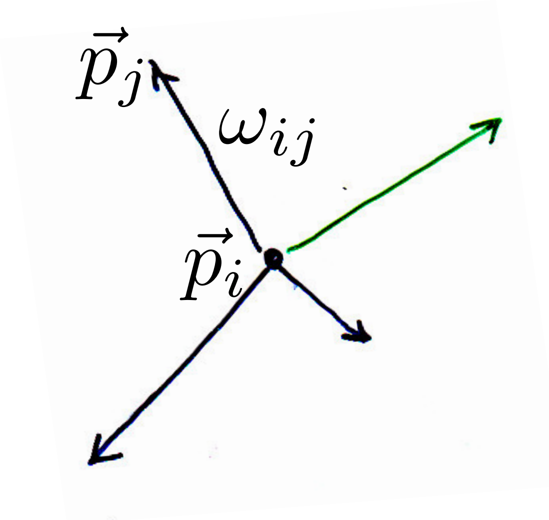

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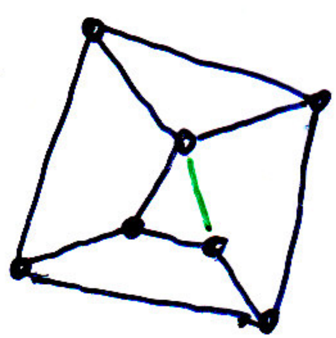
(weighted)
Laplacian matrix $L = \begin{pmatrix} \ddots & 0 & -w_{ij} \\ 0 & \ddots & 0 \\ -w_{ij} & 0 & \ddots \end{pmatrix}$

unweighted $L = -$(adjacency matrix) with degrees d_i on the main diagonal



$$L = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 4 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{pmatrix}$$

vertex degree d_i

Spanning trees

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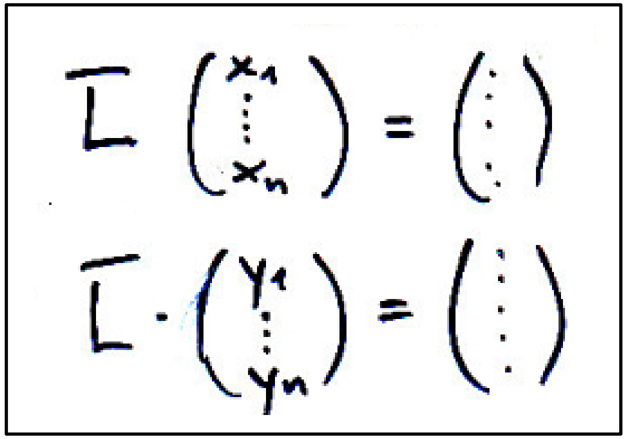

$$x_i \text{ or } y_i = \frac{D_i}{\det \bar{L}}$$

scaling by $\det \bar{L}$ gives integer coordinates (x_i, y_i)

Maxwell-Cremona correspondence gives integer coordinates z_i

$$\det \bar{L} = \text{number of tree-like structures}$$
$$< \text{number of spanning trees}$$

Easy bound on spanning trees

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$$\#T \le \prod_{v=1}^{n} d_v \qquad \text{(product of the degrees)}$$

follows from the Hadamard bound for the determinant of positive semidefinite matrices.

For planar graphs: $\#T \le \prod_{v=1}^{n} d_v \le \left(\sum_{v=1}^{n} d_v \;/\; n\right)^n < 6^n$

$$\#T \le \prod_{v=1}^{n} d_v \cdot \frac{1}{2m}\left(1 + \tfrac{1}{n-1}\right)^{n-1} \le \prod_{v=1}^{n} d_v \cdot \frac{e}{2m}$$

for graphs with m edges [Grone, Merris 1988]

The Outgoing Edge Method

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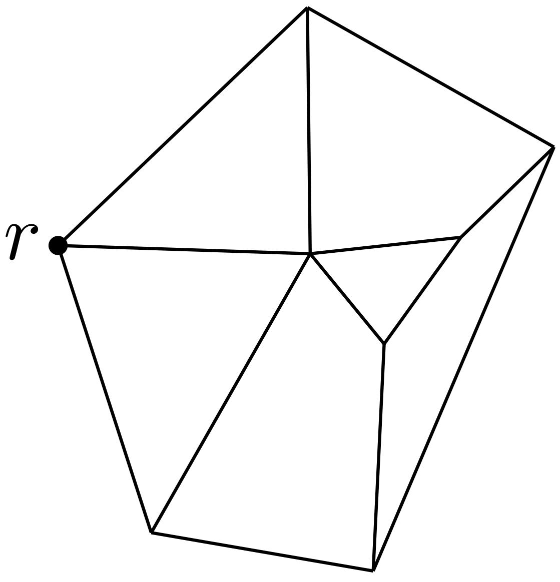
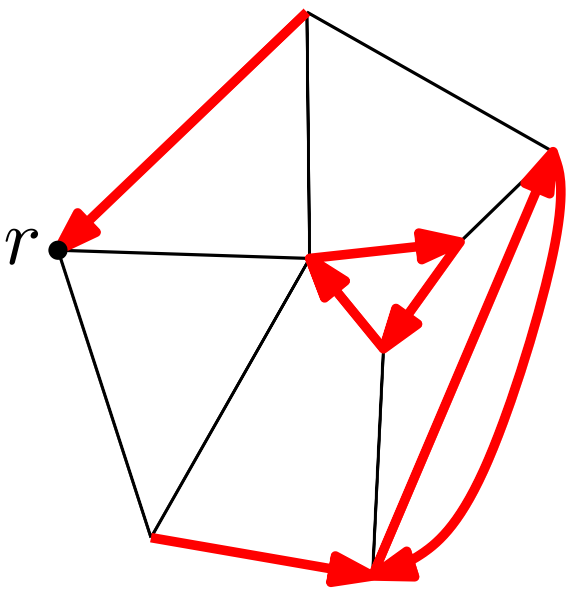


Pick a root r

The Outgoing Edge Method



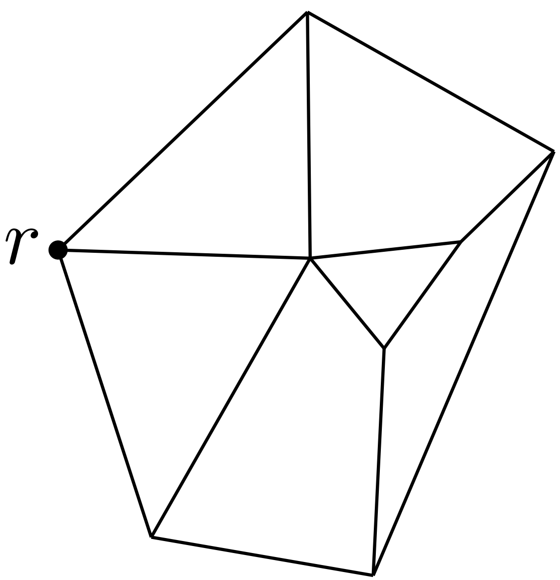
Pick a root r

Select an arbitrary outgoing edge for each vertex $v \neq r$.

$$\#\text{choices} = \prod_{v \neq r} d_v$$

The Outgoing Edge Method

Freie Universität Berlin



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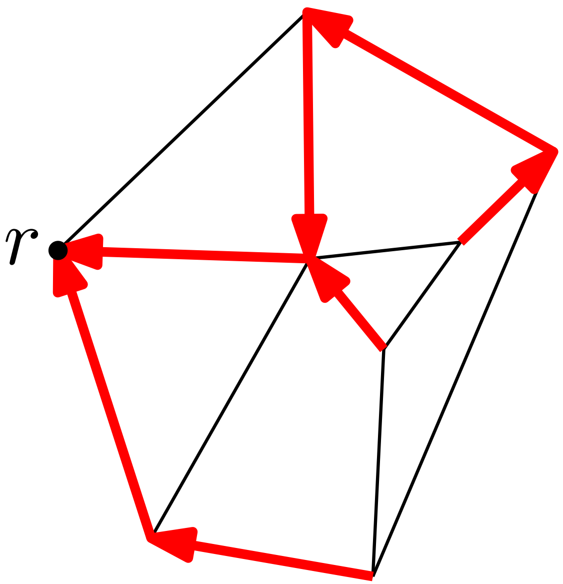

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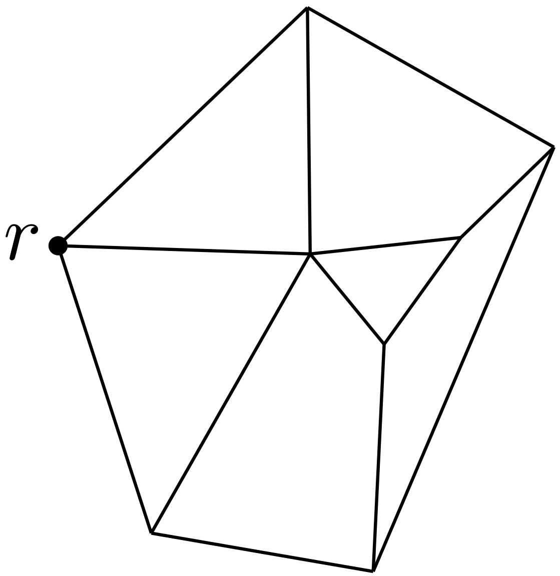

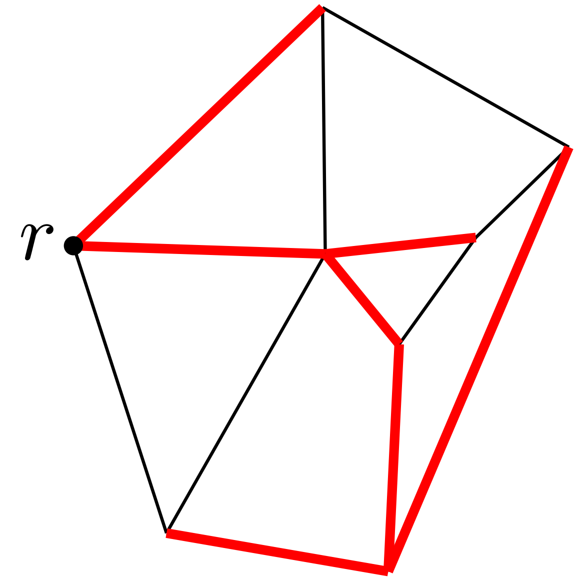

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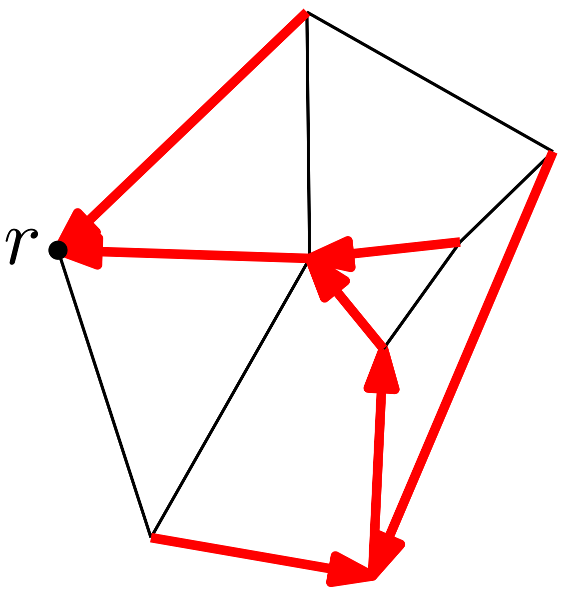
Every spanning tree arises once as a rooted directed spanning tree

$$\#T \leq \prod_{v \neq r} d_v$$

The Outgoing Edge Method



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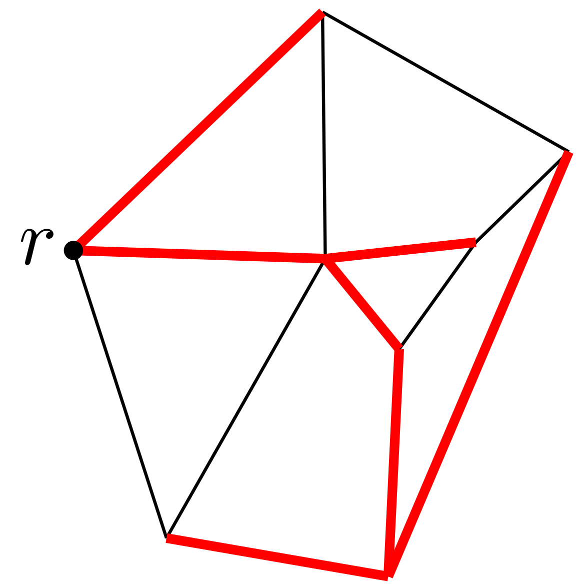

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The Outgoing Edge Method

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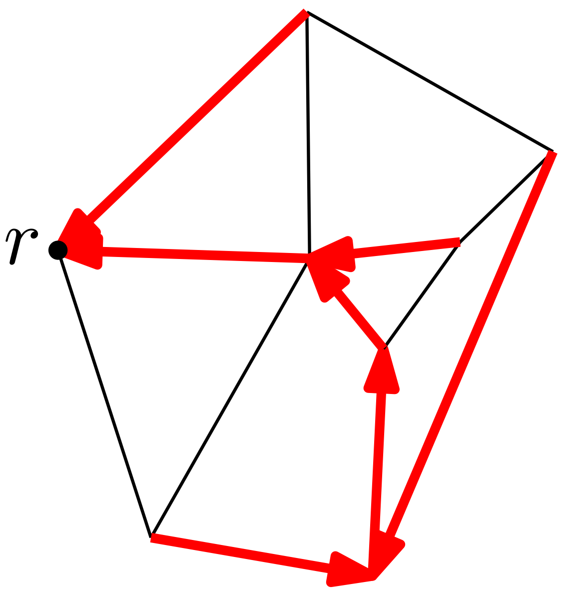

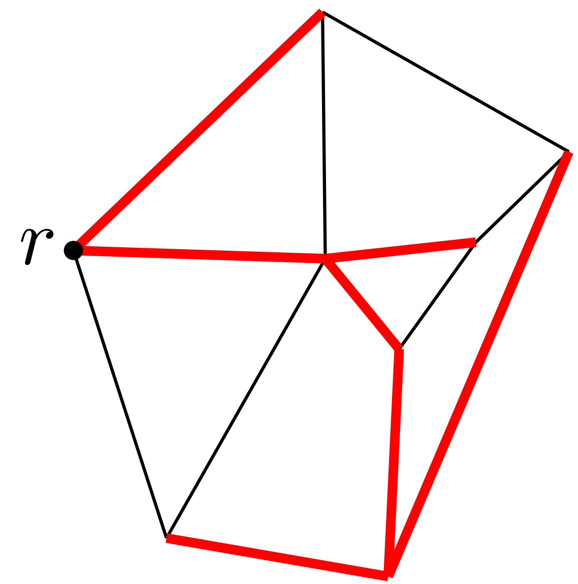

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$$\#\text{choices} = \prod_{v \neq r} d_v$$

Every spanning tree arises once as a rooted directed spanning tree

$$\#T \leq \prod_{v \neq r} d_v < 6^n$$

General bound for planar graphs

W.l.o.g., the graph is triangulated.

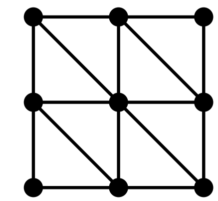
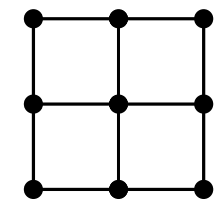
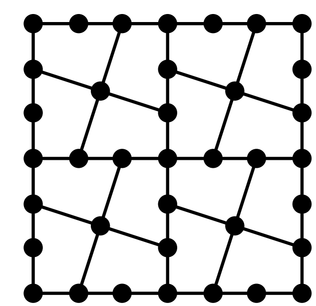
The *dual graph* has $n^* = 2n - 3$ vertices and the same number $\#T$ of spanning trees.

It is 3-regular, and therefore

$$\#T \le \frac{2\log_3 n^*}{3 \cdot n^*} \left(\frac{4}{\sqrt{3}}\right)^{n^*} \le \left(\frac{16}{3}\right)^n = 5.333\ldots^n$$

[B. McKay 1983, Chung and Yao 1999, for k-regular graphs]

#spanning trees of planar graphs

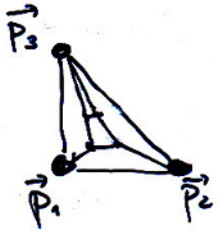
	can have at most	can have	
planar graphs with n vertices ...	$5.333\ldots^n$	5.029^n	
__ without triangles	3.530^n	3.209^n	
__ without Δ and $\Box$	2.848^n	2.561^n	

... spanning trees.

[There are recent improvements by K. Buchin and A. Schulz.]

Triangular outer face

Take $\vec{p}_1 = \binom{0}{0}$ $\vec{p}_2 = \binom{1}{0}$ $\vec{p}_3 = \binom{1}{0}$



→ All $x_i, y_i = \frac{D_i}{D}$

Multiply everything by D

→ integer coordinates in the range $[0..D]$

Triangular outer face

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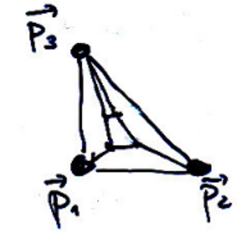


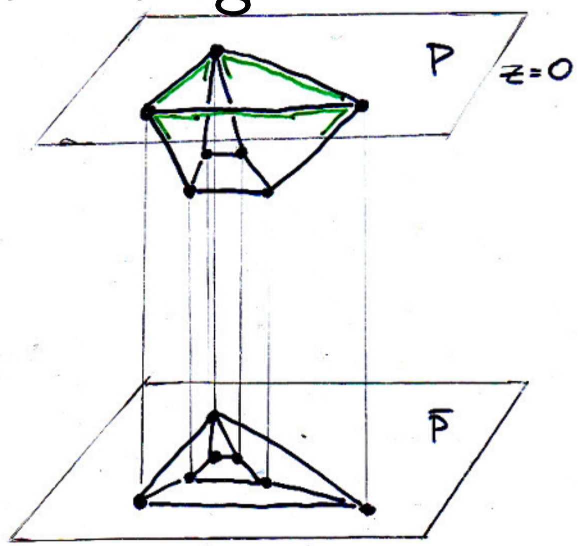
→ All $x_i, y_i = \frac{D_i}{D}$

Multiply everything by D

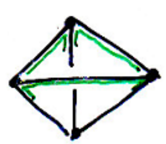
→ integer coordinates in the range $[0..D]$

Maxwell-Cremona lifting:

gradient of "boundary" faces bounded by $n.D$



→ P contained in



→ $z_i \in [-nD^2 \cdot \frac{1}{3} .. 0]$

Triangular outer face

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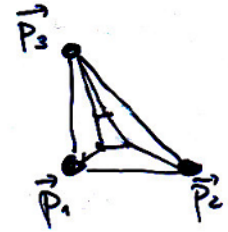


→ All $x_i, y_i = \frac{D_i}{D}$

Multiply everything by D

→ integer coordinates in the range $[0..D]$

THEOREM: A 3-polytope which contains a triangle can be realized with integer vertex coordinates (x_i, y_i, z_i) with

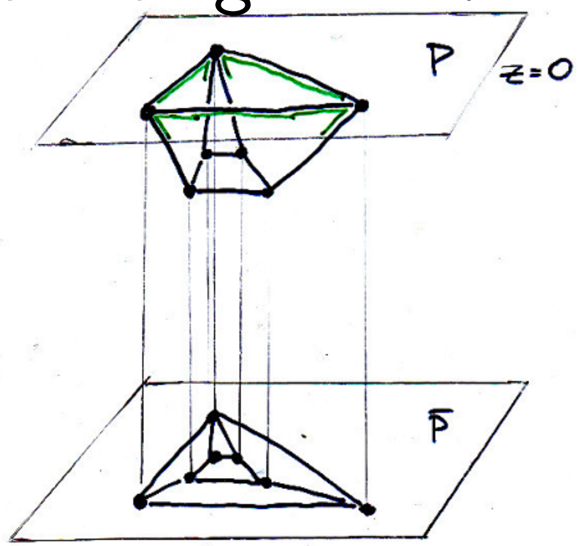
$$0 \le x_i, y_i \le (\tfrac{16}{3})^n$$

$$0 \le z_i \le 2n(\tfrac{16}{3})^{2n}$$

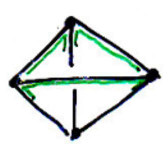
[Richter-Gebert 1996]

Maxwell-Cremona lifting:

gradient of "boundary" faces bounded by $n \cdot D$



→ P contained in



→ $z_i \in [-nD^2 \cdot \frac{1}{3} .. 0]$

No triangular outer face

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(no triangle $\Rightarrow$ P has a 3-valent vertex

use polarity: P^* has a triangle

move origin into interior

face $ax+by+cz = 1 \rightarrow$ vertex $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

a,b,c is the solution of

$$a x_1 + b y_1 + c z_1 = 1$$
$$a x_2 + b y_2 + c z_2 = 1$$
$$a x_3 + b y_3 + c z_3 = 1$$

$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$ vertices of P^*

$a = \frac{A}{D}$ $b = \frac{B}{D}$, $c = \frac{C}{D}$ $|A|, |B|, |C|, |D| \leq \sqrt{27} \cdot \max\{x_i, y_i, z_i\}^3$

Gives vertices of P $\vec{p_i} = \left(\frac{A_i}{D_i}, \frac{B_i}{D_i}, \frac{C_i}{D_i}\right)$

multiply by $\prod_{i=1}^{n} D_i \rightarrow$ integer coordinates

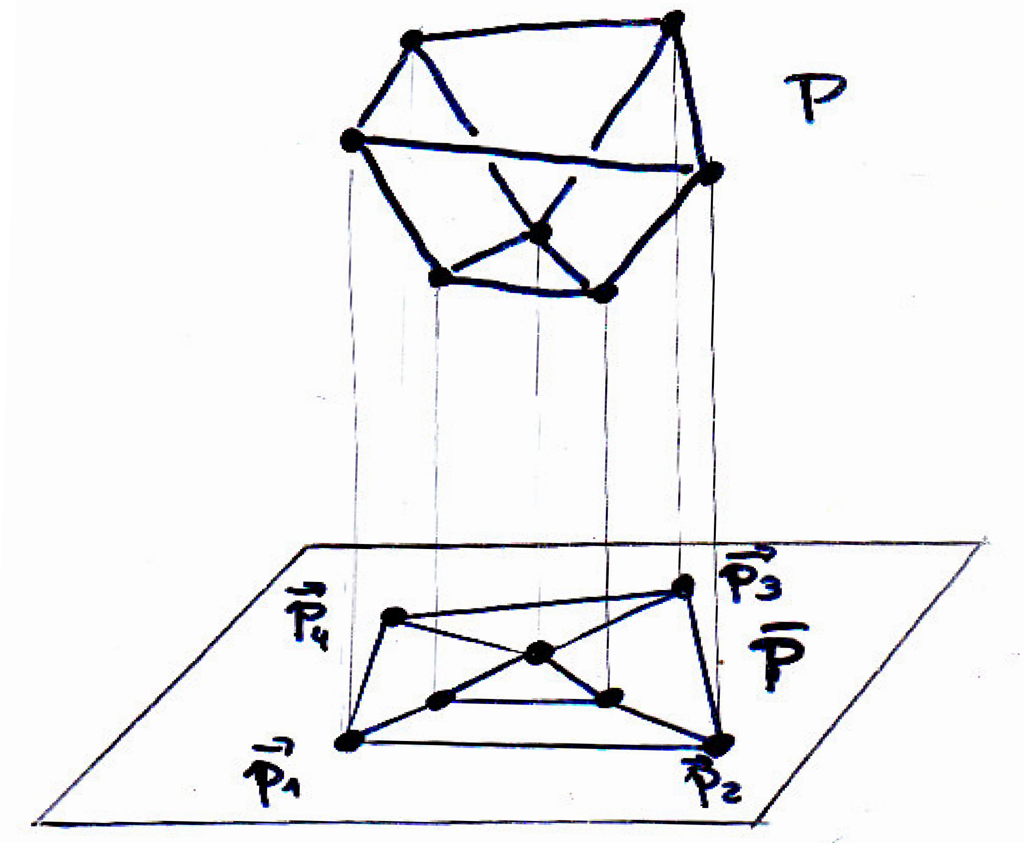
A 3-polytope can be realized with integer coordinates at most $\left(\sqrt{27}\,(n \cdot 42^n)^3\right)^{n+1}$ [Richter-Gebert 1996]

$\leq 2^{20n^2}$

4 boundary vertices

- If the graph contains a quadrilateral face:

NOT EVERY CHOICE OF $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$ LEADS TO AN EQUILIBRIUM WITH $w_{ij} \equiv 1$ FOR INTERIOR EDGES.

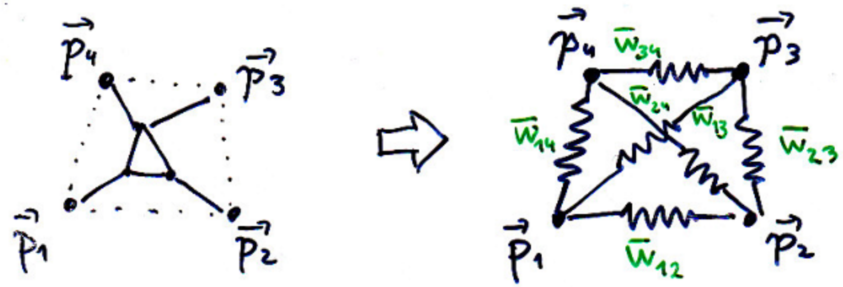


4 boundary vertices

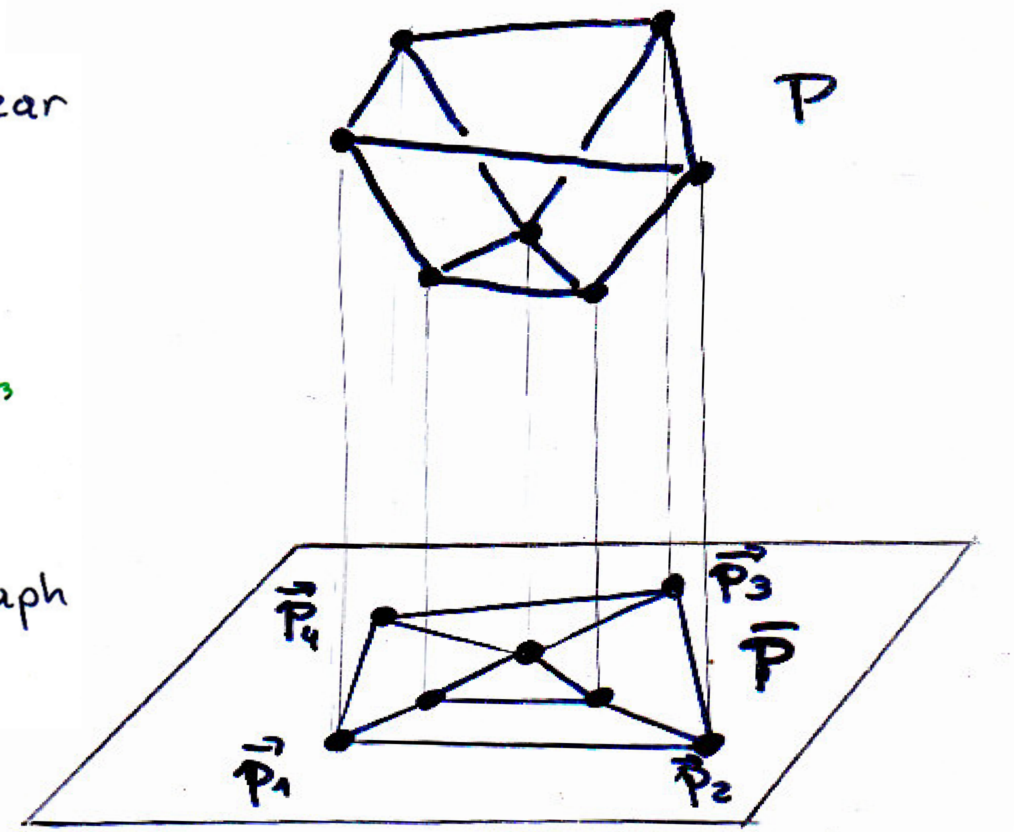
- If the graph contains a quadrilateral face:

NOT EVERY CHOICE OF $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$ LEADS TO AN EQUILIBRIUM WITH $w_{ij} \equiv 1$ FOR INTERIOR EDGES.

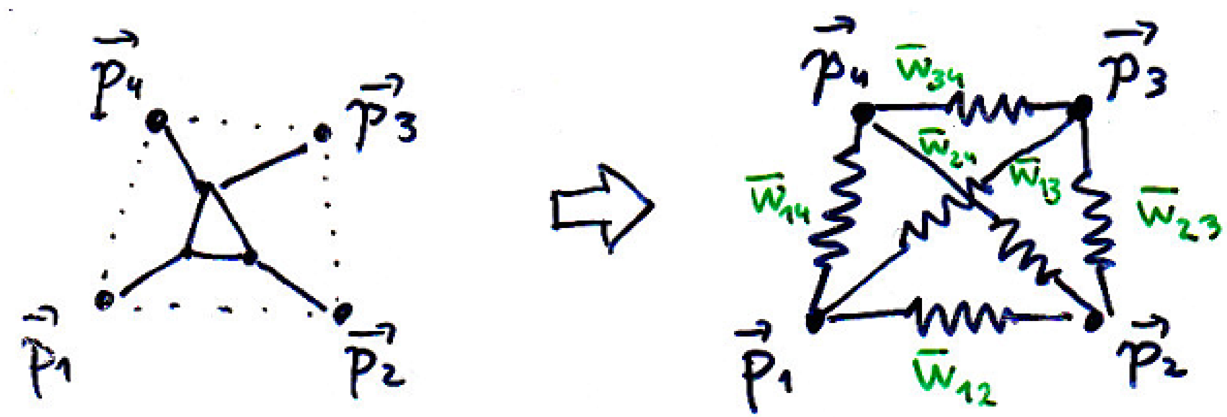
The force pulling at $\vec{p}_1$ is a linear function of $\vec{p}_2 - \vec{p}_1$, $\vec{p}_3 - \vec{p}_1$, $\vec{p}_4 - \vec{p}_1$.



can be modeled as a complete graph with 4 vertices and stresses $\bar{w}_{ij}$.



4 boundary vertices



G $\bar{G}$

The Substitution Lemma:

There are $\bar{\omega}_{ij}$, $1 \leq i < j \leq 4$, such that for all $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$, the resulting forces in G on $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$ are the same as in the sustitution graph $\bar{G}$ on the four vertices $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$ only.

4 boundary vertices

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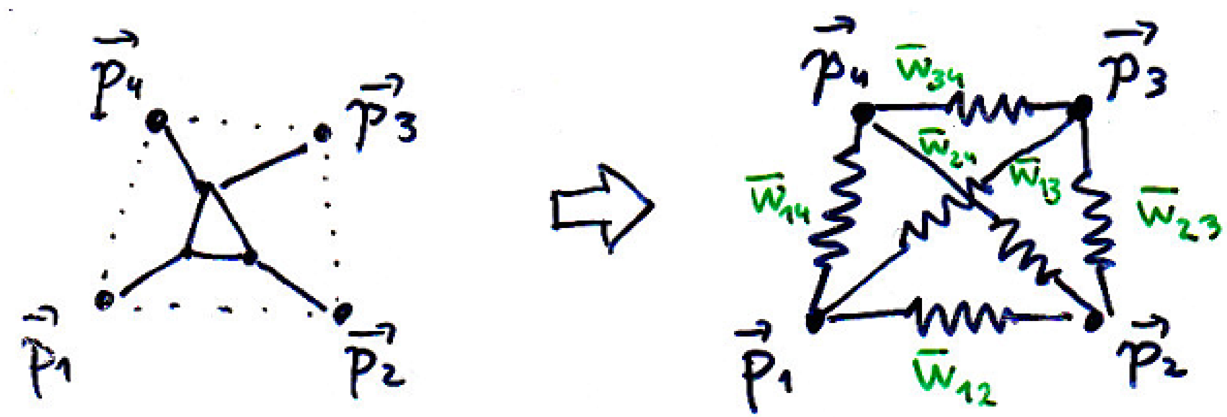


How to place $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$?

$\bar{w}_{12}, \bar{w}_{23}, \bar{w}_{34}, \bar{w}_{41}$ can be controlled by modifying $w_{12}, w_{23}, w_{34}, w_{41}$.

Only $\bar{w}_{13}$ and $\bar{w}_{24}$ matter!

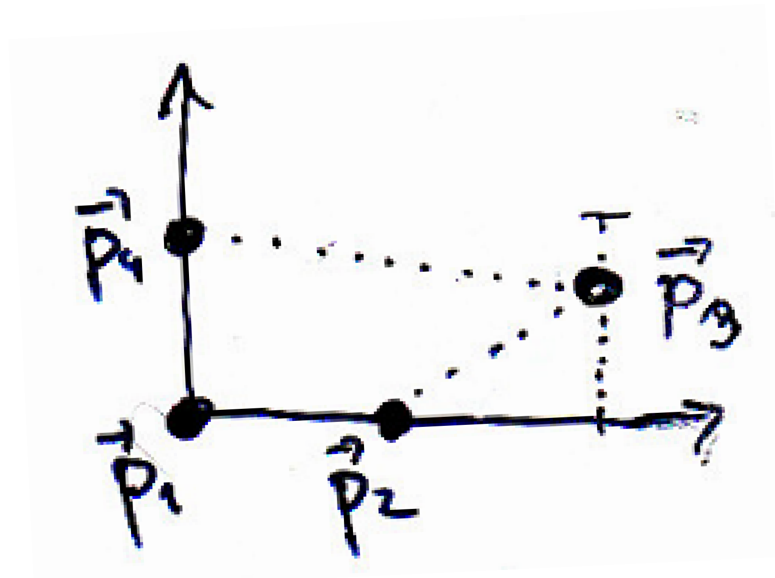
W.l.o.g. $\vec{p}_1 = \binom{0}{0}$ $\vec{p}_2 = \binom{1}{0}$ $\vec{p}_4 = \binom{0}{1}$ (affine transformation)

$\vec{p}_3 = \binom{x_3}{y_3}$: Equilibrium can be obtained if

$$\frac{\bar{w}_{13}}{\bar{w}_{24}} \cdot x_3 y_3 - x_3 - y_3 + 1 = 0$$

4 boundary vertices

ensure $\bar{w}_{13} \geq \bar{w}_{24}$ (w.l.o.g.)



$$x_3 = 2, \qquad y_3 = \frac{\bar{\omega}_{24}}{2\bar{\omega}_{13} - \bar{\omega}_{24}}$$

5 boundary vertices

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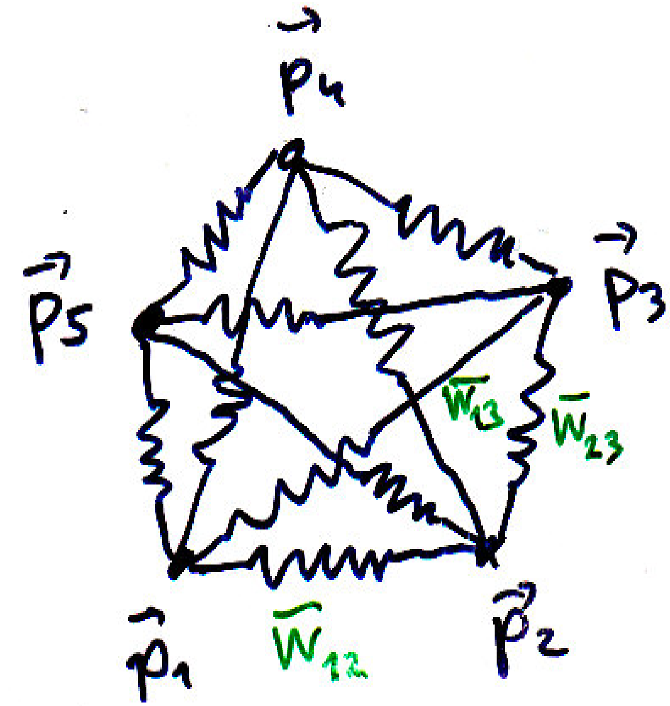
- P contains a 5-face
- compute

$\bar{w}_{13}, \bar{w}_{24}, \bar{w}_{35}, \bar{w}_{14}, \bar{w}_{25}$



5 boundary vertices

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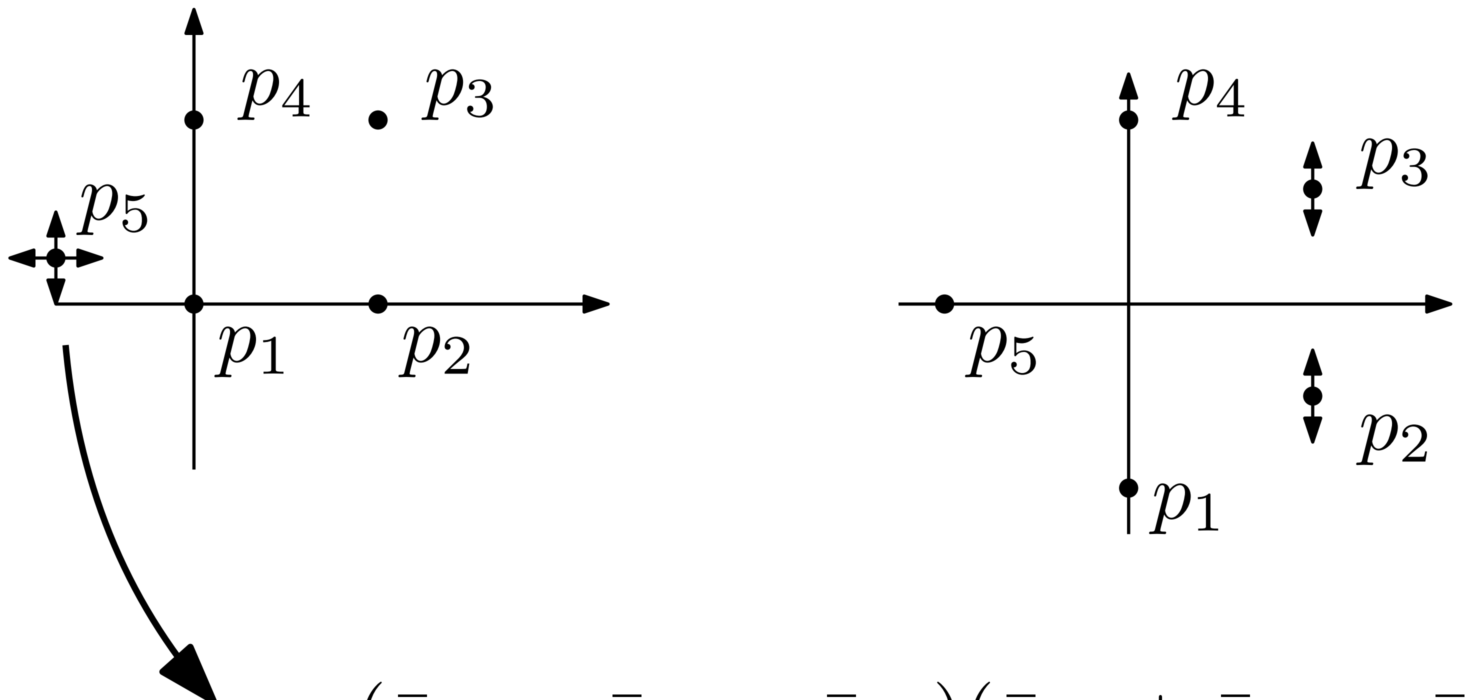
2 cases (depending on $\bar{\omega}$'s):



$$x_5 = \frac{(\bar{\omega}_{13} - \bar{\omega}_{25} - \bar{\omega}_{24})(\bar{\omega}_{35} + \bar{\omega}_{13} - \bar{\omega}_{24})}{\bar{\omega}_{35}\bar{\omega}_{14} + \bar{\omega}_{14}\bar{\omega}_{25} + \bar{\omega}_{25}\bar{\omega}_{24} + \bar{\omega}_{13}\bar{\omega}_{35} - \bar{\omega}_{35}\bar{\omega}_{25}}$$

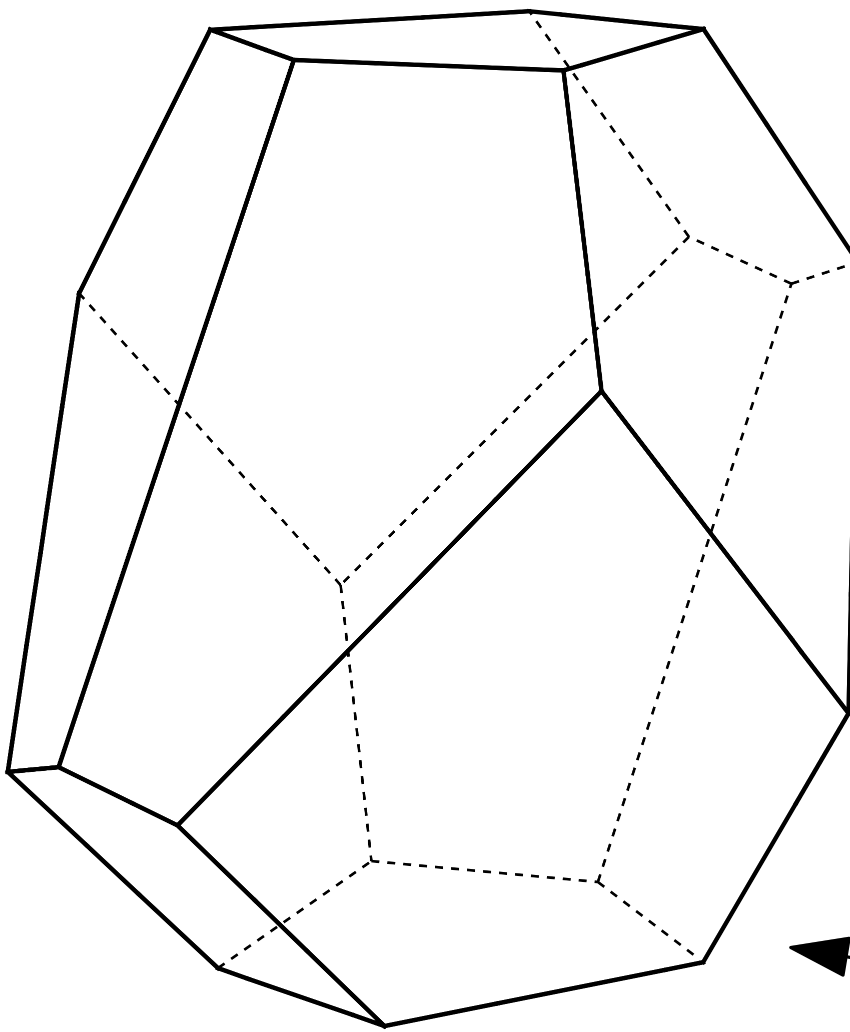
$$y_5 = \frac{\bar{\omega}_{35} + \bar{\omega}_{13} - \bar{\omega}_{24}}{\bar{\omega}_{35} + \bar{\omega}_{25}}$$

Putting everything together

Every 3-connected planar graphs has a triangle, a quadrilateral, or a pentagon.

Theorem (Ribó Mor, Rote, Schulz).
Every 3-polytope with n vertices can be embedded with coordinates $0 \le x_i \le 9^n$, $0 \le y_i \le 24^n$, $0 \le z_i \le 188^n$.

The dodecahedron



Algorithm gives

$$z \leq 1.11 \times 10^{25}$$

(general bound $\approx 10^{47}$)

remove common factors

$\Longrightarrow \quad 0 \leq x_i \leq 1374$

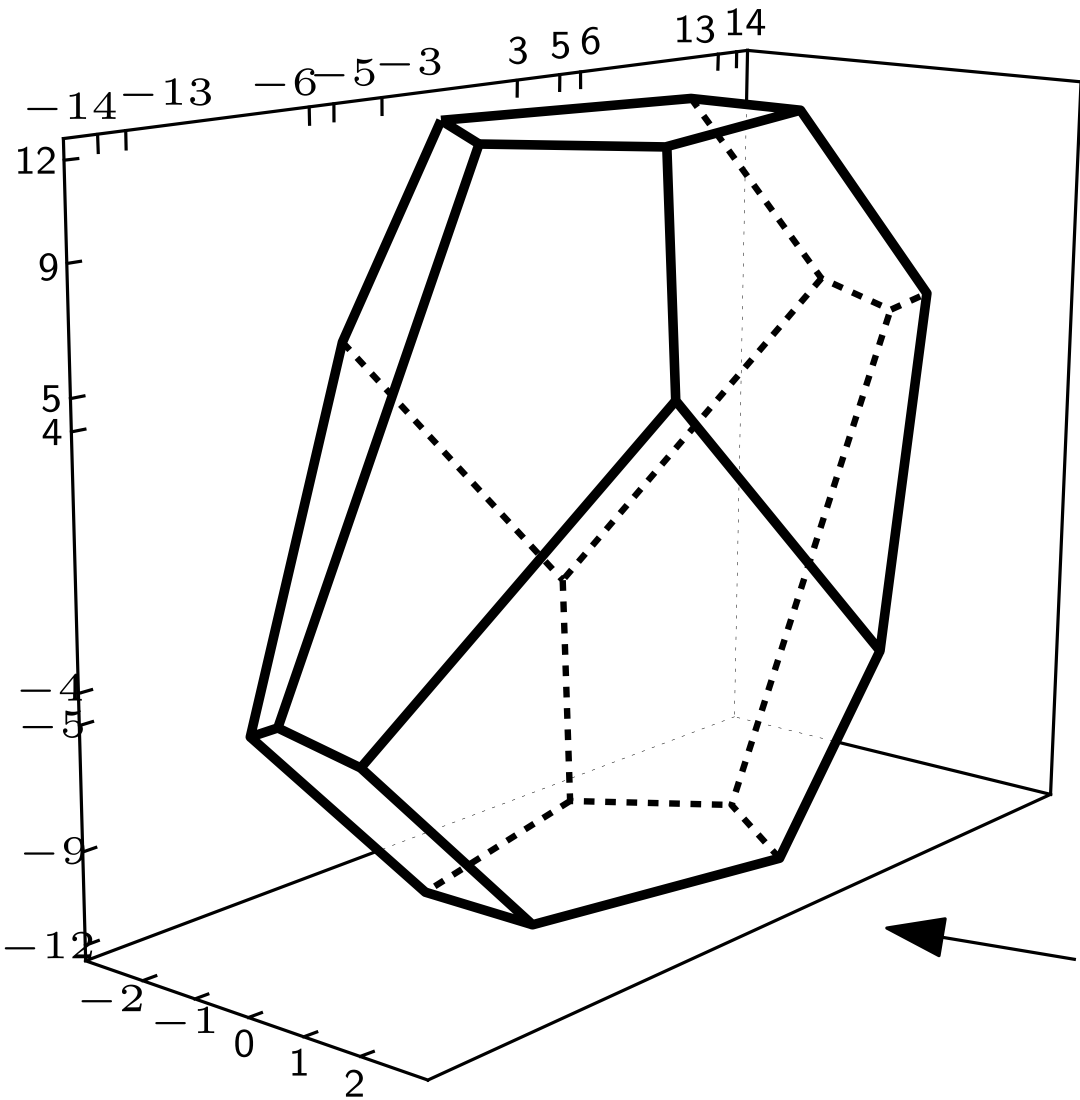
$0 \leq y_i \leq 898$

$0 \leq z_i \leq 406.497$

$\longleftarrow$ in a $4 \times 24 \times 28$ box

(done by hand)

The dodecahedron



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$$0 \le z_i \le 406.497$$

$\longleftarrow$ in a $4 \times 24 \times 28$ box

(done by hand)

Lower bounds

Klatte (1982)

Acketa and Žunić (1995)

Thiele (1991) [Jarník 1922]

An n-gon needs an integer grid of side length

$$\frac{2\pi}{12^{3/2}} \cdot n^{3/2} + O(n \log n)$$

Lower bounds

Andrews (1961, 1963) Fleiner, Kaibel, Rote (1999)

Any d-polytope with n vertices/facets needs an integer grid of side length at least

$$n^{\frac{d+1}{d(d-1)}} \cdot \frac{e}{2} \cdot \frac{1}{d} \underbrace{(1+o(1))}_{\text{as } d\to\infty}$$

tight for fixed d (Bárány and Larman, 1998+): take the convex hull of the integer points in a sphere of appropriate radius.

$d = 3$: $\Omega(n^{4/3})$

Various constructions

Zickfeld (2007):
Certain classes of stacked polytopes need only a polynomial-size grid.

Bárány and Rote (2006):
Strictly convex drawings on an $O(n^2) \times O(n^2)$ grid.

Fixing the planar projection and then minimizing z is not a good idea.

Other uses of Tutte embeddings

Pach and Tóth (2002):
Monotone drawings of planar graphs (by induction and case analysis)

Chrobak, Goodrich, and Tamassia (1996):
Polytopes with given x-coordinates (for example, $1, 2, 3, \ldots$).

Ribó (2006):
$\longrightarrow$ perturbation of self-touching linkages

Inductive method (Das&Goodrich)

for triangulated polytopes.

Find a large independent set of degree ≤ 8.
Contract an incident edge for each vertex (in parallel), maintaining 3-connectivity.
$\longrightarrow$ linear-time algorithm, fast parallel algorithm

$O(\log n)$ rounds; in each round the bit-size is multiplied by a constant factor.
$\longrightarrow$ bit-size $=$ poly(n)