Pseudotriangulations and the Expansion Polytope

A *pointed pseudotriangulation* of a set of points in the plane is a partition of the convex hull into pseudotriangles: polygons with three convex corners and an arbitrary number of reflex vertices. This geometric structure arises naturally in the context of rigidity of frameworks and *expansive motions*: motions of points in the plane where no pairwise distance decreases. The set of expansive infinitesimal motions is a polyhedron. By perturbing its facets, one arrives at a polytope whose vertices are in one-to-one correspondence with the pointed pseudotriangulations. The expansion polytope can also be considered in one dimension. It leads to the well-known associahedron in this case.

The expansion polytope provides an indirect existence proof of infinitesimal expansive motions for a polygonal chain, which is a crucial step in the solution of the Carpenter's Rule Problem: Every planar polygonal chain can be straightened without self-intersections.

Pseudotriangulations and the Expansion Polytope

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PLANE GEOMETRY:

1. Pseudotriangulations: basic definitions and properties

RIGIDITY AND KINEMATICS:

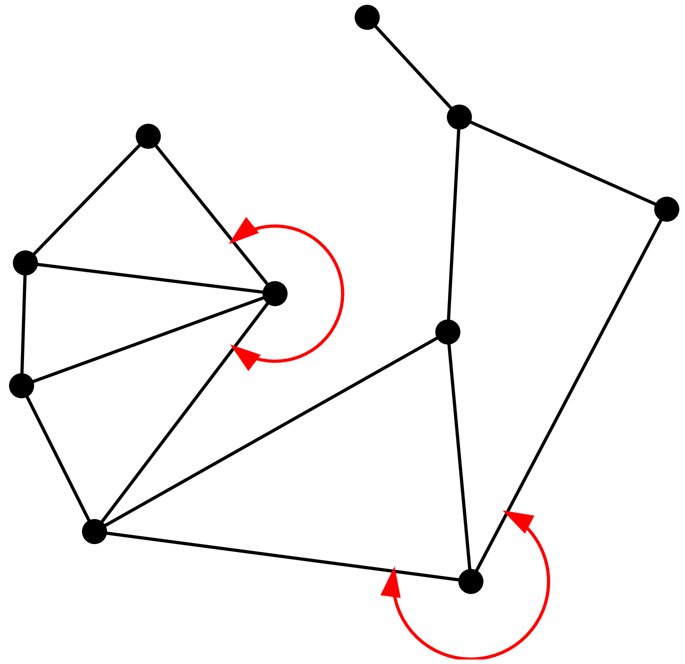
2. The Carpenter's Rule Problem

POLYTOPES:

3. The expansion cone and the pseudotriangulation polytope

Pointed Vertices

A *pointed* vertex is incident to an angle $> 180^\circ$ (a *reflex* angle or *big* angle).



A straight-line graph is pointed if all vertices are pointed.

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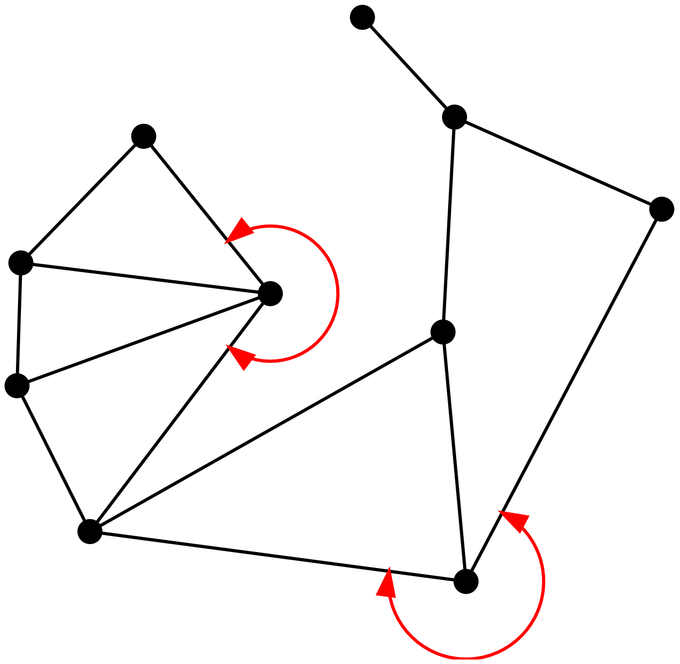

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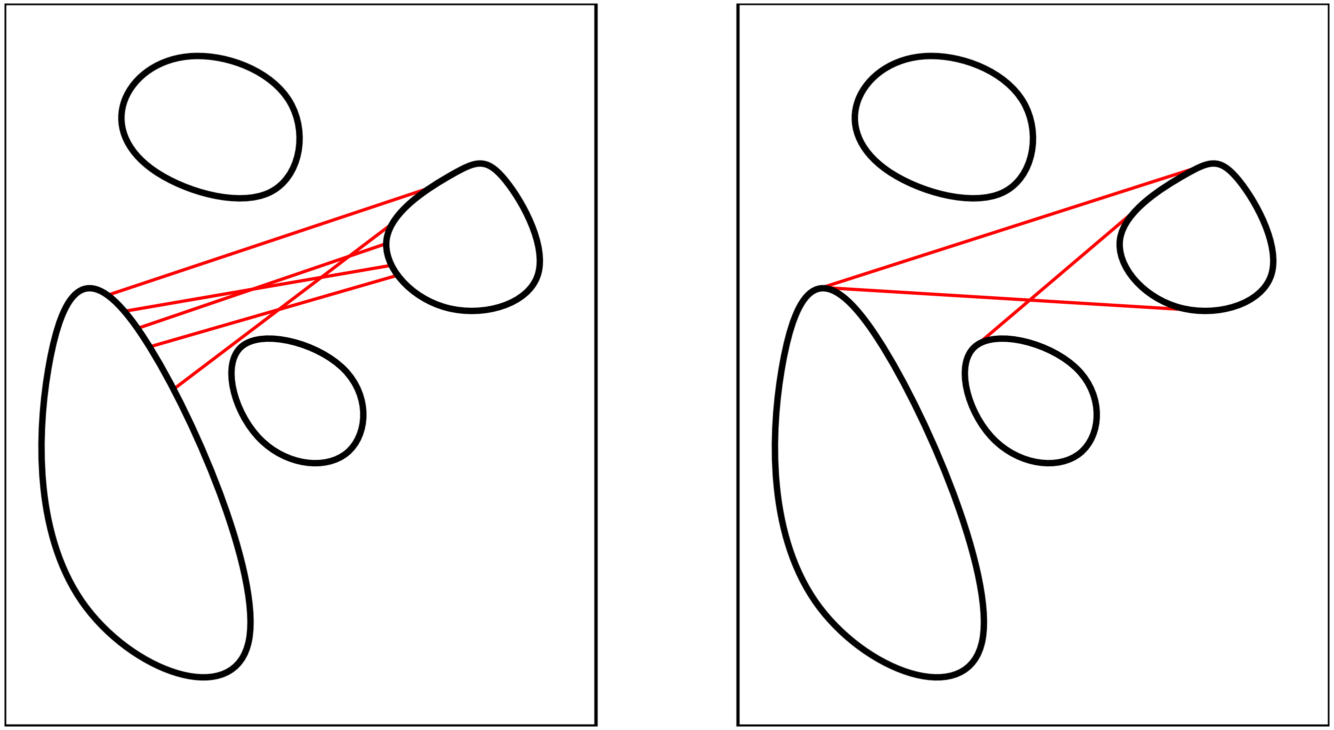
Where do pointed vertices arise?

Visibility among convex obstacles

Equivalence classes of *visibility segments*. Extreme segments are *bitangents* of convex obstacles.

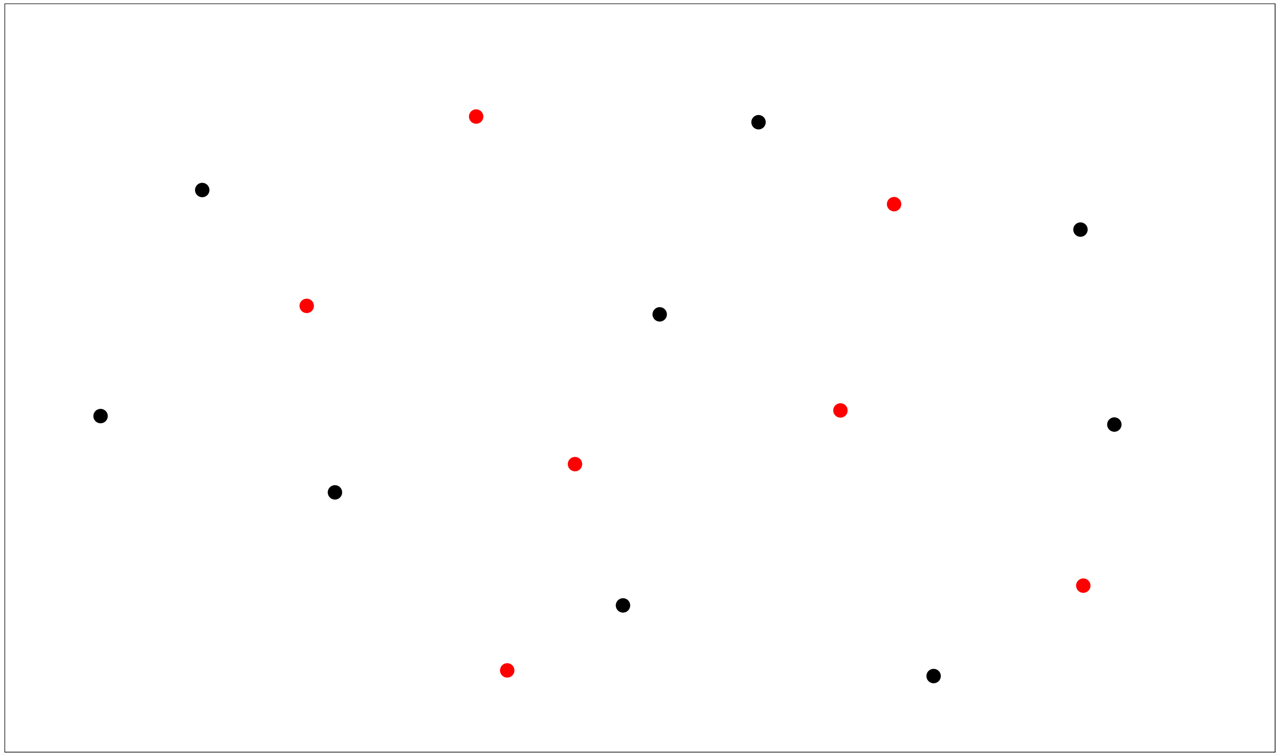


[Pocchiola and Vegter 1996]

Pseudotriangulations

Given: A set V of vertices, a subset $V_p \subseteq V$ of *pointed vertices*.

A *pseudotriangulation* is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in V_p pointed.



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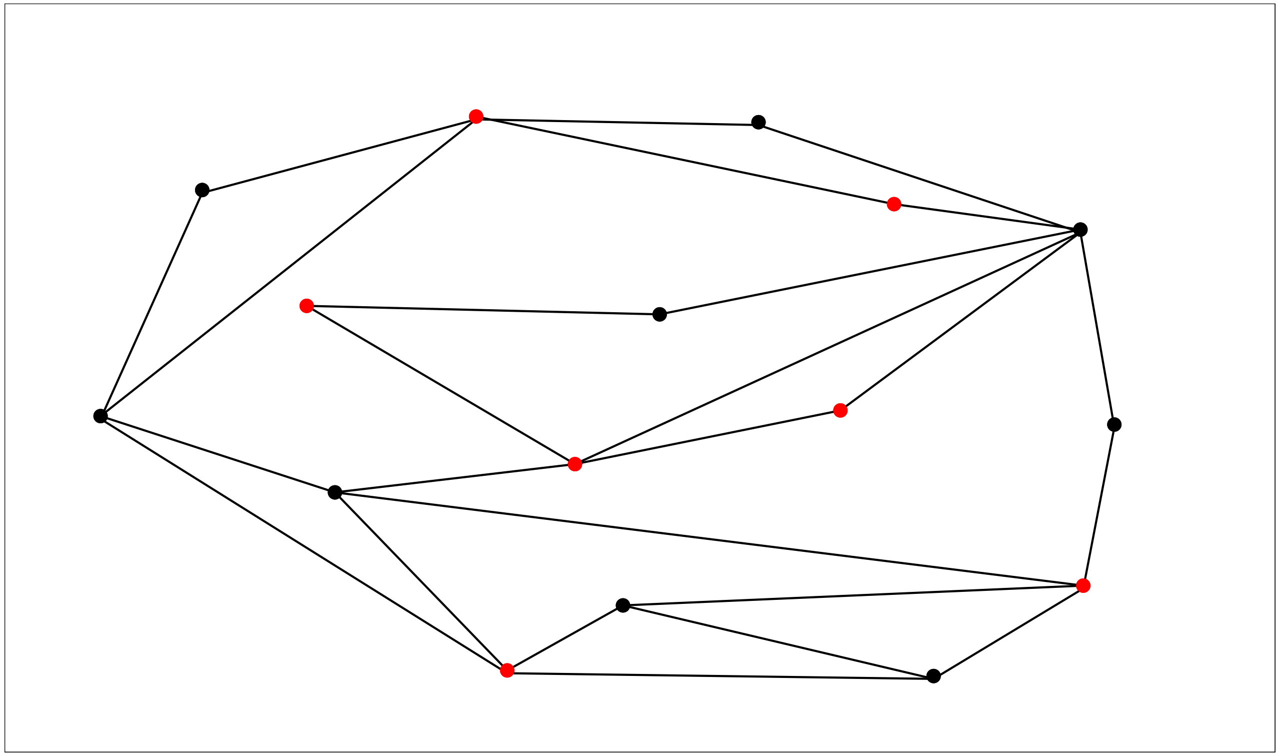

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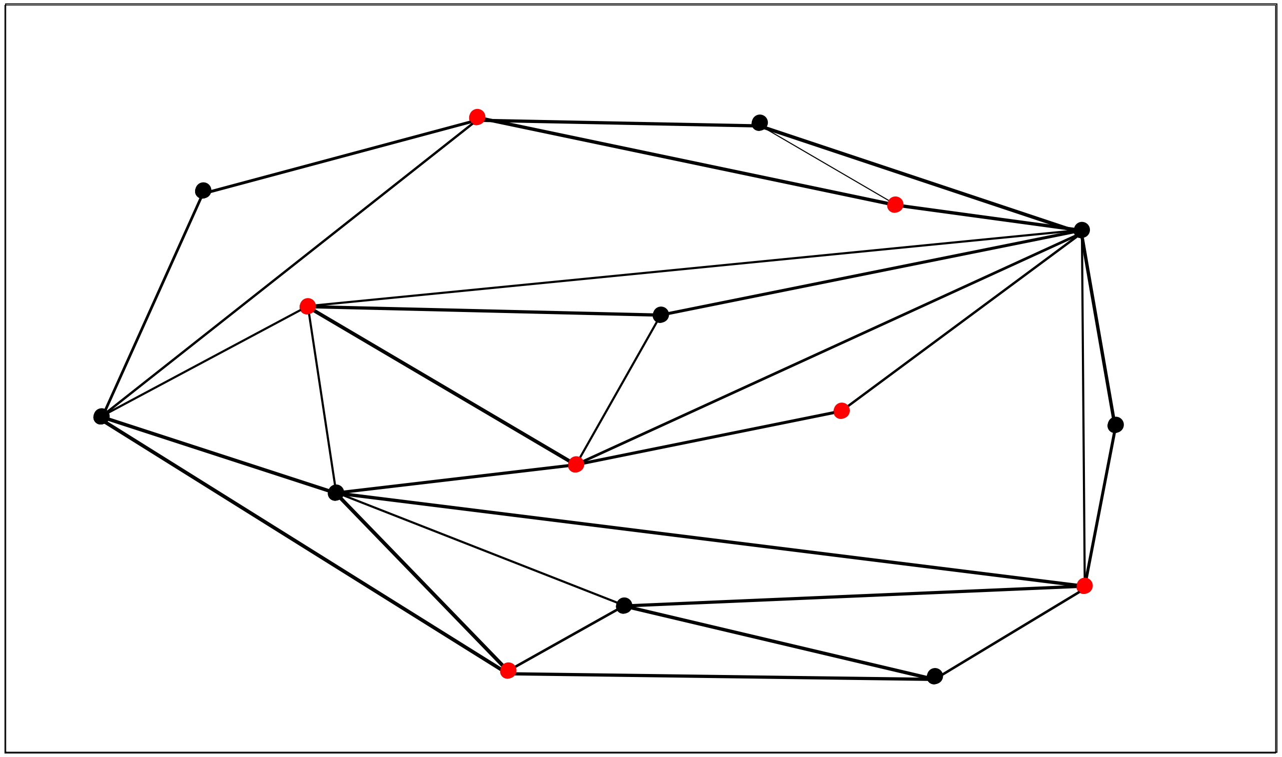

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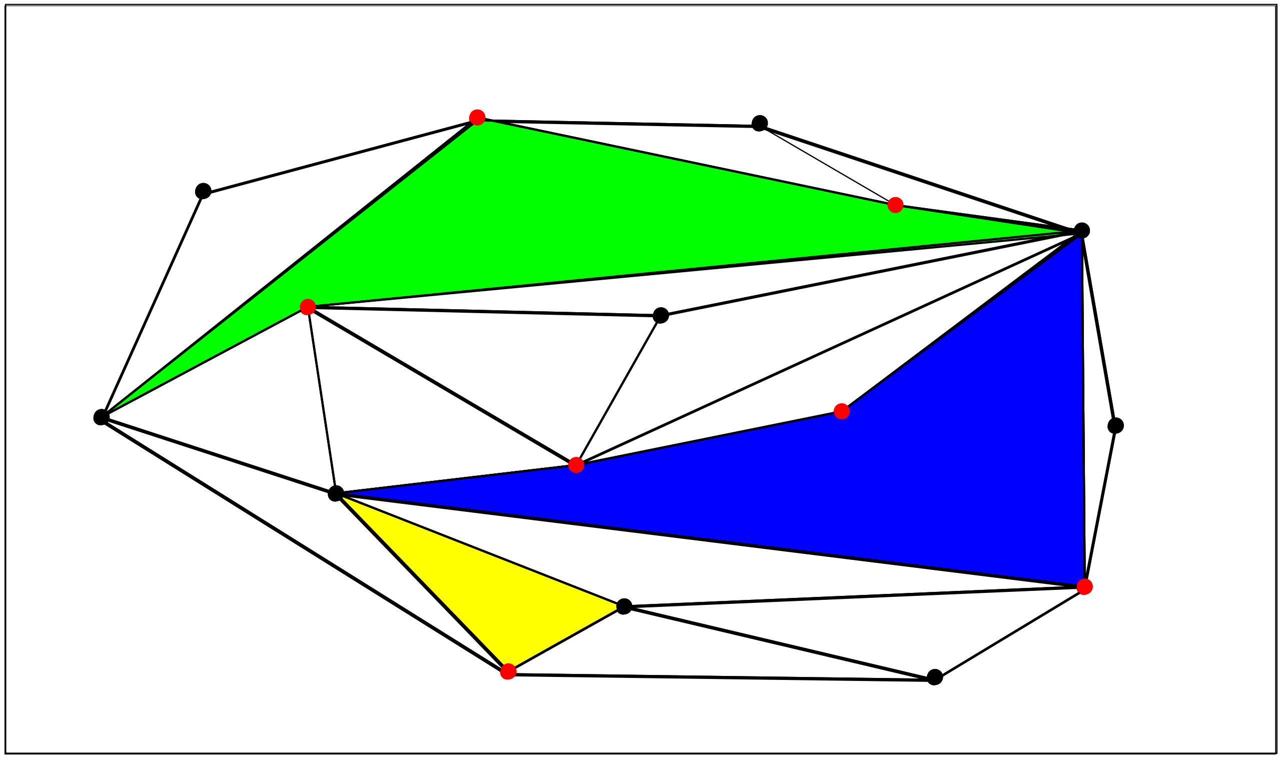
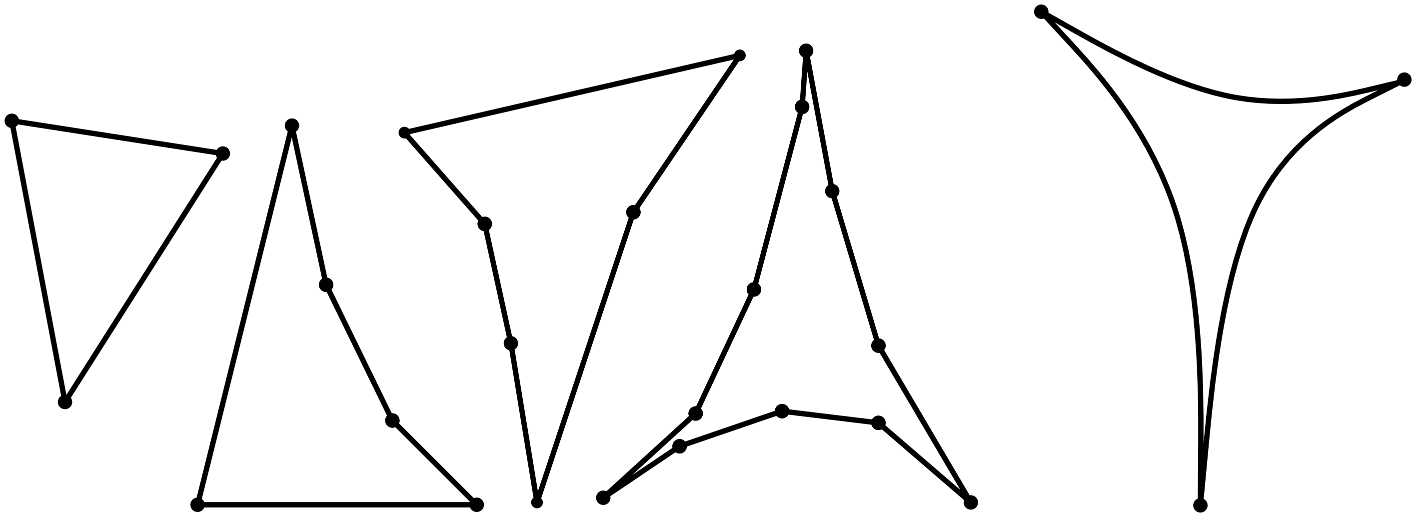


Pseudotriangles

A pseudotriangle has three convex *corners* and an arbitrary number of reflex vertices ($> 180^\circ$).



Pseudotriangulations

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Proof. $(1) \implies (2)$ All convex hull edges are in E.
$\rightarrow$ decomposition of the polygon into faces.
Need to show: If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Characterization of pseudotriangulations

Lemma. *If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.*

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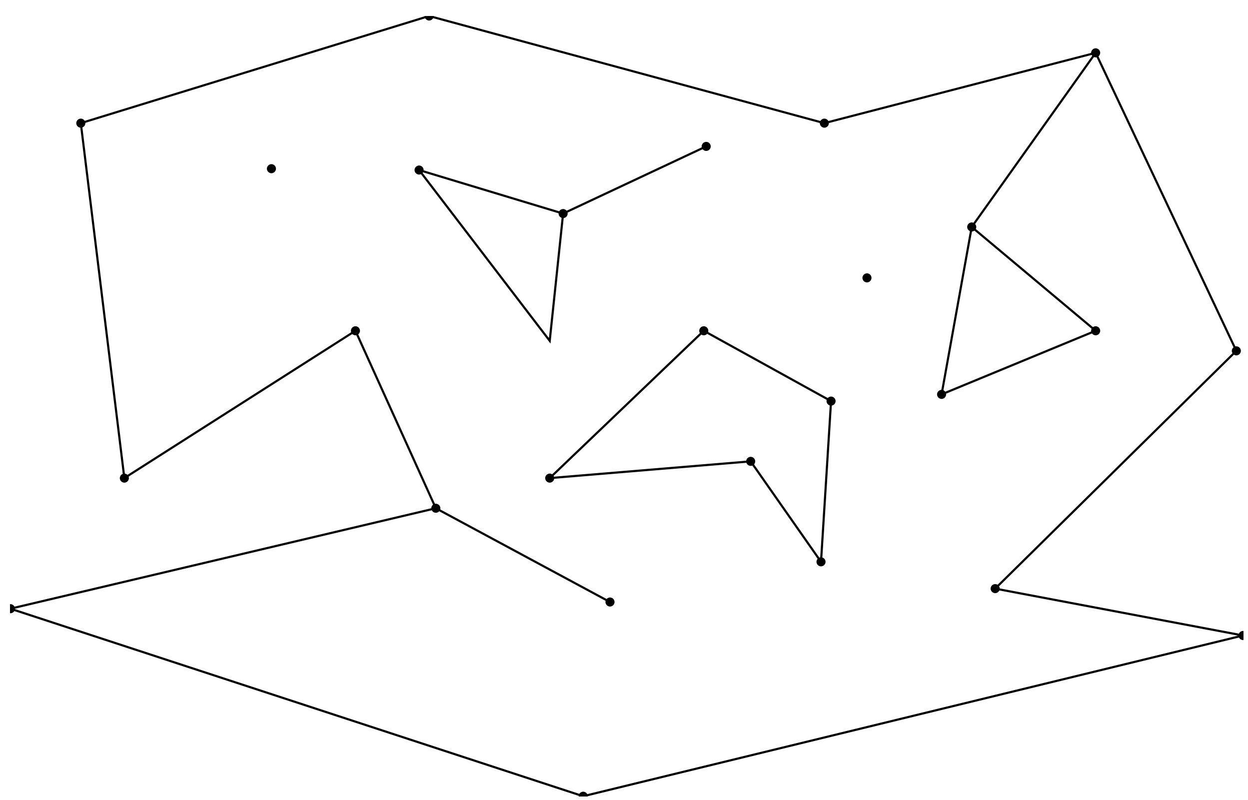
Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.



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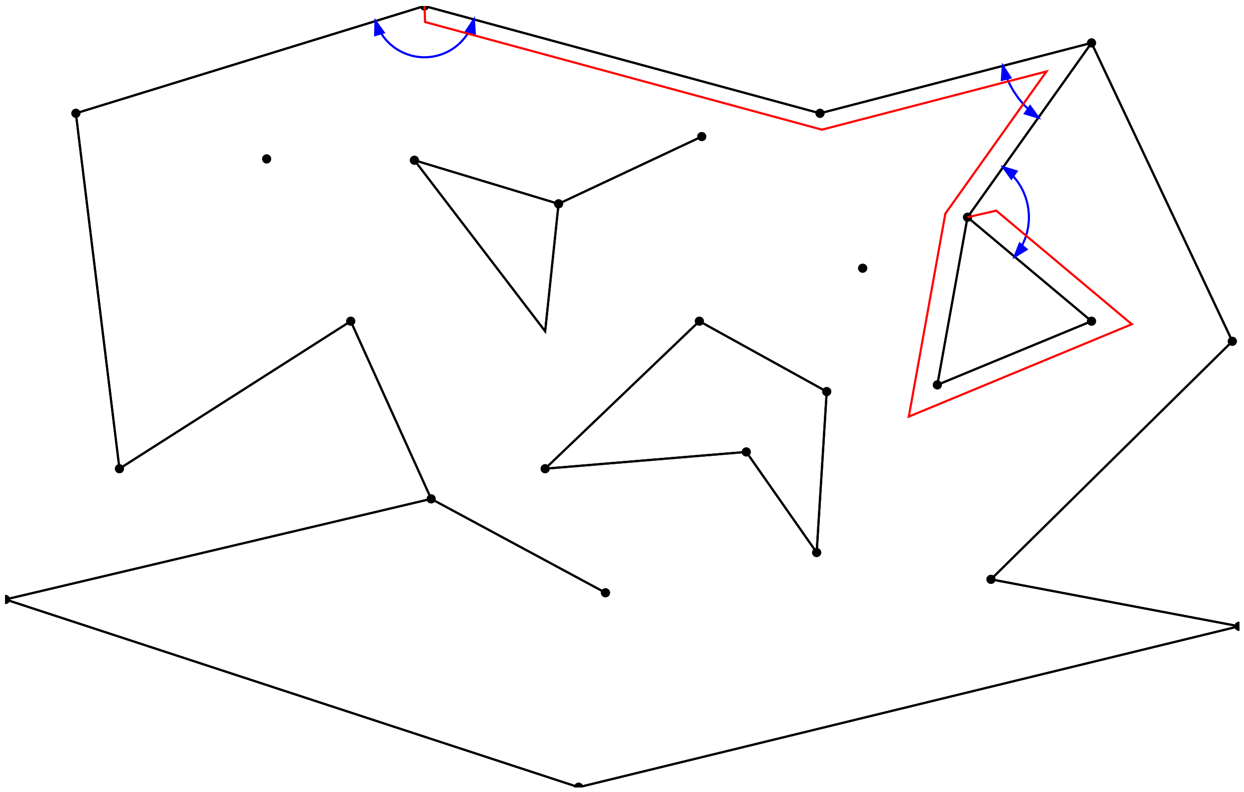

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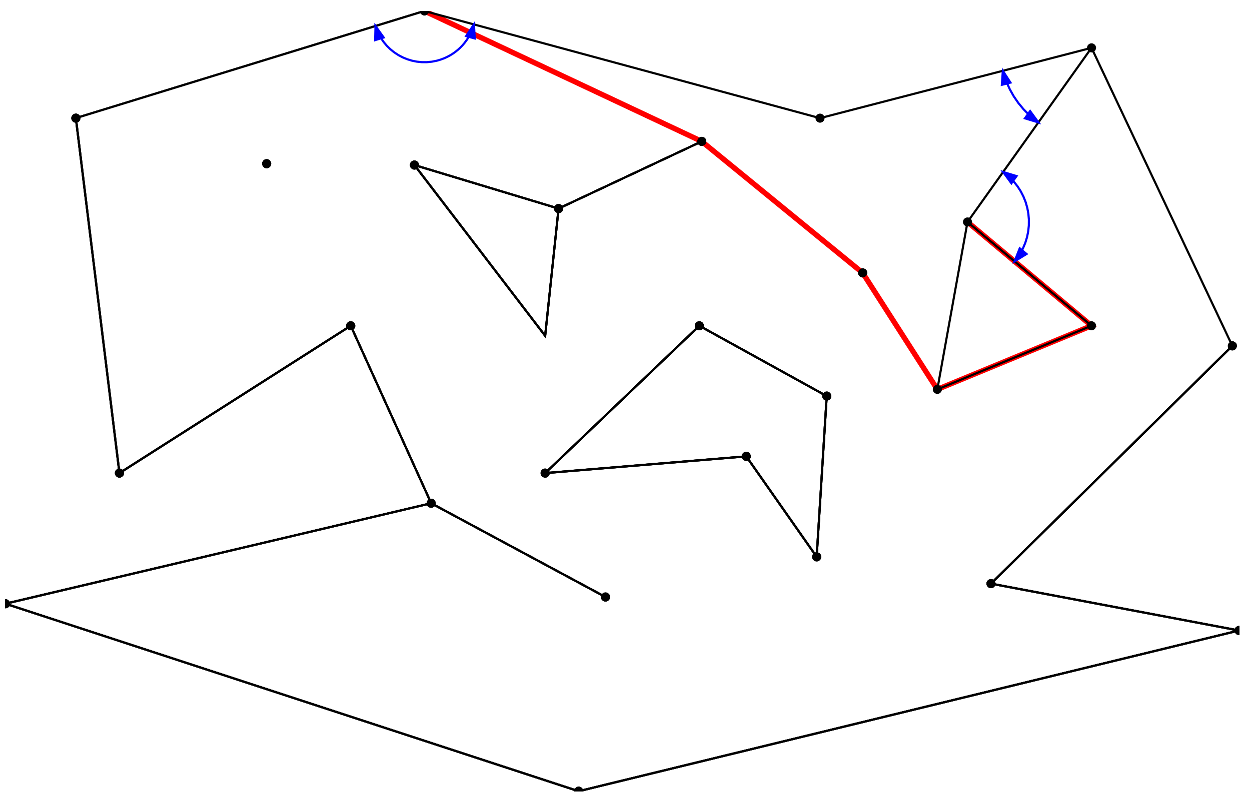
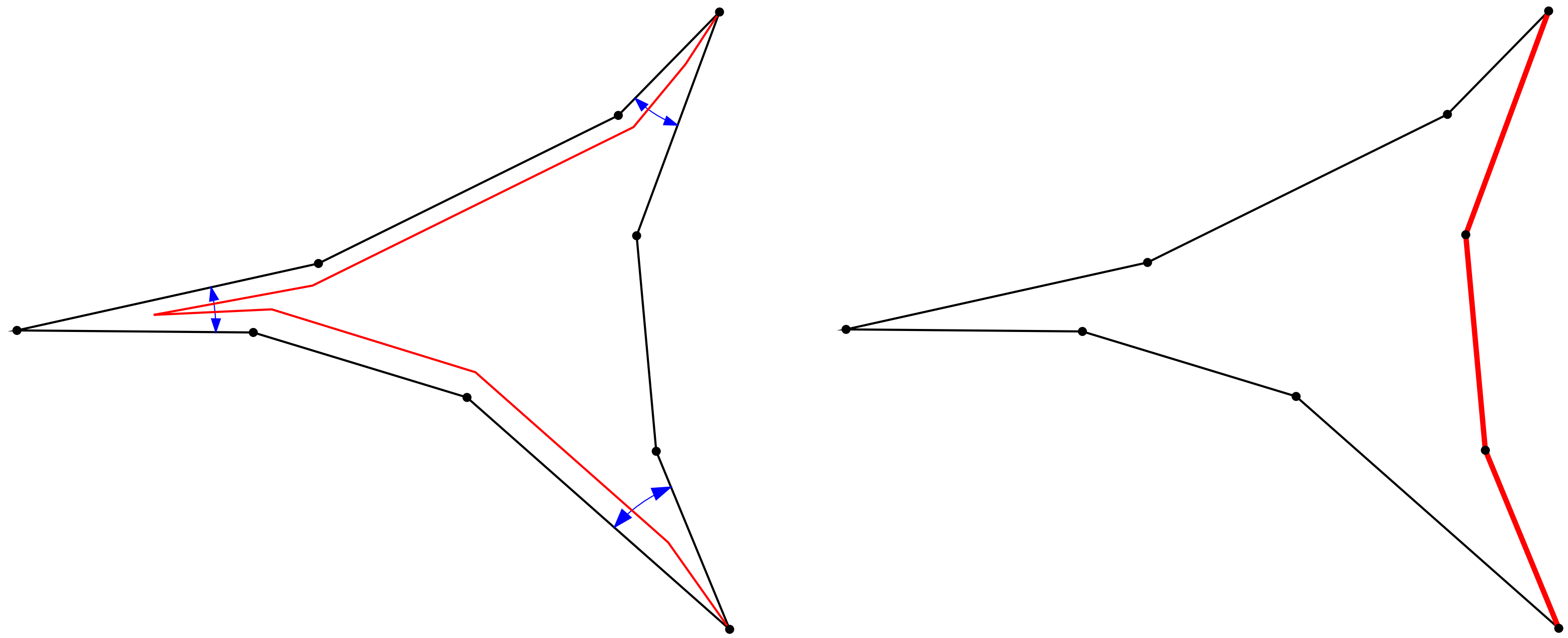


Characterization of pseudotriangulations continued

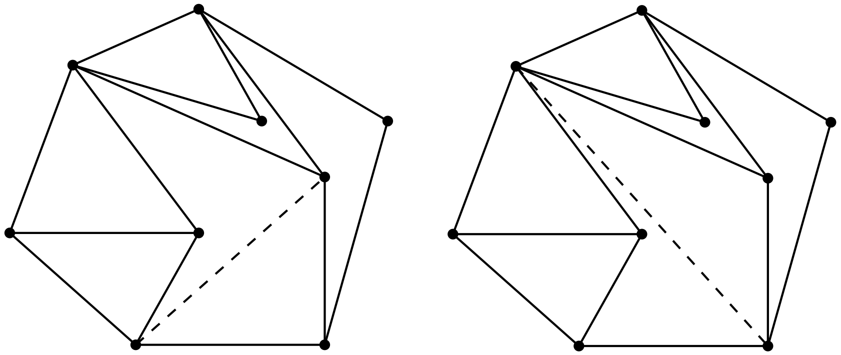
A new edge is always added, unless the face is already a pseudotriangle (without inner obstacles).



[Rote, C. A. Wang, L. Wang, Xu 2003]

Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.



before after

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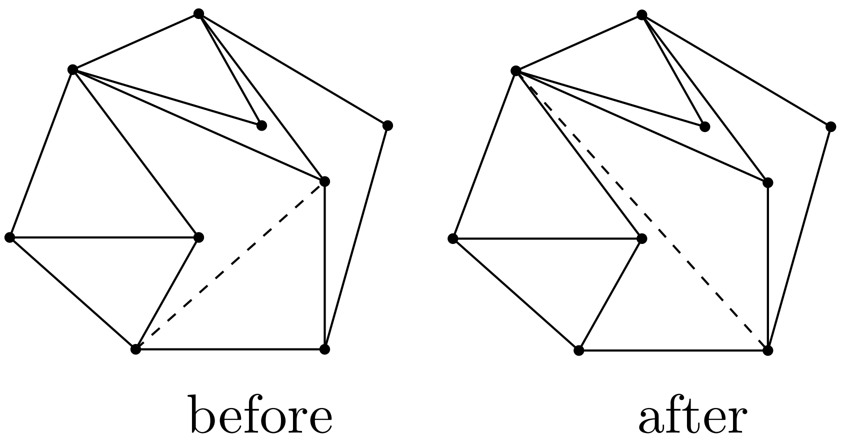
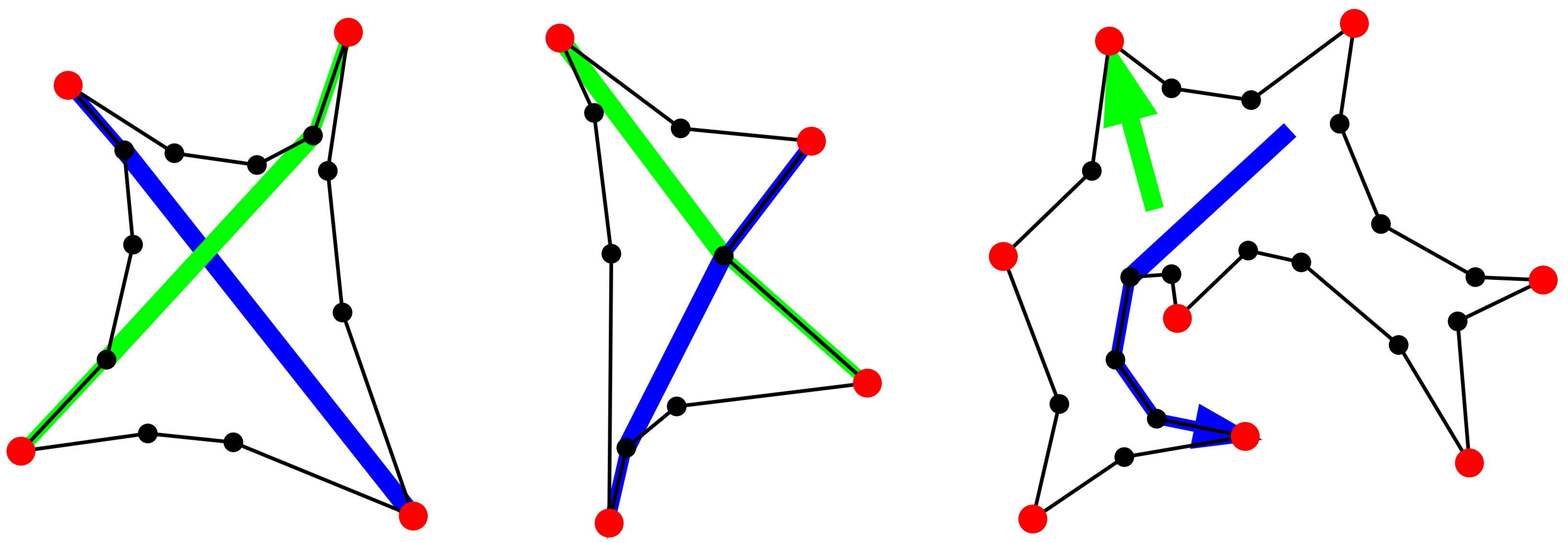


The flip graph is connected.
Its diameter is $O(n \log n)$. [Bespamyatnikh 2003]

Flipping

Every pseudoquadrangle has precisely two diagonals, which cut it into two pseudotriangles.

[*Proof.* Every *tangent ray* can be continued to a geodesic path running along the boundary to a corner, in a unique way.]



Vertex and face counts

Lemma. *A pseudotriangulation with* x *nonpointed and* y *pointed vertices has* $e = 3x + 2y - 3$ *edges and* $2x + y - 2$ *pseudotriangles.*

Corollary. *A pointed pseudotriangulation with* n *vertices has* $e = 2n - 3$ *edges and* $n - 2$ *pseudotriangles.*

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$$\underbrace{\sum_t k_t + k_{\text{outer}}}_{2e} - 3|T| = y$$

$$e + 2 = (|T| + 1) + (x + y) \qquad \text{(Euler)}$$

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Corollary. *A non-crossing pointed graph with* $n \geq 2$ *vertices has at most* $2n - 3$ *edges.*

Pseudotriangulations/ Geodesic Triangulations

Applications:

- kinetics of bar frameworks, robot motion planning, the "Carpenter's Rule Problem" [Streinu 2000]
- data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]
- kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002]
- art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]

2A. RIGIDITY, PLANAR LAMAN GRAPHS

Infinitesimal motions — rigid frameworks

A *framework* is a set of movable joints (vertices) connected by rigid *bars* (edges) of fixed length.

n points $p_1, \ldots, p_n$.

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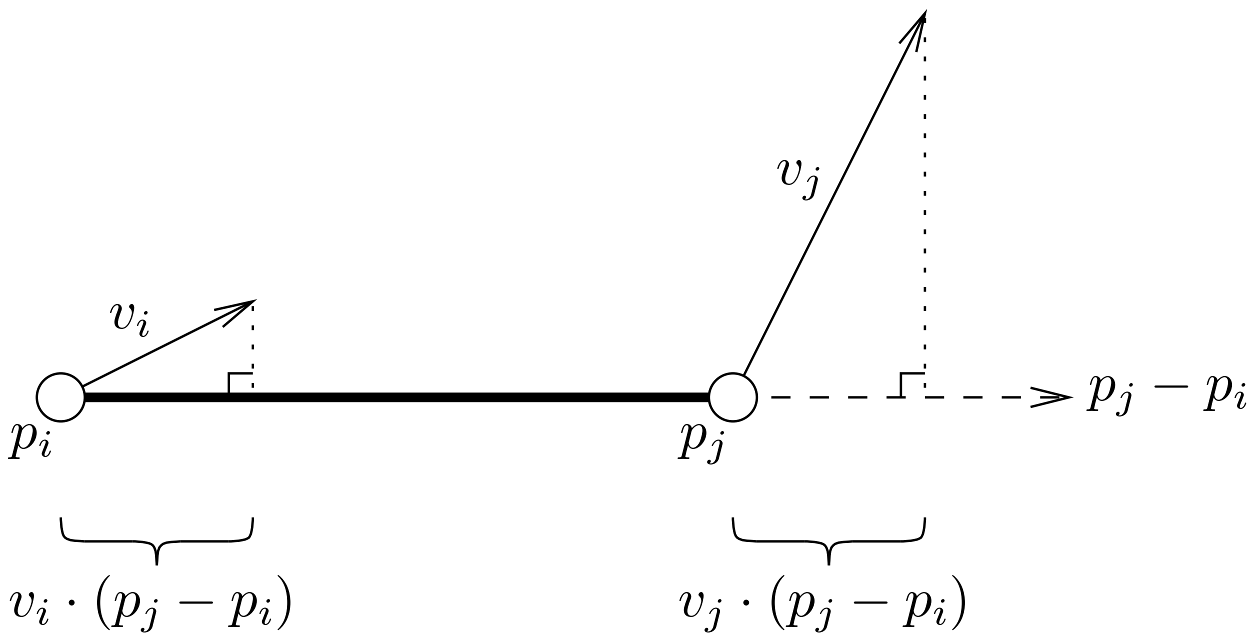
   velocity vectors $v_1, \ldots, v_n$.

3. constraints:
   $|p_i(t) - p_j(t)|$ is constant for every edge (bar) ij.

Expansion

$$\tfrac{1}{2} \cdot \frac{d}{dt}|p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle =: \exp_{ij}$$



expansion (or *strain*) $\exp_{ij}$ of the segment ij

The rigidity map

of a framework $((V,E),(p_1,\ldots,p_n))$:

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The rigidity matrix:

$$M = \underbrace{\begin{pmatrix} \text{the} \\ \text{rigidity} \\ \text{matrix} \end{pmatrix}}_{2|V|} \left.\vphantom{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}\right\} E$$

Infinitesimally rigid frameworks

A framework is *infinitesimally rigid* if

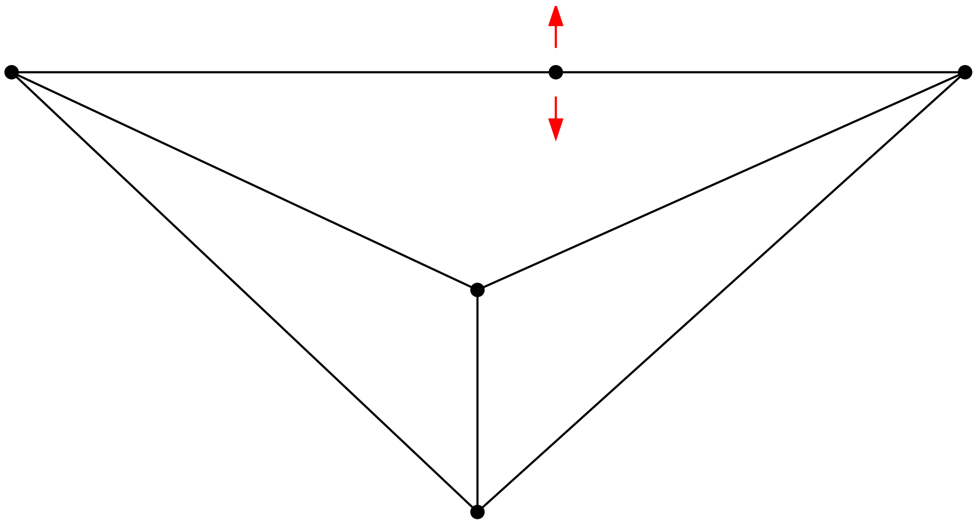
$$M(v) = 0$$

has only the trivial solutions: translations and rotations of the framework as a whole.

Rigid frameworks

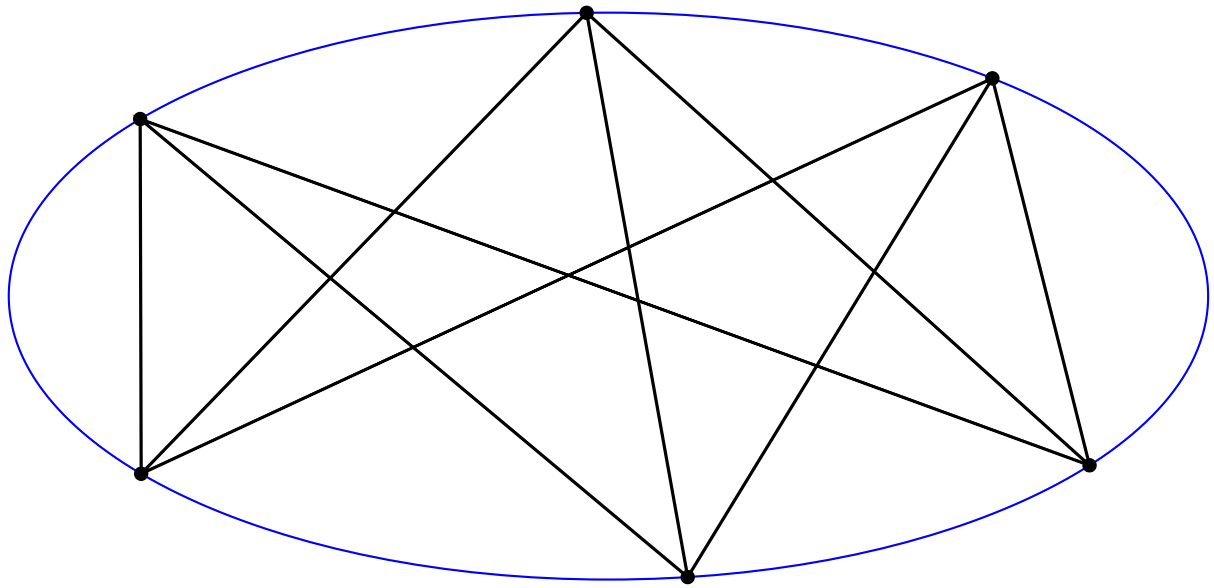
An infinitesimally rigid framework is rigid.

This framework is rigid, but not infinitesimally rigid:



Generically rigid frameworks

A given graph can be rigid in most embeddings, but it may have special non-rigid embeddings:



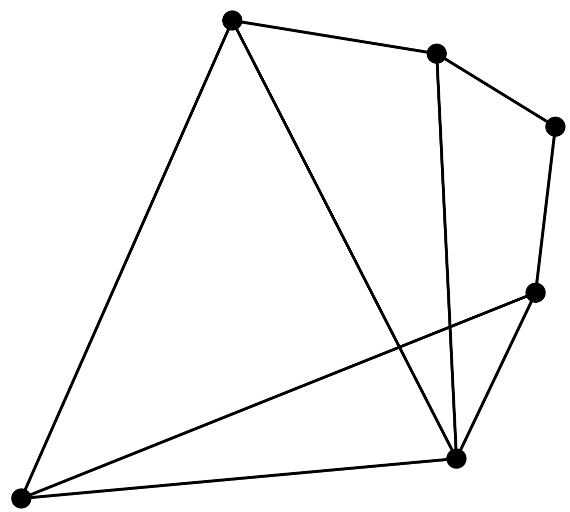
A graph is *generically rigid* if it is infinitesimally rigid in almost all embeddings.

This is a *combinatorial property* of the graph.

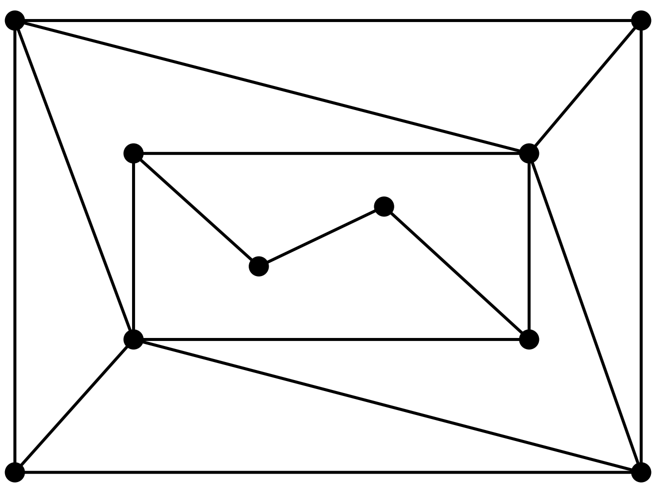
Minimally rigid frameworks

Theorem. A graph with n vertices is *minimally rigid* in the plane (with respect to $\subseteq$) iff it has the *Laman property*:

- It has $2n - 3$ edges.
- Every subset of $k \geq 2$ vertices spans at most $2k - 3$ edges.



$n = 6,\ e = 9$



$n = 10,\ e = 17$

[Laman 1961]

A pointed pseudotriangulation is a Laman graph

Proof: Every subset of $k \geq 2$ vertices is pointed and has therefore at most $2k - 3$ edges.

[Streinu 2001]

Every planar Laman graph is a pointed pseudotriangulation

Theorem. *Every planar Laman graph has a realization as a pointed pseudotriangulation. The outer face can be chosen arbitrarily.*

[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]

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Proof I: Induction, using *Henneberg constructions*

Proof II: via Tutte embeddings for directed graphs

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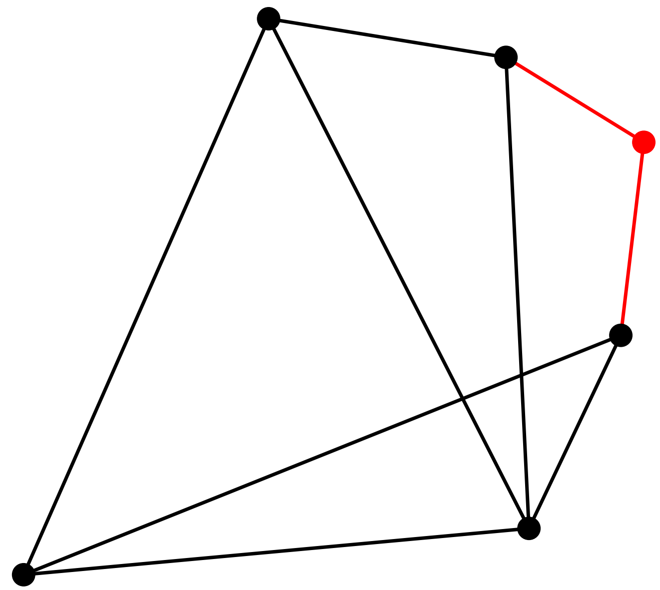
Proof I: Induction, using *Henneberg constructions*

Proof II: via Tutte embeddings for directed graphs

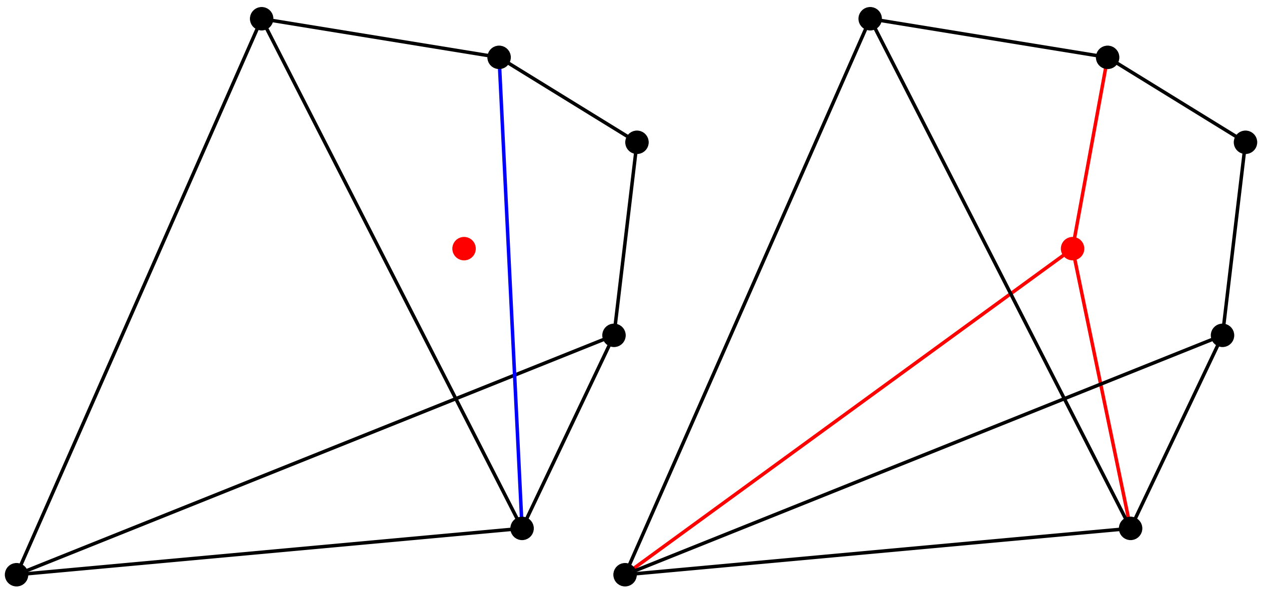
Theorem. *Every rigid planar graph has a realization as a pseudotriangulation (not necessarily pointed).*

[Orden, Santos, B. Servatius, H. Servatius 2003]

Henneberg constructions



Type I

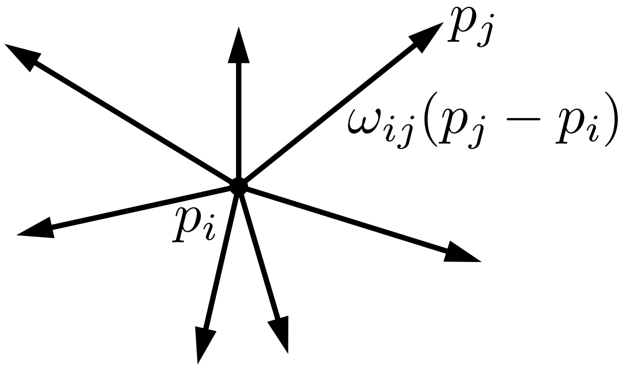


Type II

Self-stresses

Assign a *stress* $\omega_{ij} = \omega_{ji} \in \mathbb{R}$ to each edge.

Equilibrium of forces in every vertex i:



$$\sum_j \omega_{ij}(p_j - p_i) = 0$$

$$M^{\mathrm{T}}\omega = 0$$

$$\exp = Mv$$

2B. RIGIDITY AND KINEMATICS
Unfolding of polygons — expansive motions

Theorem. Every polygonal arc in the plane can be brought into straight position, without self-overlap.

Every polygon in the plane can be unfolded into convex position. [Connelly, Demaine, Rote 2000], [Streinu 2000]

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Proof outline:

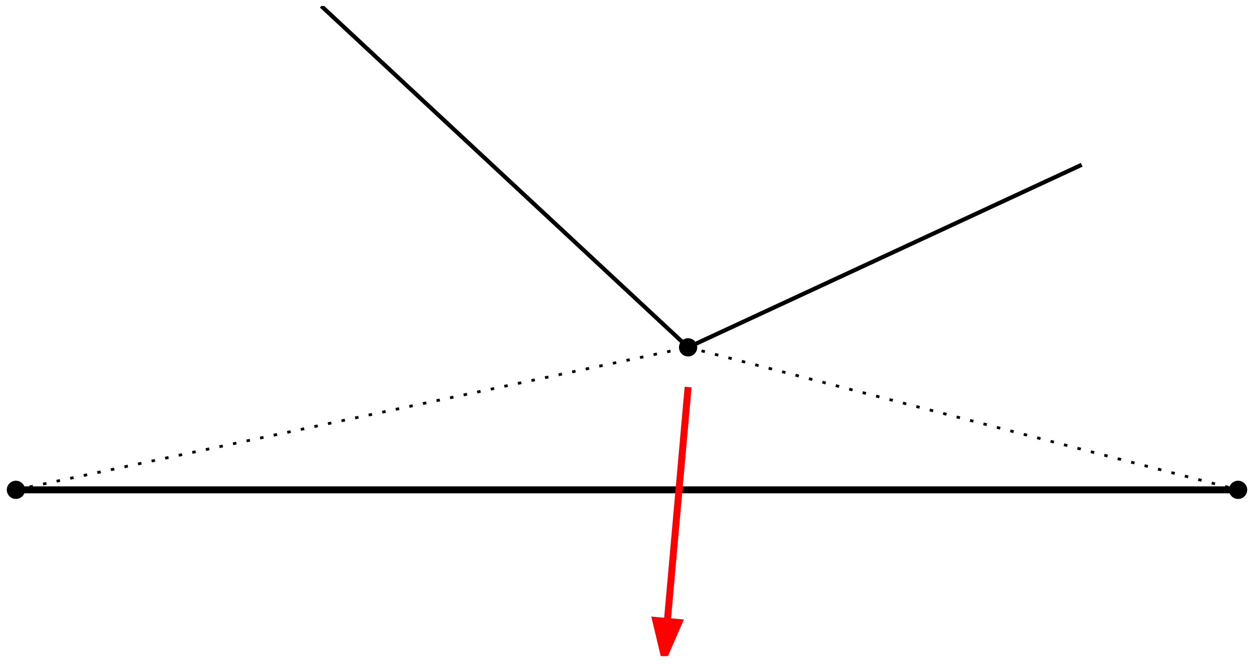
1. Find an *expansive* infinitesimal motion.

2. Find a global motion.

Expansive Motions

No distance between any pair of vertices decreases.

Expansive motions cannot overlap.



Expansive motions

$\exp_{ij} = 0$ for all *bars* ij

(preservation of length)

$\exp_{ij} \geq 0$ for all other pairs (*struts*) ij

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(expansiveness)

. . . need to show that an expansive motion exists . . .

Every Polygon has an Expansive Motion

Proof I: (Outline)

Existence of an expansive motion

$\Updownarrow$ (duality)

Self-stresses (rigidity)
Self-stresses on planar frameworks

$\Updownarrow$ (Maxwell-Cremona correspondence)

polyhedral terrains

[Connelly, Demaine, Rote 2000]

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Proof II: via pseudotriangulations and the Pseudotriangulation Polytope

[Streinu 2000] [Rote, Santos, Streinu 2003]

3. A polyhedron for pointed pseudotriangulations

Theorem. *For every set S of points in general position, there is a convex $(2n-3)$-dimensional polyhedron X whose vertices correspond to the pointed pseudotriangulations of S.*

[Rote, Santos, Streinu 2003]

There is one inequality for each pair of points.

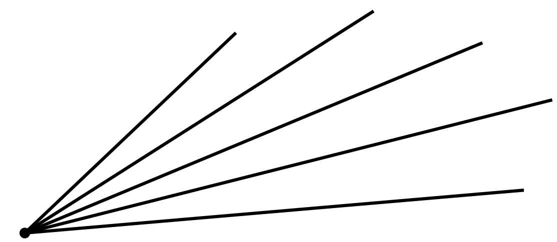
At a vertex of X:

tight inequalities $\leftrightarrow$ edges of a pointed pseudotriangulation.

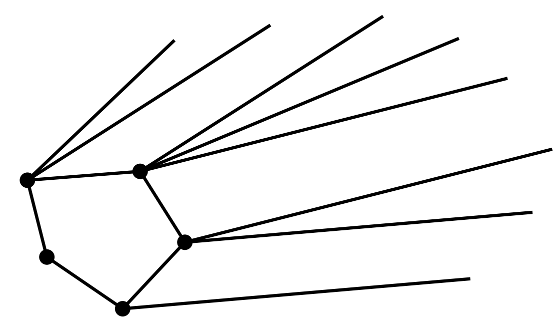
Cones and polytopes

[Rote, Santos, Streinu 2002]

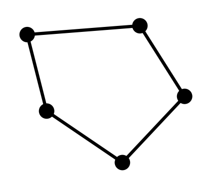
- The *expansion cone*
  $\bar{X}_0 = \{\, \exp_{ij} \geq 0 \,\}$



- The *perturbed expansion cone*
  = the *PPT polyhedron*
  $\bar{X}_f = \{\, \exp_{ij} \geq f_{ij} \,\}$

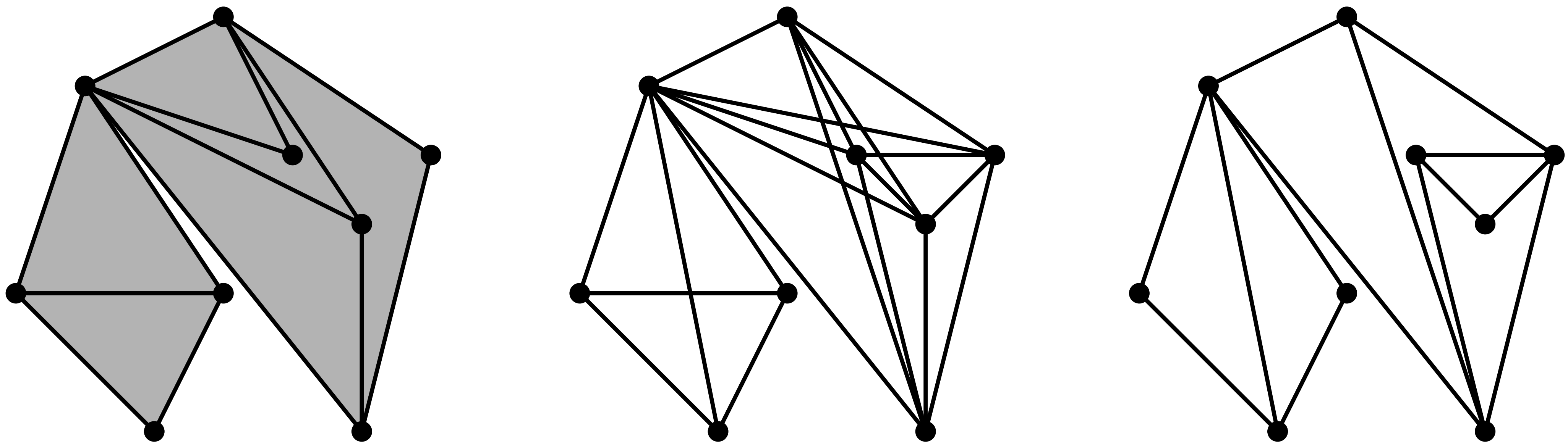


- The *PPT polytope*
  $X_f = \{\, \exp_{ij} \geq f_{ij},$
  $\exp_{ij} = f_{ij}$ for ij on boundary $\}$



Extreme rays of the expansion cone

Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]
Rigid substructures can be identified.



The Dimension of the Polyhedra: Pinning of Vertices

Trivial Motions: Motions of the point set as a whole (translations, rotations).

Normalization: Pin a vertex and a direction. ("tie-down")

$$v_1 = 0$$

$$v_2 \parallel p_2 - p_1$$

This eliminates 3 degrees of freedom.

The polyhedra "live in" $2n - 3$ dimensions.
(plus a 3-dimensional lineality space).

A polyhedron for pseudotriangulations

With a suitable perturbation of the constraints "$\exp_{ij} \geq 0$" to "$\exp_{ij} \geq f_{ij}$", the vertices are in 1-1 correspondence with the pointed pseudotriangulations.

$\longrightarrow$ the PPT-polyhedron

$$\bar{X}_f = \{\, (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \,\}$$

$\longrightarrow$ an independent proof that expansive motions exist

Tight edges

For $v = (v_1, \ldots, v_n) \in \bar{X}_f$,

$$E(v) := \{ ij \mid \exp_{ij} = f_{ij} \}$$

is the *set of tight edges* at v.

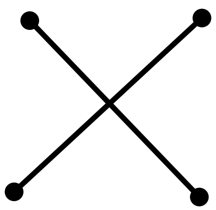
Maximal sets of tight edges $\equiv$ vertices of $\bar{X}_f$.

What are good values of f_{ij}?

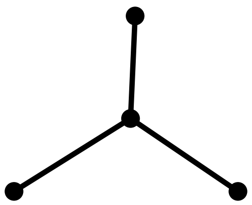
Which configurations of edges can occur in a set of tight edges?

We want:

- no crossing edges



- no 3-star with all angles $\leq 180^\circ$



The PPT-polyhedron

$\longrightarrow$ For every vertex v, $E(v)$ is non-crossing and pointed.

$\longrightarrow$ $|E(v)| \leq 2n-3$

$\longrightarrow$ $|E(v)| = 2n-3$ and $\bar{X}_f$ is a simple polyhedron.

Every vertex is incident to $2n-3$ edges.

Edge $\equiv$ removing a segment from $E(v)$.

Removing an interior segment leads to an adjacent pseudotriangulation (flip).

Removing a hull segment is an extreme ray. $\square$

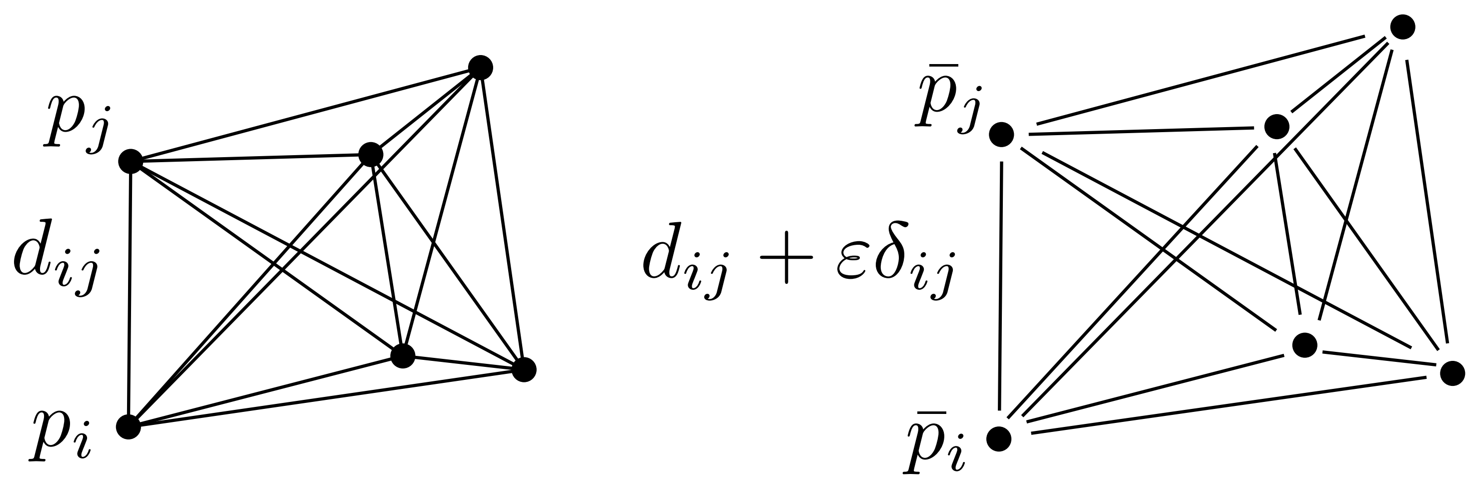
Increasing the distances

$$d_{ij} := \|p_i - p_j\|$$

Find new locations $\bar{p}_i$ such that

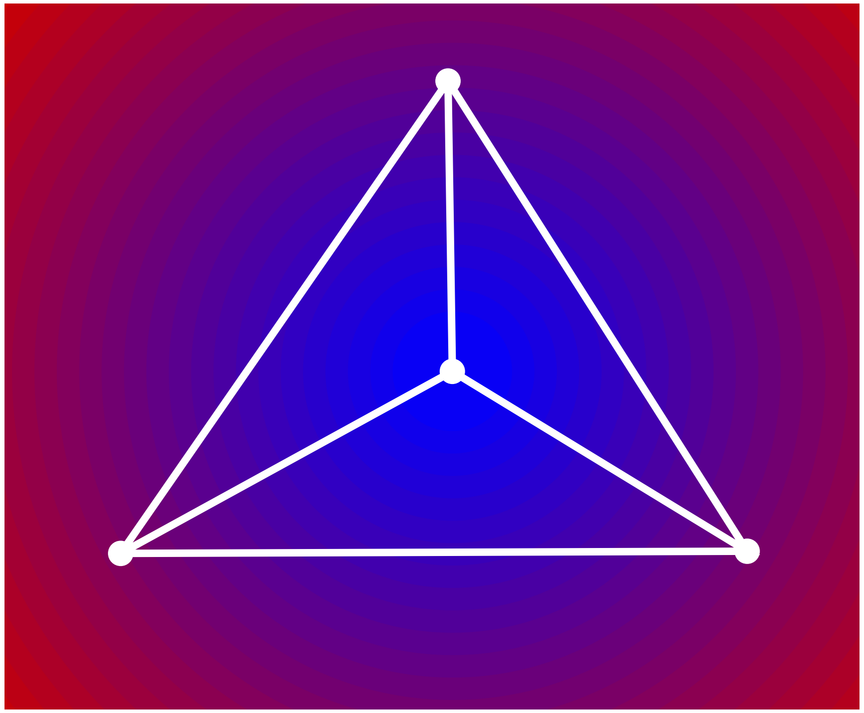
$$\|\bar{p}_i - \bar{p}_j\| \geq d_{ij} + \varepsilon\delta_{ij}$$

for very small (infinitesimal) ε and appropriate numbers δ_{ij}.



If the new distances $d_{ij} + \varepsilon\delta_{ij}$ are generic, the maximal sets of tight inequalities will correspond to minimally rigid graphs.

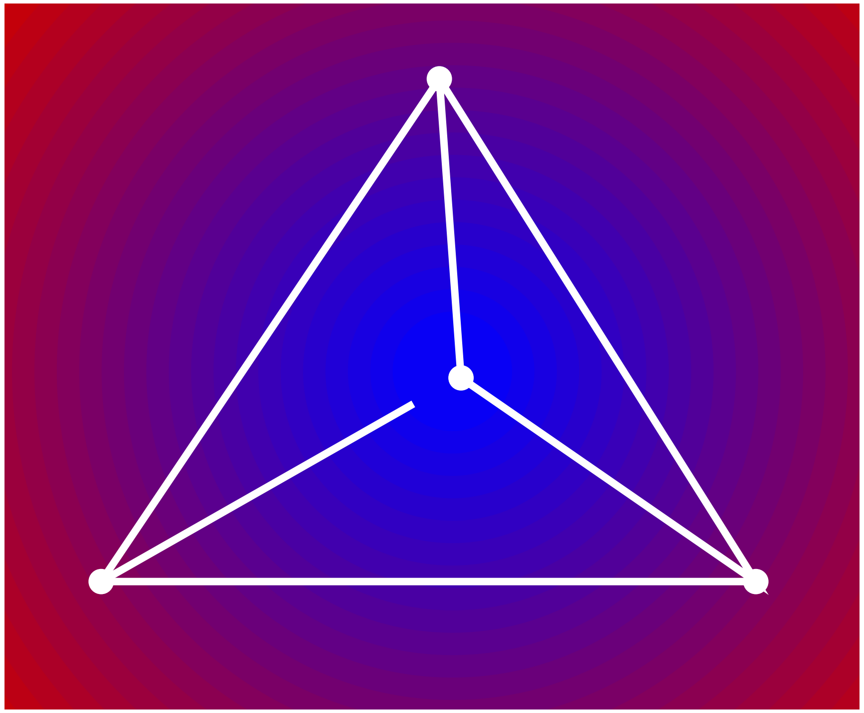
Heating up the bars



$$\Delta T = |x|^2$$

$$\text{Length increase } \geq \int_{x \in p_i p_j} |x|^2 \, ds$$

Heating up the bars



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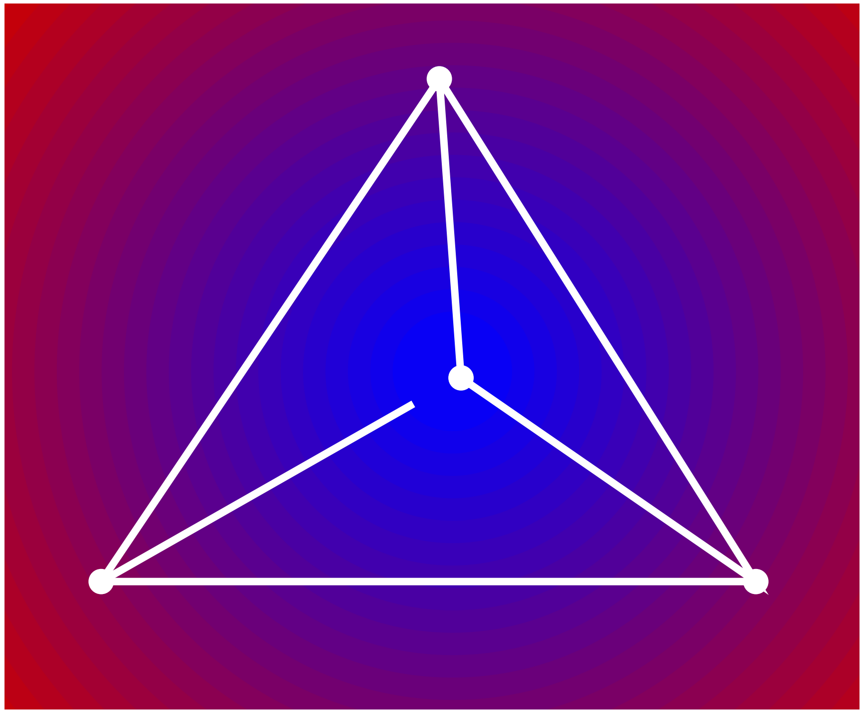

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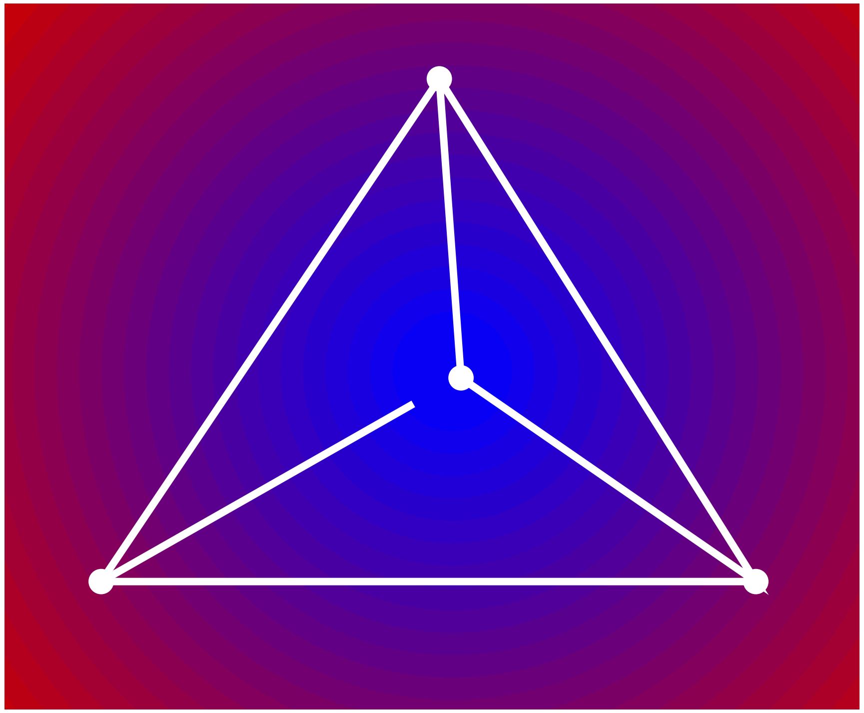
Heating up the bars



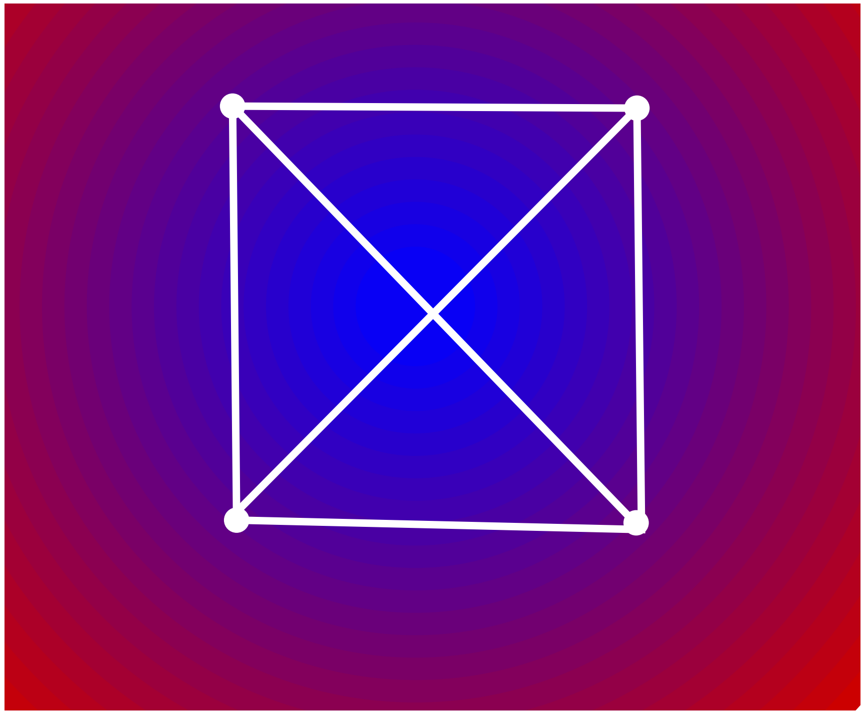
$$\Delta T = |x|^2$$

$$\text{Length increase } \geq \int_{x \in p_i p_j} |x|^2 \, ds$$

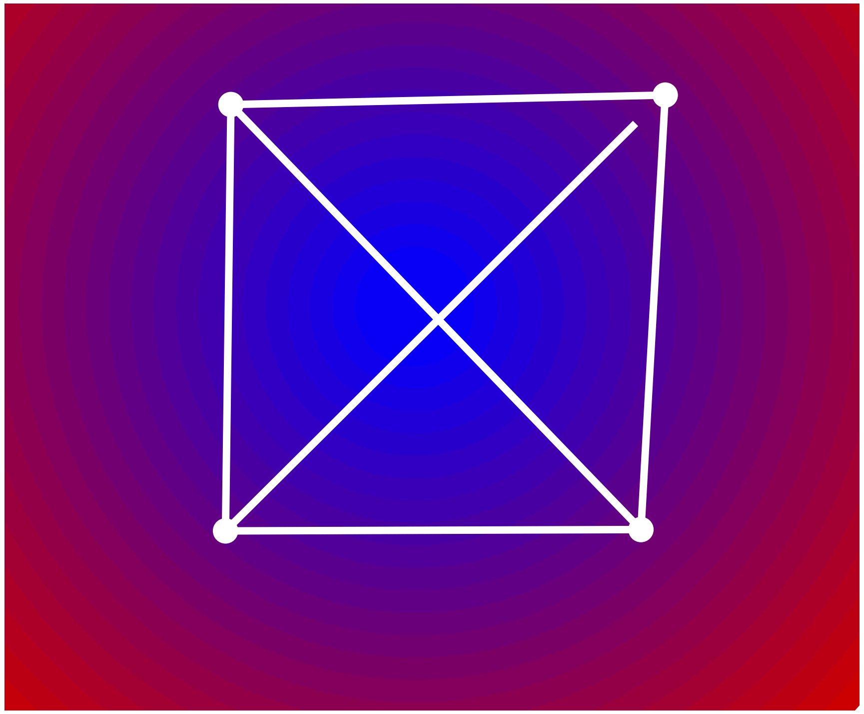
$$\delta_{ij} = \int_{x \in p_i p_j} |x|^2 \, ds$$

$$\delta_{ij} = |p_i - p_j| \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2) \cdot \tfrac{1}{3}$$

Heating up the bars — points in convex position



$\Rightarrow$



The PPT polyhedron

$$\bar{X}_f = \{ (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \}$$

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$

The PPT polyhedron

$$\bar{X}_f = \{ (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \}$$

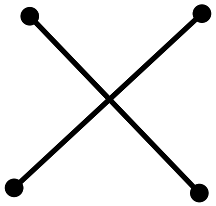
- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- Alternative definition that leads to an equivalent polytope.
  $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

  $[x, y, z] =$ signed area of the triangle xyz
  a, b: two arbitrary points.

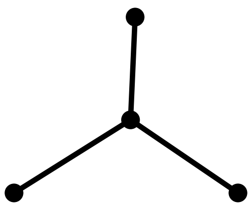
Good values f_{ij} for 4 points

In a set of tight edges, we want:

- no crossing edges

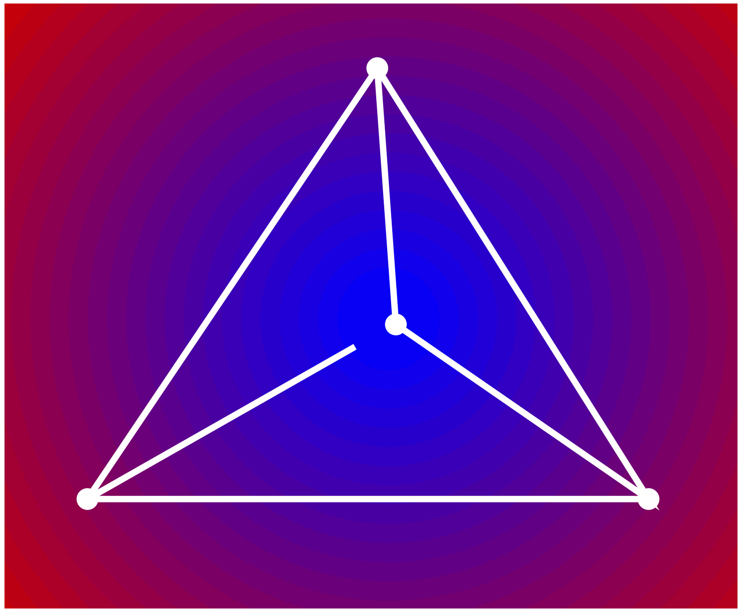


- no 3-star with all angles $\leq 180^\circ$



It is sufficient to look at 4-point subsets.

Good values f_{ij} for 4 points

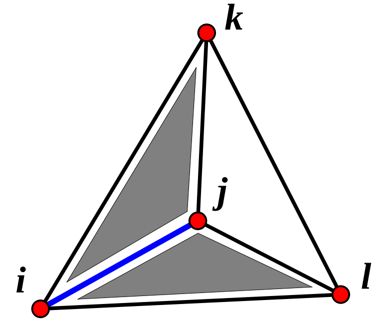


f_{ij} is given on six edges.
Any five values $\exp_{ij}$ determine the last one.
Check if the resulting value $\exp_{ij}$ of the last edge is feasible ($\exp_{ij} \geq f_{ij}$)
$\longrightarrow$ checking the sign of an expression.

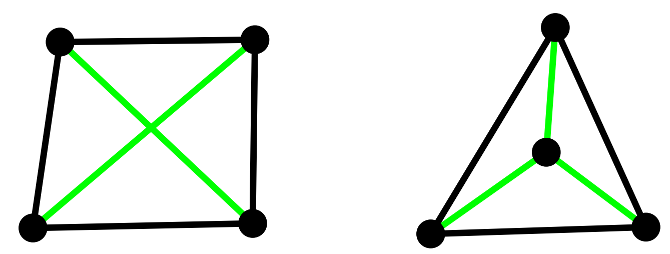
Good Values f_{ij} for 4 points

A 4-tuple p_1, p_2, p_3, p_4 has a unique self-stress (up to a scalar factor).

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \text{ for all } 1 \le i < j \le 4$$



$\omega_{ij} > 0$ for boundary edges.
$\omega_{ij} < 0$ for interior edges.



Why the stress?

If the *equation*

$$\sum_{1\le i<j\le 4} \omega_{ij} f_{ij} = 0$$

holds, then f_{ij} are the expansion values $\exp_{ij}$ of a motion (v_1, v_2, v_3, v_4).

Actually, “if and only if”.

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Actually, "if and only if".

[$M^{\mathrm{T}}\omega = 0$, $f = \exp = Mv$]

Good perturbations

We need

$$\omega_{12} f_{12} + \omega_{13} f_{13} + \omega_{14} f_{14} + \omega_{23} f_{23} + \omega_{24} f_{24} + \omega_{34} f_{34} > 0$$

for all 4-tuples of points p_1, p_2, p_3, p_4, with

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \qquad f_{ij} = [a, p_i, p_j][b, p_i, p_j]$$

Good perturbations

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$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} = 1$$

Good perturbations

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$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} > 0$$

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$$\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} = 1 > 0$$

What is the meaning of $\sum_{1\leq i<j\leq 4}\omega_{ij}f_{ij} = 1$?

"I believe there is some underlying homology in this situation. Given the fact that motions and stresses also fit into a setting of cohomology and homology as well, the authors might, at least, mention possible homology descriptions."

[a referee, about the definition of ω_{ij}]

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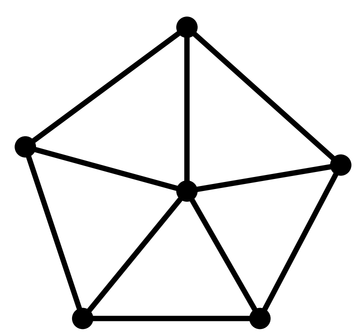
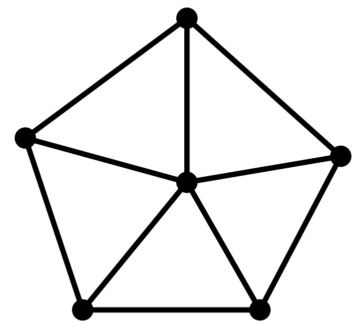
[a referee, about the definition of ω_{ij}]

One can define a similar formula for ω for the k-wheel.



$\sum_{ij\in E}\omega_{ij}f_{ij}=1$ for the k-wheel

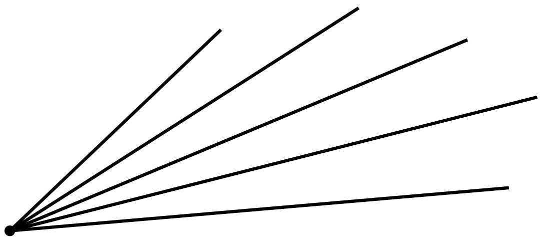


$$\omega_{i,i+1}=\frac{1}{[p_i,p_{i+1},p_0]\cdot[p_1,p_2,\ldots,p_k]}$$

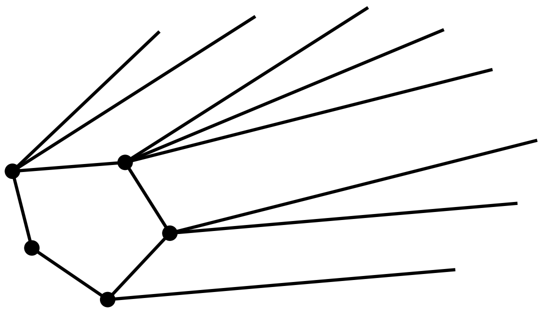
$$\omega_{0i}=\frac{1}{[p_{i-1},p_i,p_0]\cdot[p_i,p_{i+1},p_0]}\cdot\frac{[p_{i-1},p_i,p_{i+1}]}{[p_1,p_2,\ldots,p_k]}$$

Cones and polytopes

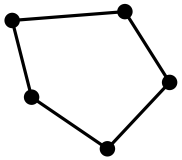
- The *expansion cone*
  $\bar{X}_0 = \{ \exp_{ij} \geq 0 \}$



- The *perturbed expansion cone*
  $=$ the *PPT polyhedron*
  $\bar{X}_f = \{ \exp_{ij} \geq f_{ij} \}$



- The *PPT polytope*
  $X_f = \{ \exp_{ij} \geq f_{ij},$
  $\exp_{ij} = f_{ij}$ for ij on boundary $\}$



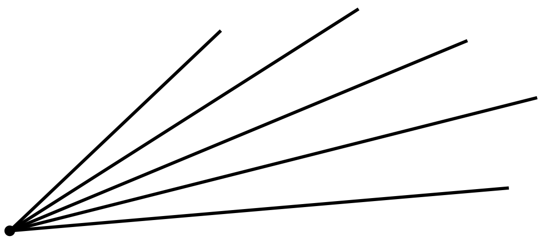
The PPT polytope

Cut out all rays:
Change $\exp_{ij} \geq f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

Theorem. *For every set S of points in general position, there is a convex $(2n-3)$-dimensional polytope whose vertices correspond to the pointed pseudotriangulations of S.*

Cones and polytopes

- The *expansion cone*
  $\bar{X}_0 = \{\, \exp_{ij} \geq 0 \,\}$



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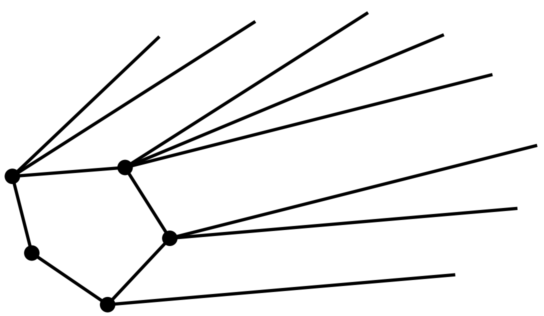

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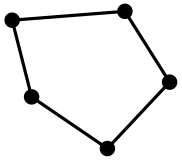
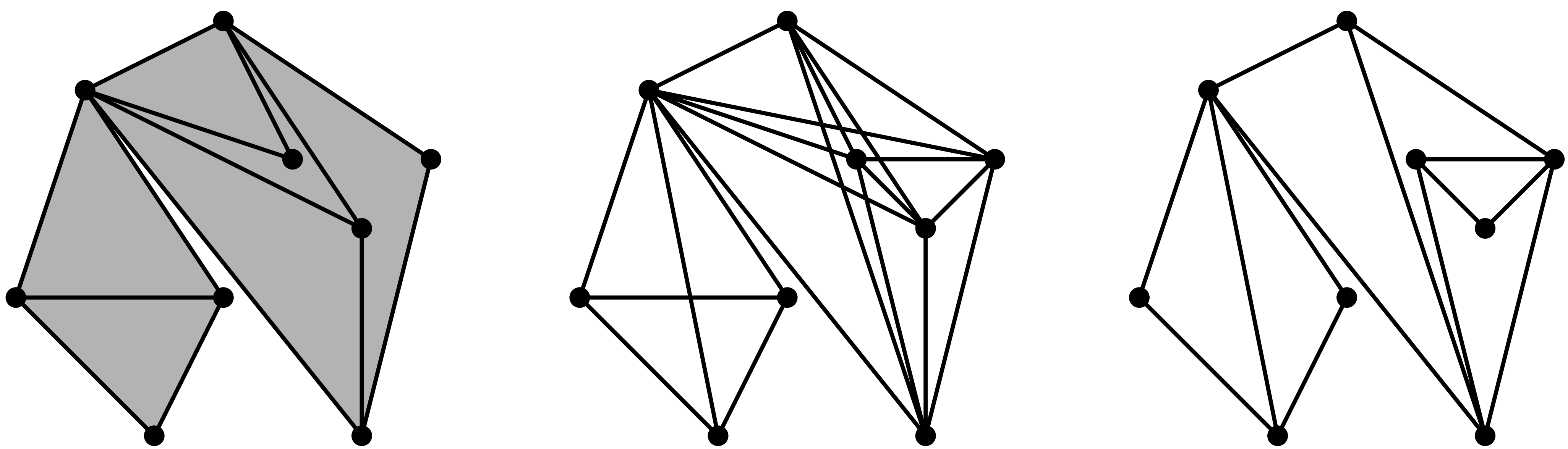


Extreme rays of the expansion cone

The Expansion Cone $\bar{X}_0$:

collapse parallel rays into one ray. $\rightarrow$ pseudotriangulations minus one hull edge. Rigid subcomponents are identified.

Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]



Expansive motions for a chain (or a polygon)

- Add edges to form a pseudotriangulation
- Remove a convex hull edge
- $\rightarrow$ expansive mechanism $\square$

Theorem. Every polygonal arc in the plane can be brought into straight position, without self-overlap.

Every polygon in the plane can be unfolded into convex position.

[Connelly, Demaine, Rote 2000], [Streinu 2000]

The PT polytope

Vertices correspond to *all* pseudotriangulations, pointed or not.

Change inequalities $\exp_{ij} \geq f_{ij}$ to

$$\exp_{ij} + (s_i + s_j)\|p_j - p_i\| \geq f_{ij}$$

with a "slack variable" s_i for every vertex.
$s_i = 0$ indicates that vertex i is pointed.

A "flip" may insert an edge, changing a vertex from pointed to non-pointed, or vice versa.

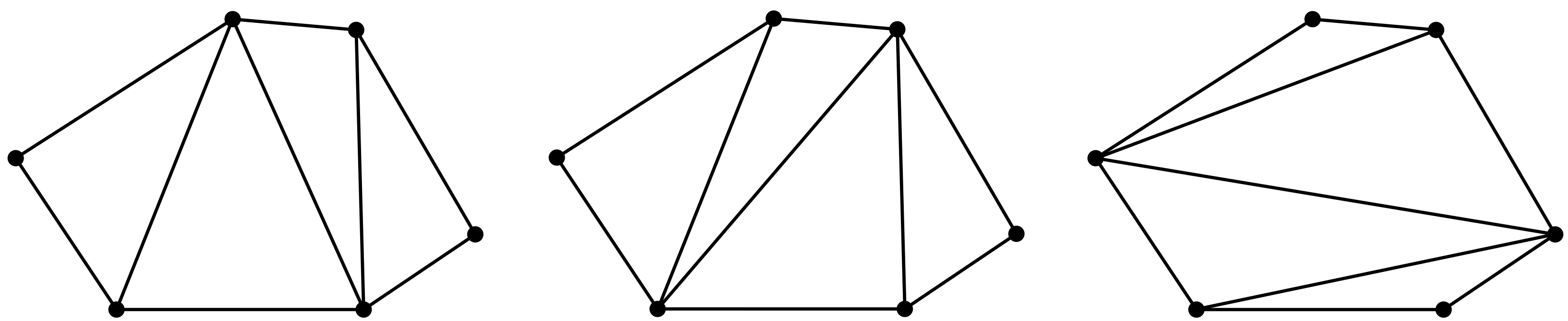
Faces are in one-to-one correspondence with all non-crossing graphs.

[Orden, Santos 2002]

Canonical pseudotriangulations

Maximize/minimize $\sum_{i=1}^{n} c_i \cdot v_i$ over the PPT-polytope.

$c_i := p_i$:



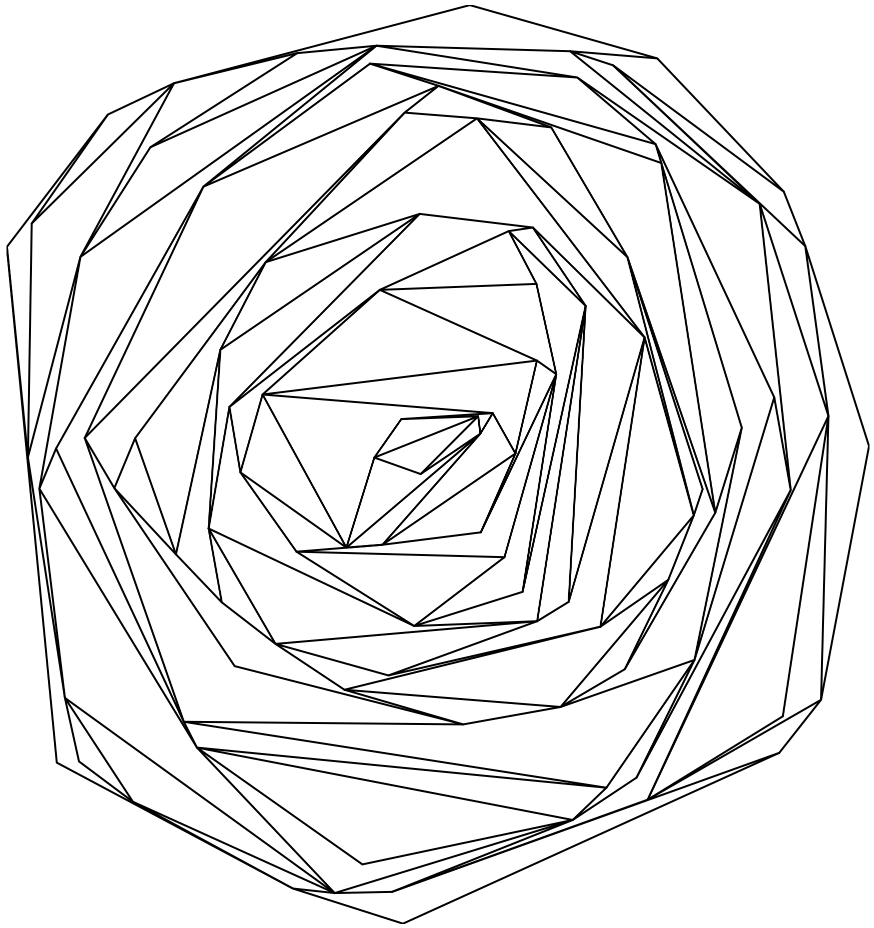
(a) (b) (c)

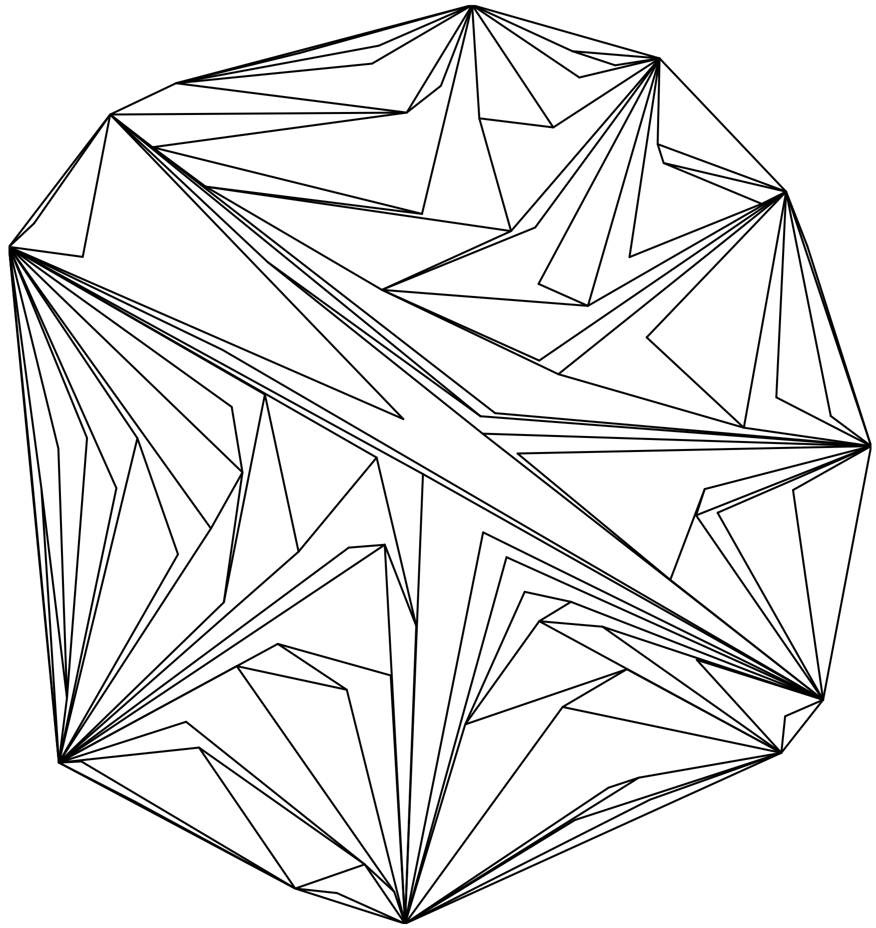
Delaunay triangulation

Max/Min $\sum p_i \cdot v_i$
(not affinely invariant)

(Can be constructed as the lower/upper convex hull of lifted points.) [André Schulz 2005]

Two pseudotriangulations for 100 random points





Which f_{ij} to choose?

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

Go to the space of the $(\exp_{ij})$ variables instead of the (v_i) variables.

$$\exp = Mv$$

Characterization of the space $(\exp_{ij})_{i,j}$

A set of values $(\exp_{ij})_{1\le i<j\le n}$ forms the expansion vector of a motion $(v_1,\ldots,v_n)$: $\exp = Mv$
if and only if the vector $(\exp_{ij})_{1\le i<j\le n}$ is orthogonal to all self-stresses $(\omega_{ij})_{1\le i<j\le n}$:

$\omega\cdot\exp = 0$ for all ω with $M^{\mathrm{T}}\omega = 0$

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$$\omega \cdot \exp = 0 \text{ for all } \omega \text{ with } M^{\mathrm{T}}\omega = 0$$

if and only if the equation

$$\sum_{1\le i<j\le 4} \omega_{ij} \exp_{ij} = 0$$

holds for all 4-tuples.

SKIP

A canonical representation

$$\sum_{1\leq i<j\leq 4} \omega_{ij}\,\mathrm{exp}_{ij} = 0, \text{ for all 4-tuples}$$

$$\mathrm{exp}_{ij} \geq f_{ij}, \text{ for all pairs } i, j$$

A canonical representation

$$\sum_{1\leq i<j\leq 4} \omega_{ij}\exp_{ij} = 0, \text{ for all 4-tuples}$$

$$\exp_{ij} \geq f_{ij}, \text{ for all pairs } i,j$$

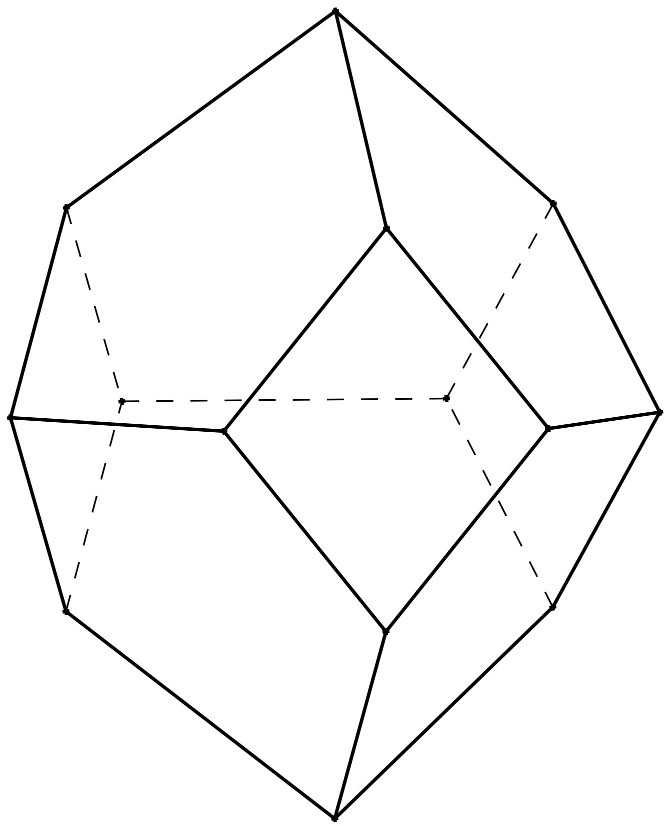
$$\sum_{1\leq i<j\leq 4} \omega_{ij} f_{ij} = 1, \text{ for all 4-tuples}$$

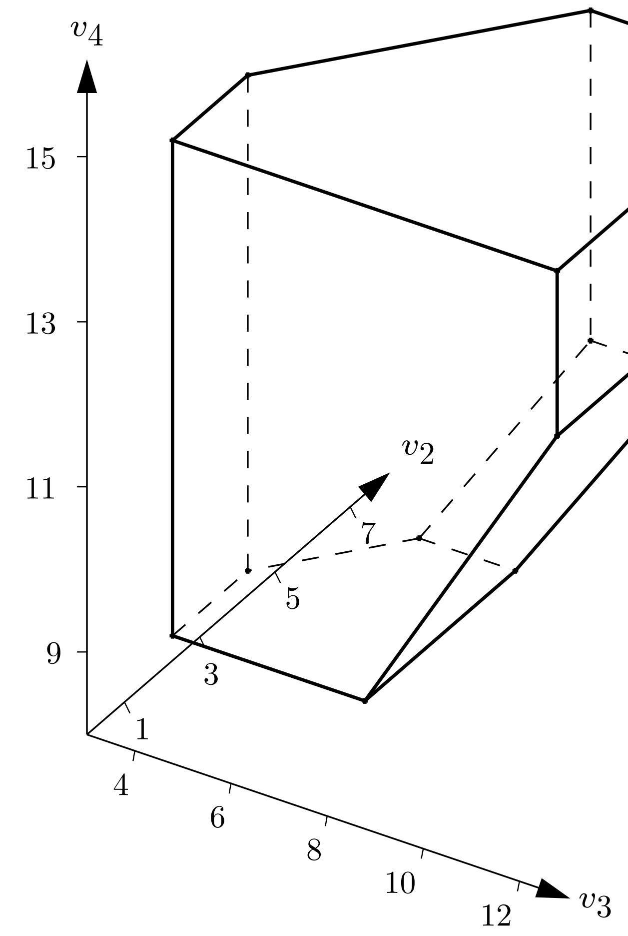
Substitute $d_{ij} := \exp_{ij} - f_{ij}$:

$$\textstyle\sum_{1\leq i<j\leq 4} \omega_{ij} d_{ij} = -1, \text{ for all 4-tuples} \quad (1)$$

$$d_{ij} \geq 0, \text{ for all } i,j \quad (2)$$

The associahedron





Catalan structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation
- $(a * (b * (c * d))) * e$ / $((a * b) * (c * d)) * e$
- . . . .

The secondary polytope

Triangulation T of a point set $\{p_1, \ldots, p_n\}$:
$T \mapsto (a_1, \ldots, a_n)$.

$$a_i := \text{total area of all triangles incident to } p_i$$

The secondary polytope :=

$$\text{conv}\{\, (a_1, \ldots, a_n)(T) \mid T \text{ is a triangulation} \,\}$$

vertices $\equiv$ regular triangulations of $(p_1, \ldots, p_n)$

$(p_1, \ldots, p_n)$ in convex position:
pseudotriangulations $\equiv$ triangulations $\equiv$ regular triangulations.

$\longrightarrow$ two realizations of the associahedron.

These two associahedra are affinely equivalent.

Expansive motions in one dimension

$$\{\,(v_i) \in \mathbb{R}^n \mid v_j - v_i \geq f_{ij} \text{ for } 1 \leq i < j \leq n\,\}$$

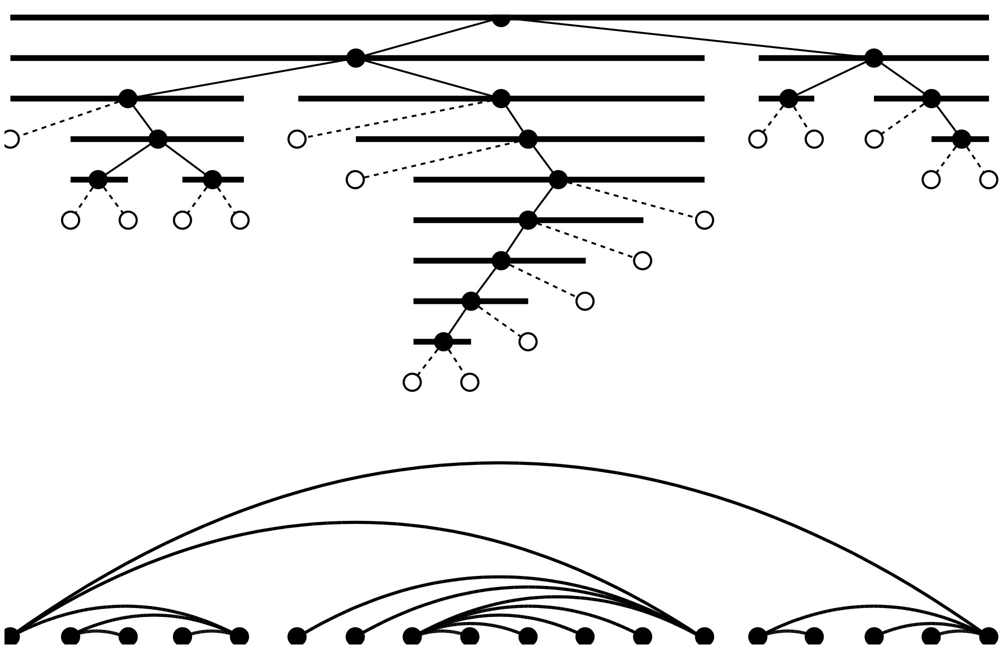
$$f_{il} + f_{jk} > f_{ik} + f_{jl}, \text{ for all } i < j < k < l.$$
$$f_{il} > f_{ik} + f_{kl}, \text{ for all } i < k < l.$$

For example, $f_{ij} := (i-j)^2$

related to the *Monge Property*.

$\longrightarrow$ gives rise to *different* realizations of the associahedron.

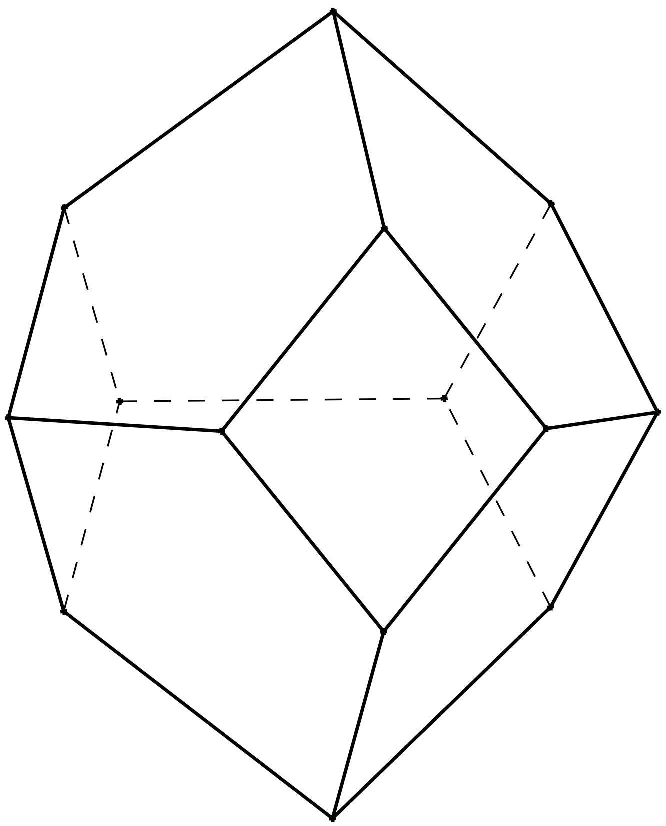
Non-crossing alternating trees

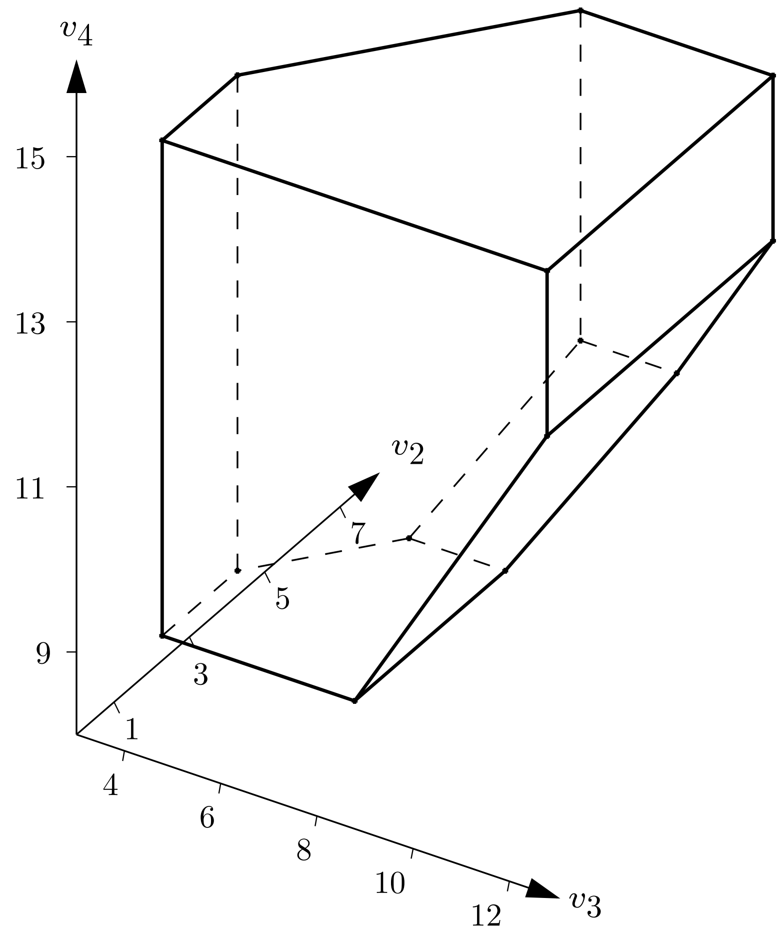


non-crossing: no two edges ik, jl with $i < j < k < l$.
alternating: no two edges ij, jk with $i < j < k$.

[Gelfand, Graev, and Postnikov 1997], in a dual setting.
[Postnikov 1997], [Zelevinsky ?], [Stasheff 1997]

The associahedron





OPEN: Pseudotriangulations in 3-space?

Rigid graphs are not well-understood in 3-space.

Alternative approach: Pseudotriangulation of the interior of a *polygon* via *locally convex functions*

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

This can be extended to 3-polytopes.

[Aurenhammer, Krasser 2005]

INPUT-A NO INPUT