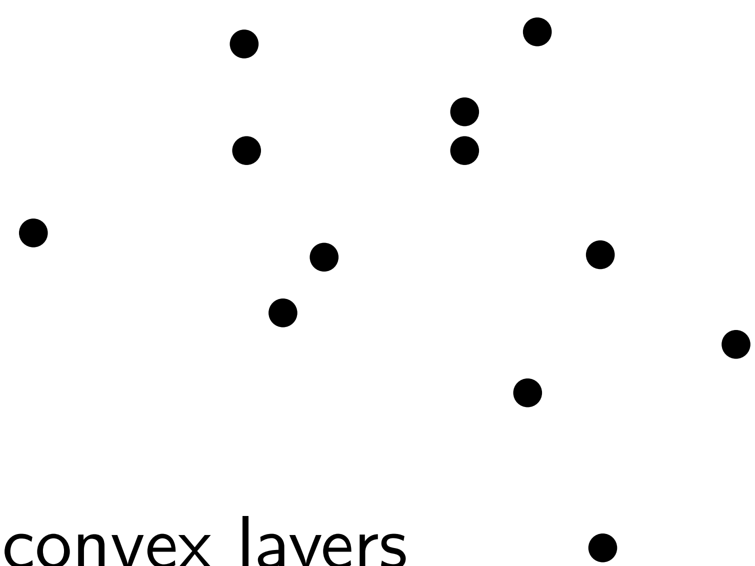
# Grid Peeling and the Affine Curve-Shortening Flow (ACSF)

Günter Rote, Moritz Rüber, and Morteza Saghafian
Freie Universität Berlin / ISTA



convex layers
onion layers

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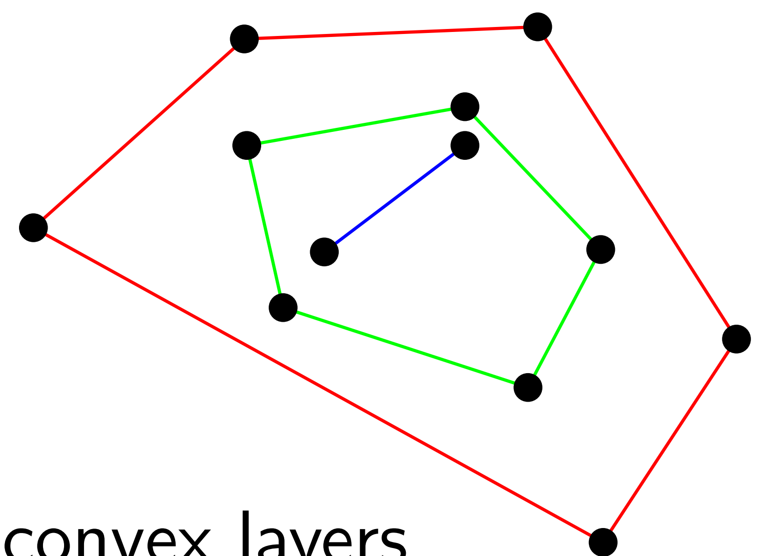

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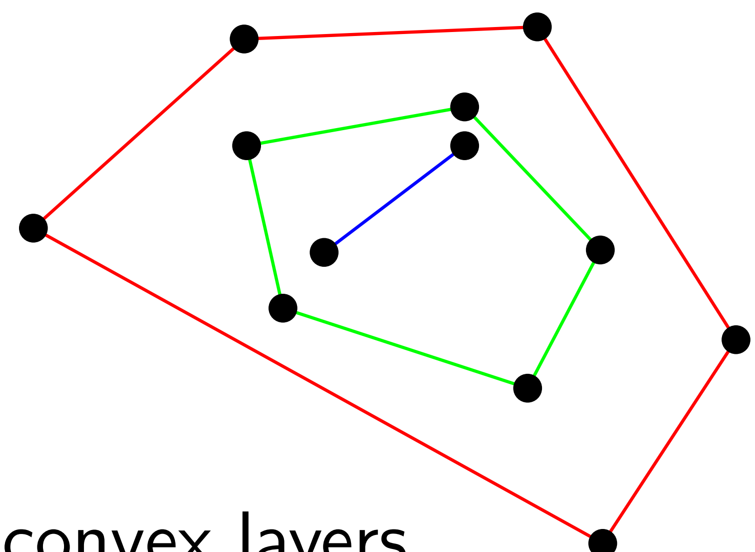

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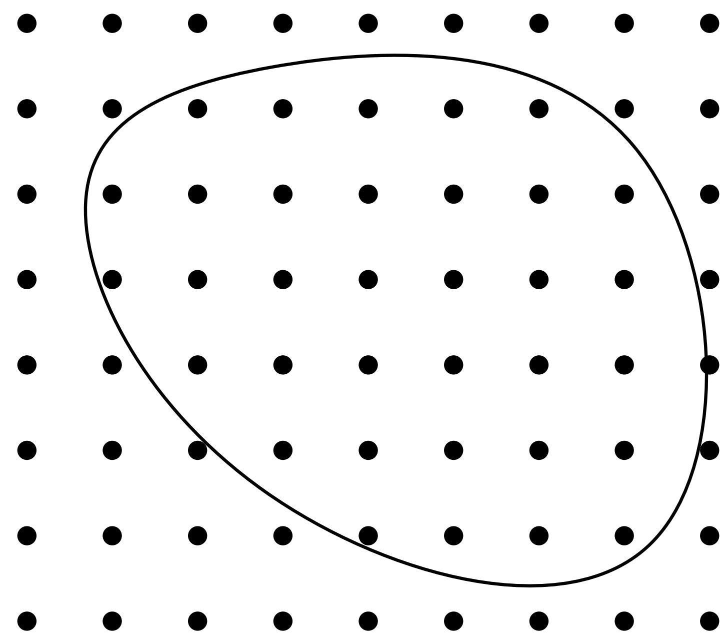

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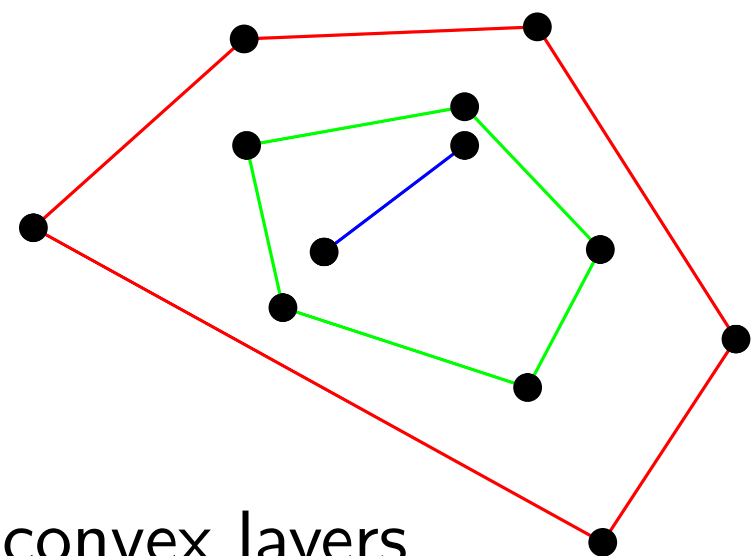

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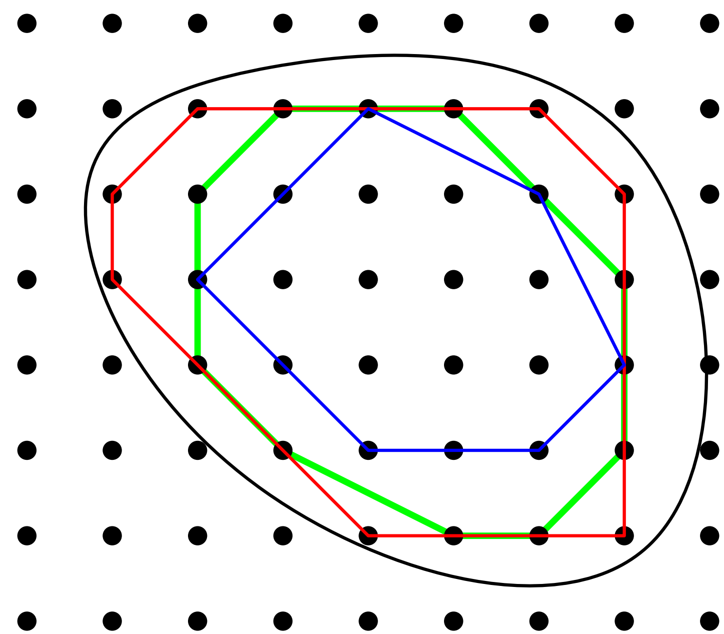
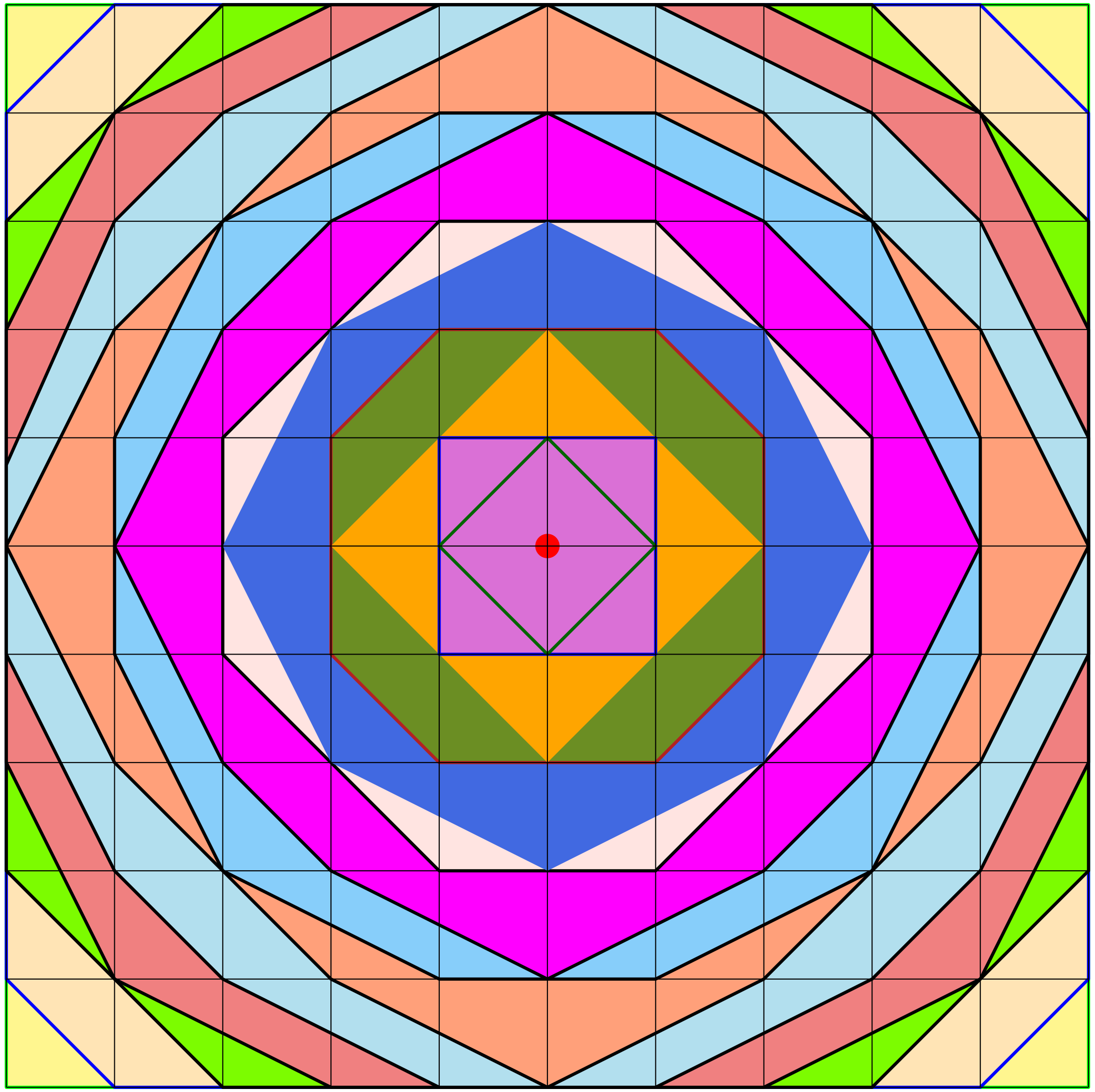


grid peeling

# Grid Peeling of the Square

[ Sariel Har-Peled and Bernard Lidický 2013 ]

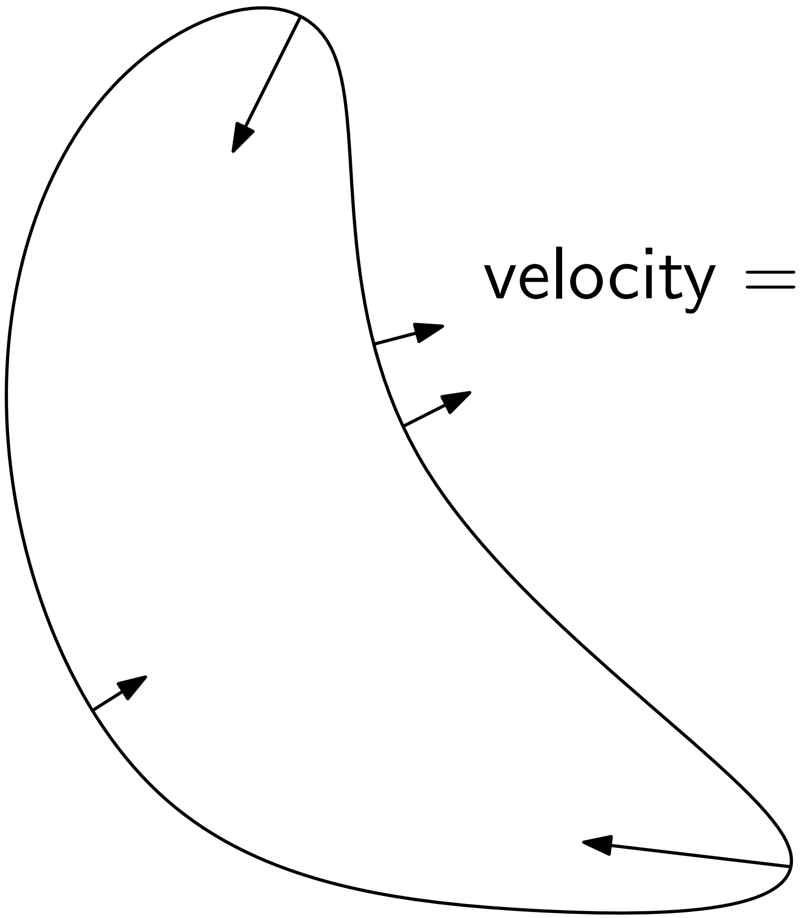


The $n \times n$ grid has $\Theta(n^{4/3})$ convex layers.

# Affine Curve-Shortening Flow (ACSF)

[ L. Alvarez, F. Guichard, P.-L. Lions, J.-M. Morel:
"Axioms and fundamental equations of image processing" 1993 ]

[ G. Sapiro and A. Tannenbaum:
"Affine invariant scale-space." Int. J. Computer Vision 1993 ]



velocity $= \kappa^{1/3}$ ($\kappa =$ curvature)

invariant under area-preserving
affine transformations!

# Peeling and the ACSF

Freie Universität Berlin

## Conjecture:

David Eppstein, Sariel Har-Peled, and Gabriel Nivasch. Grid peeling and the affine curve shortening flow. Experimental Mathematics **29** (2020), 306–316

As the grid is more and more refined, grid peeling approaches the ACSF.

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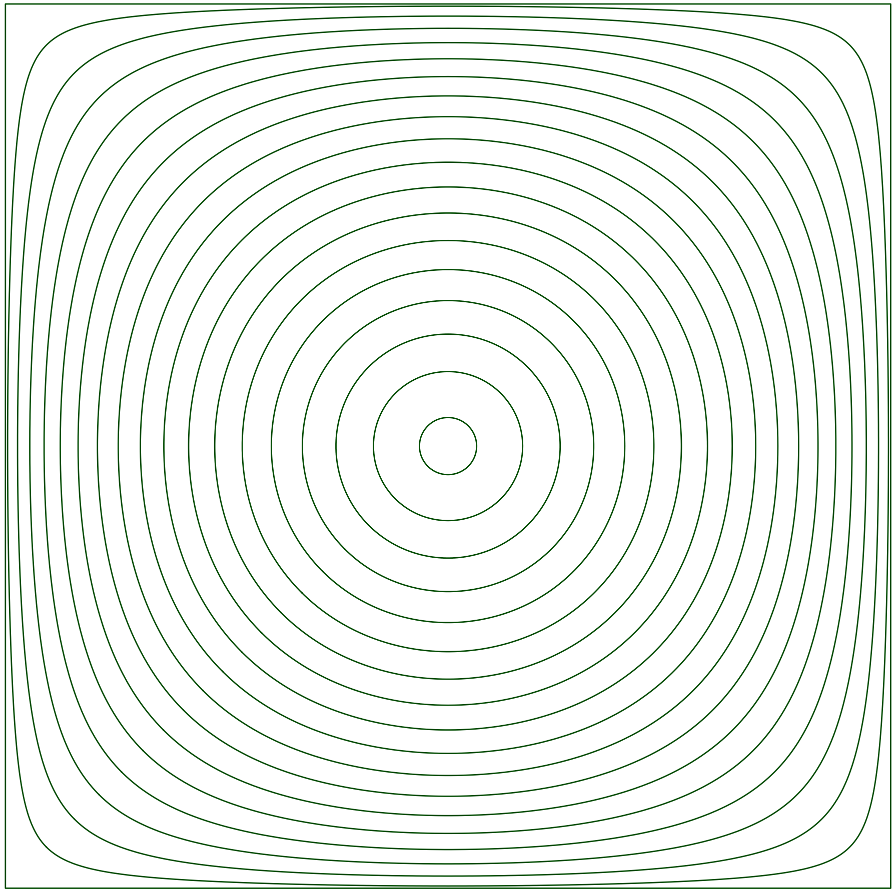

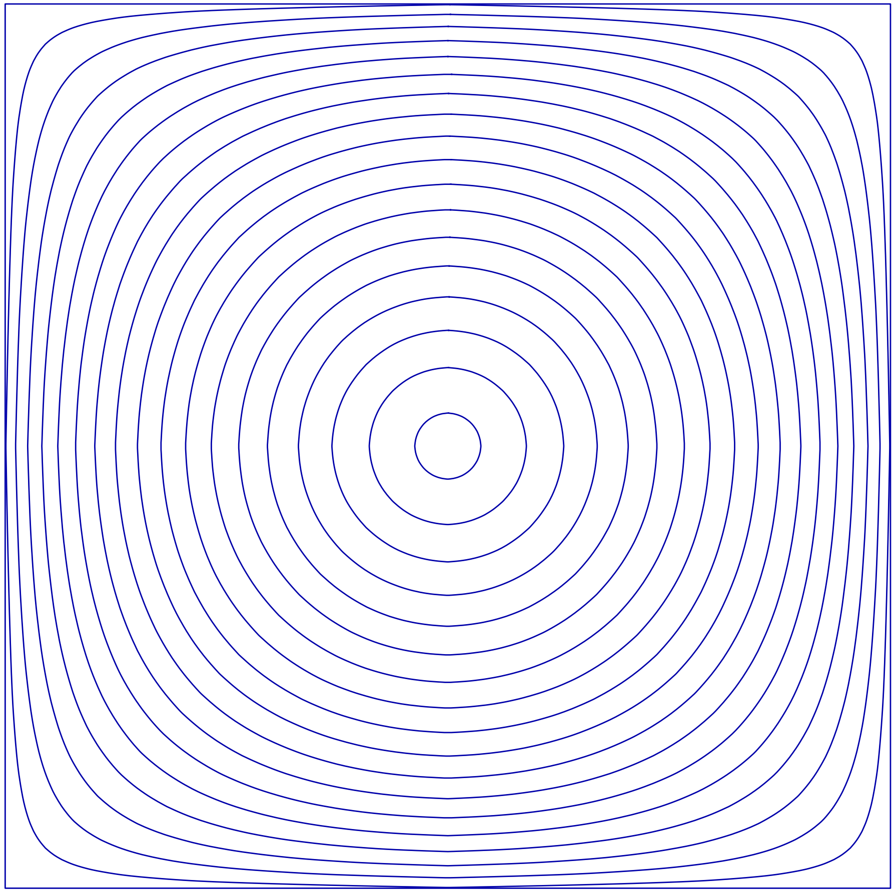

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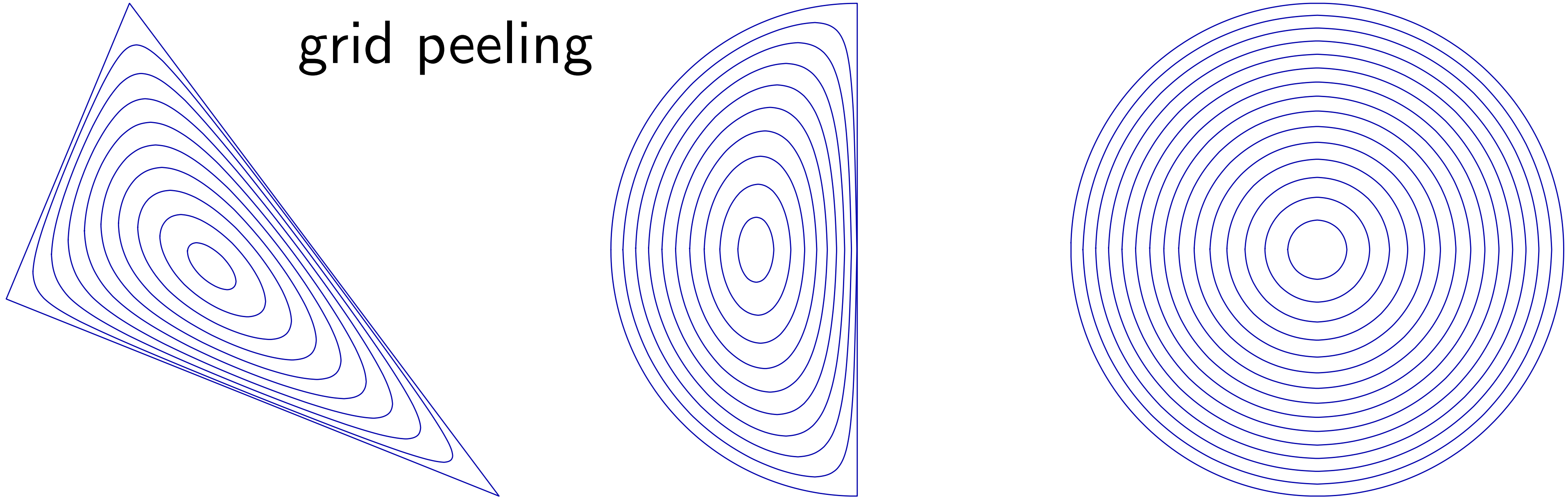


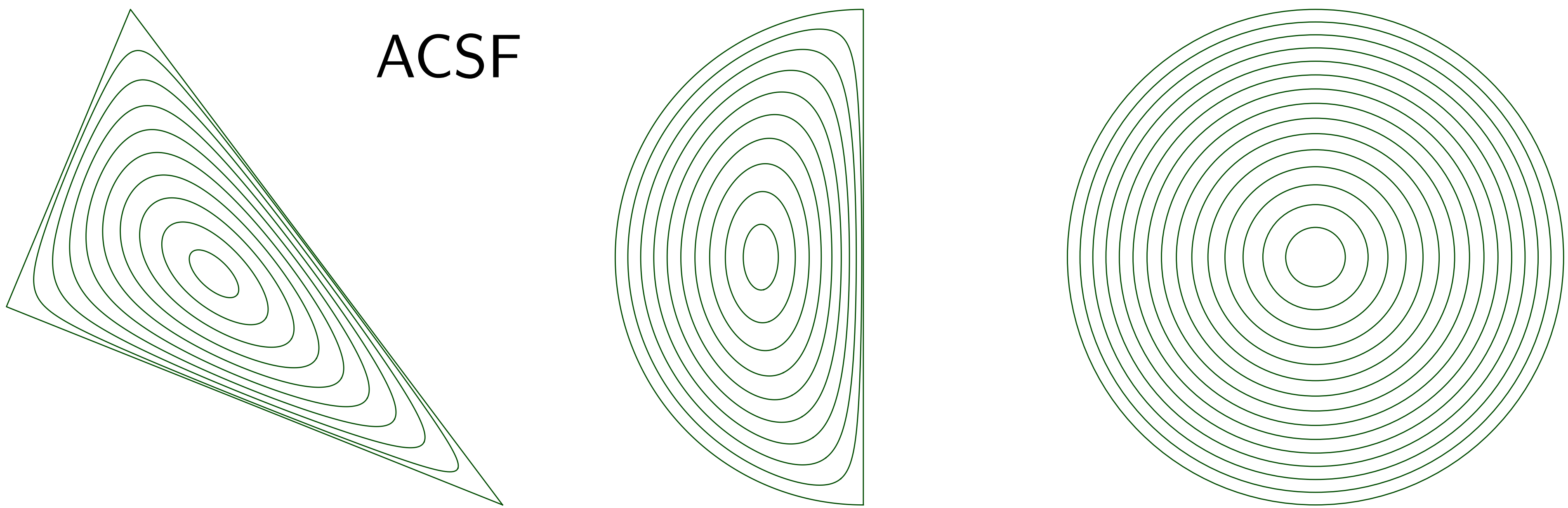

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ACSF at time $t \approx$ Grid peeling on $\frac{1}{n}$-grid after $C_g t n^{4/3}$ steps.

Conjecture: (Moritz Rüber and Günter Rote)

$$C_g = \sqrt[3]{\frac{\pi^2}{2\zeta(3)}} \approx 1.60120980542577$$

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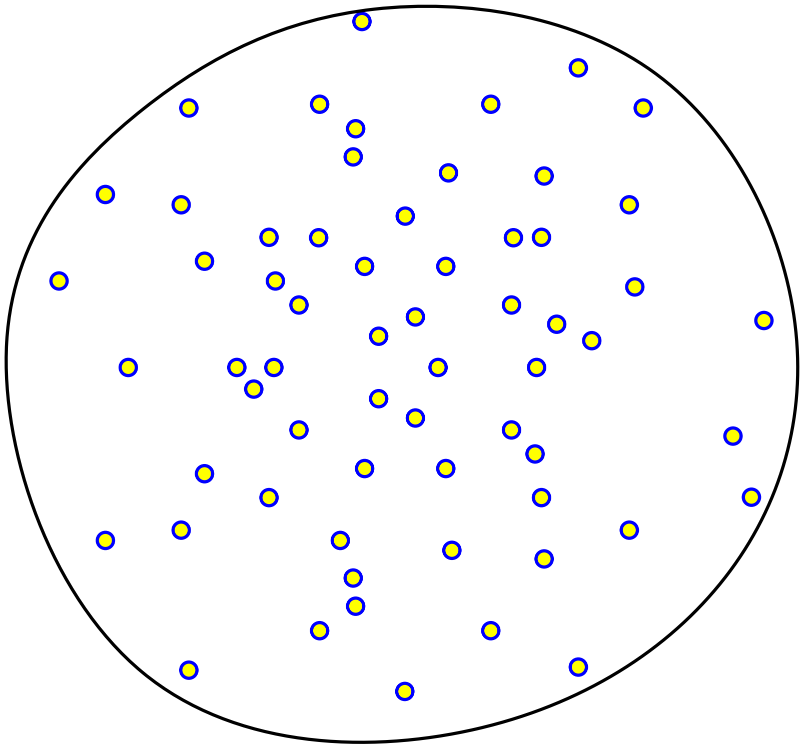
→ Jeff Calder and Charles K Smart. The limit shape of convex hull peeling. Duke Math. J. (2020)

random points



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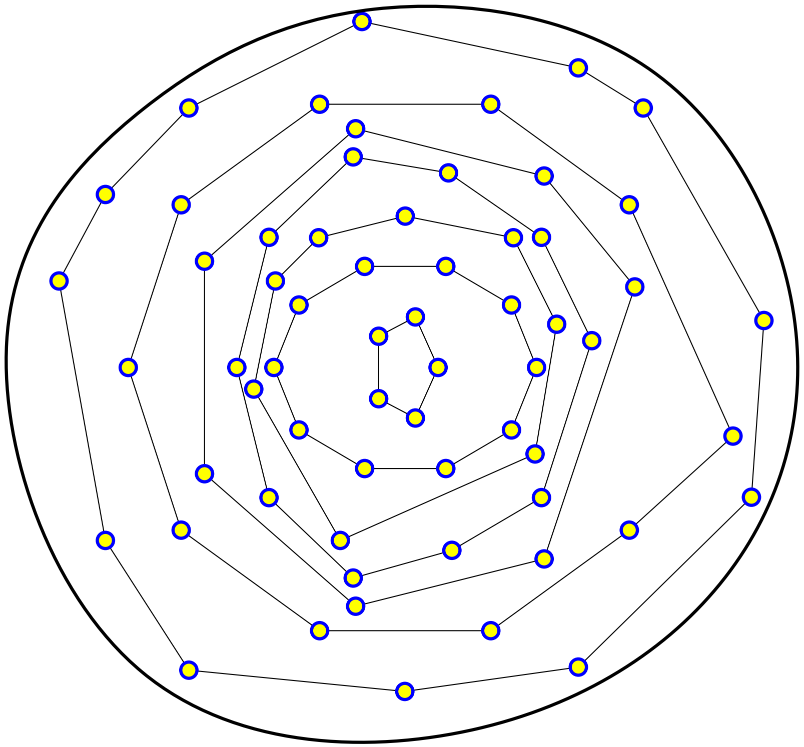
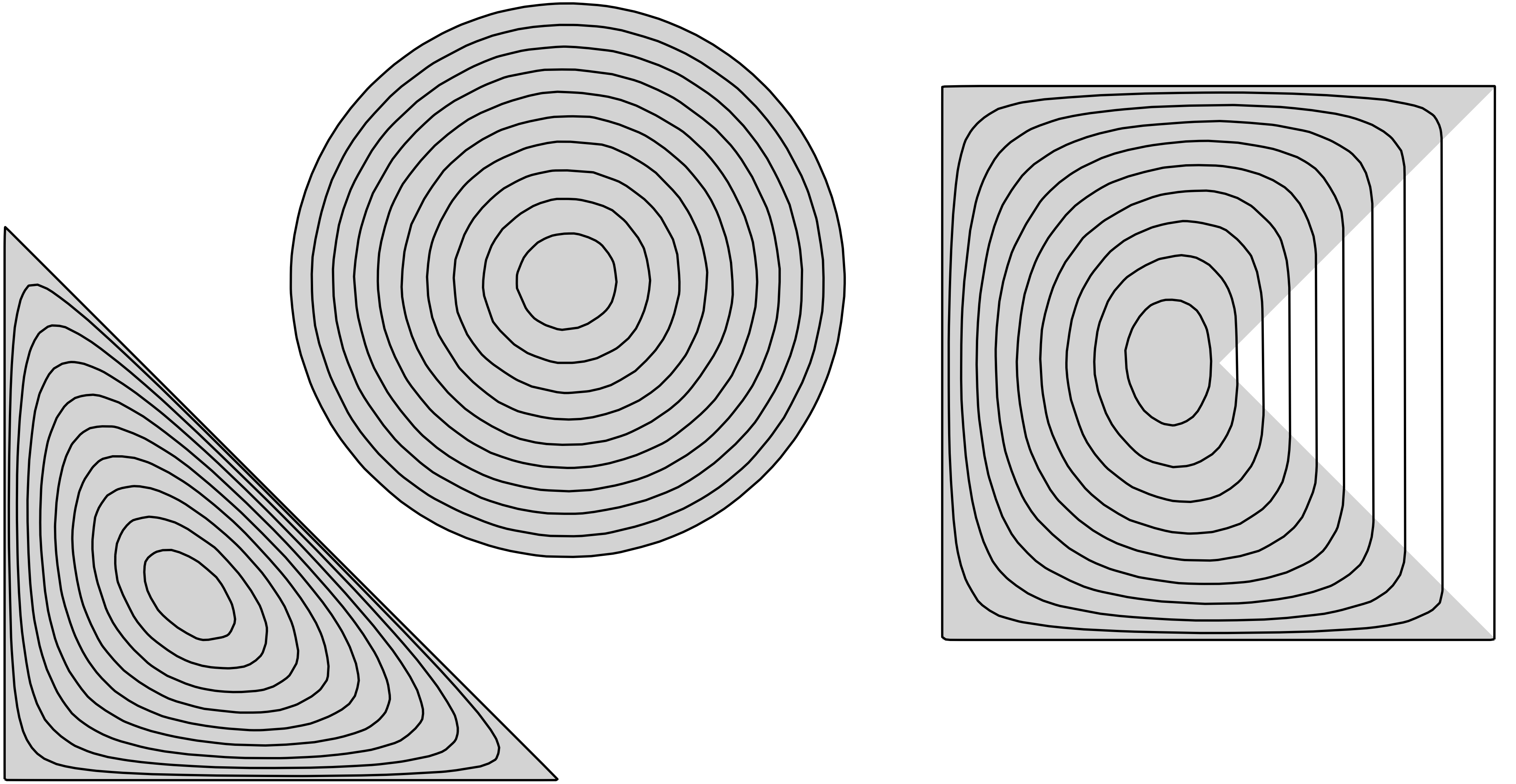


# Peeling and the ACSF

Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020

10000 random points in the shaded region



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Jeff Calder and Charles K Smart. The limit shape of convex hull peeling. Duke Math. J. **169** (2020)

Theorem:

ACSF at time $t \approx$ Peeling on density-$n^2$ set after $C_r t n^{4/3}$ steps.

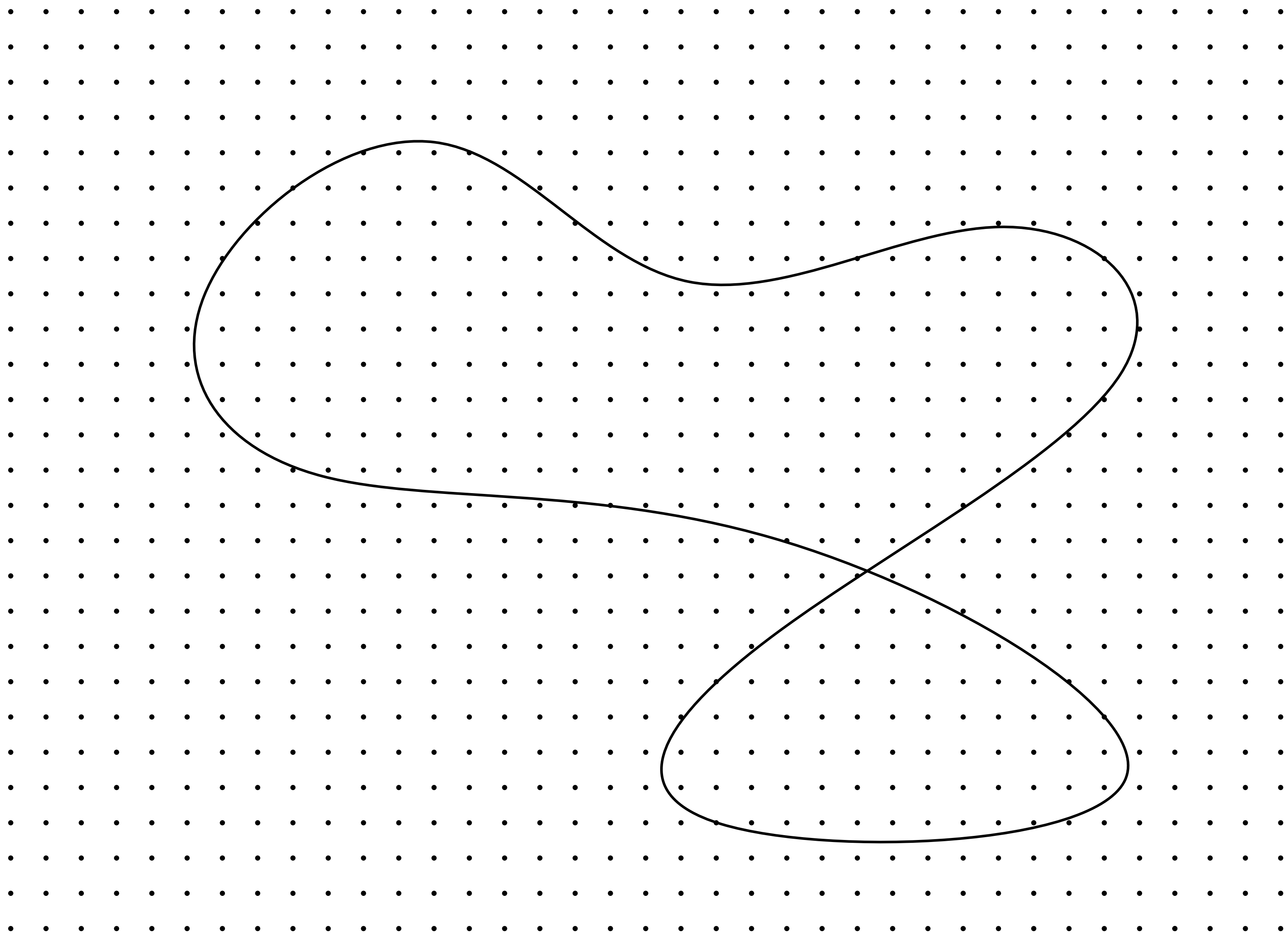
$$C_g \approx 1.6, \;\; C_r \approx 1.3$$

- Invariant under affine transformations?

# Homotopic peeling

Freie Universität Berlin

[ Sergey Avvakumov and Gabriel Nivasch 2019 ]



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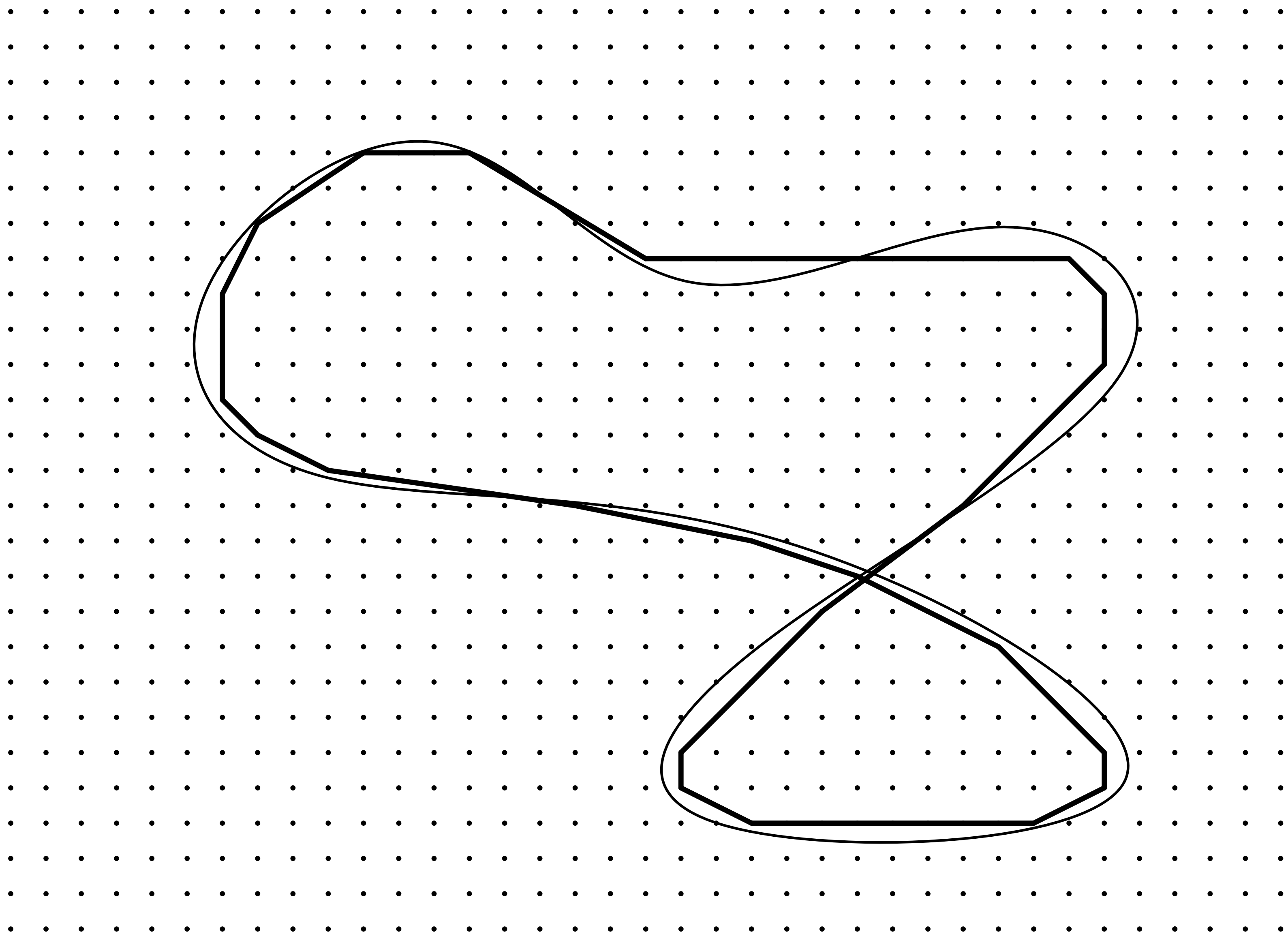

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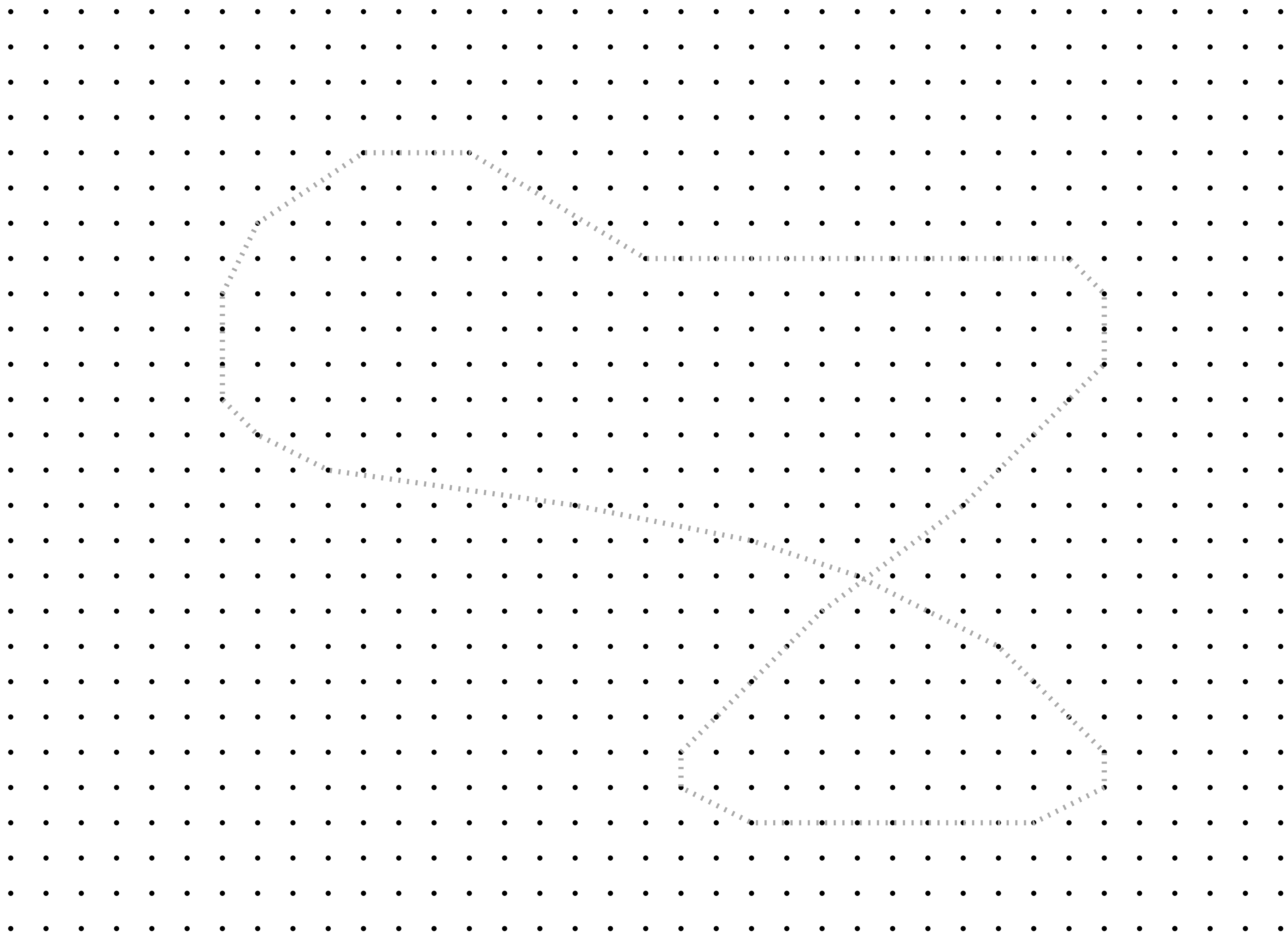

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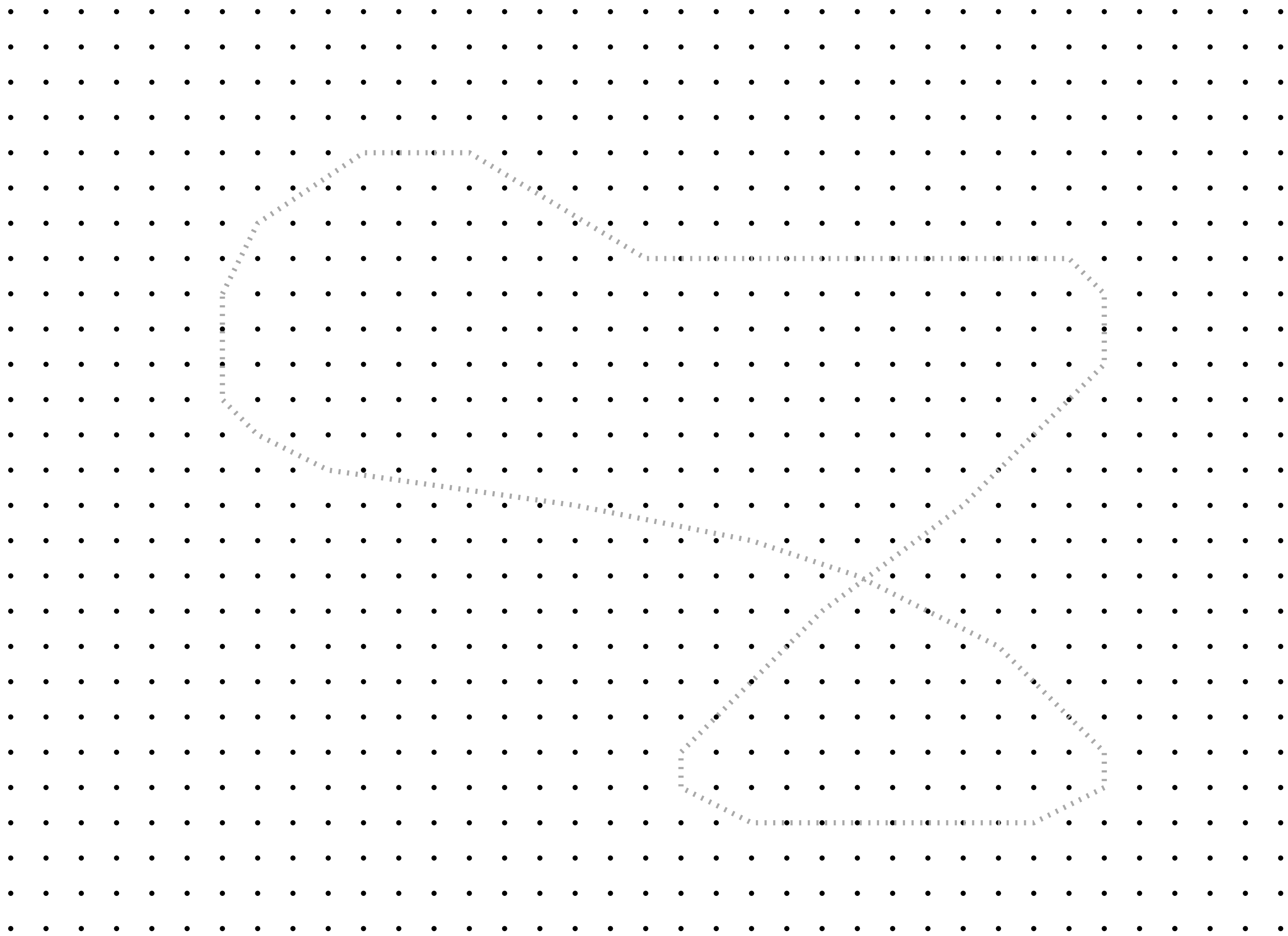

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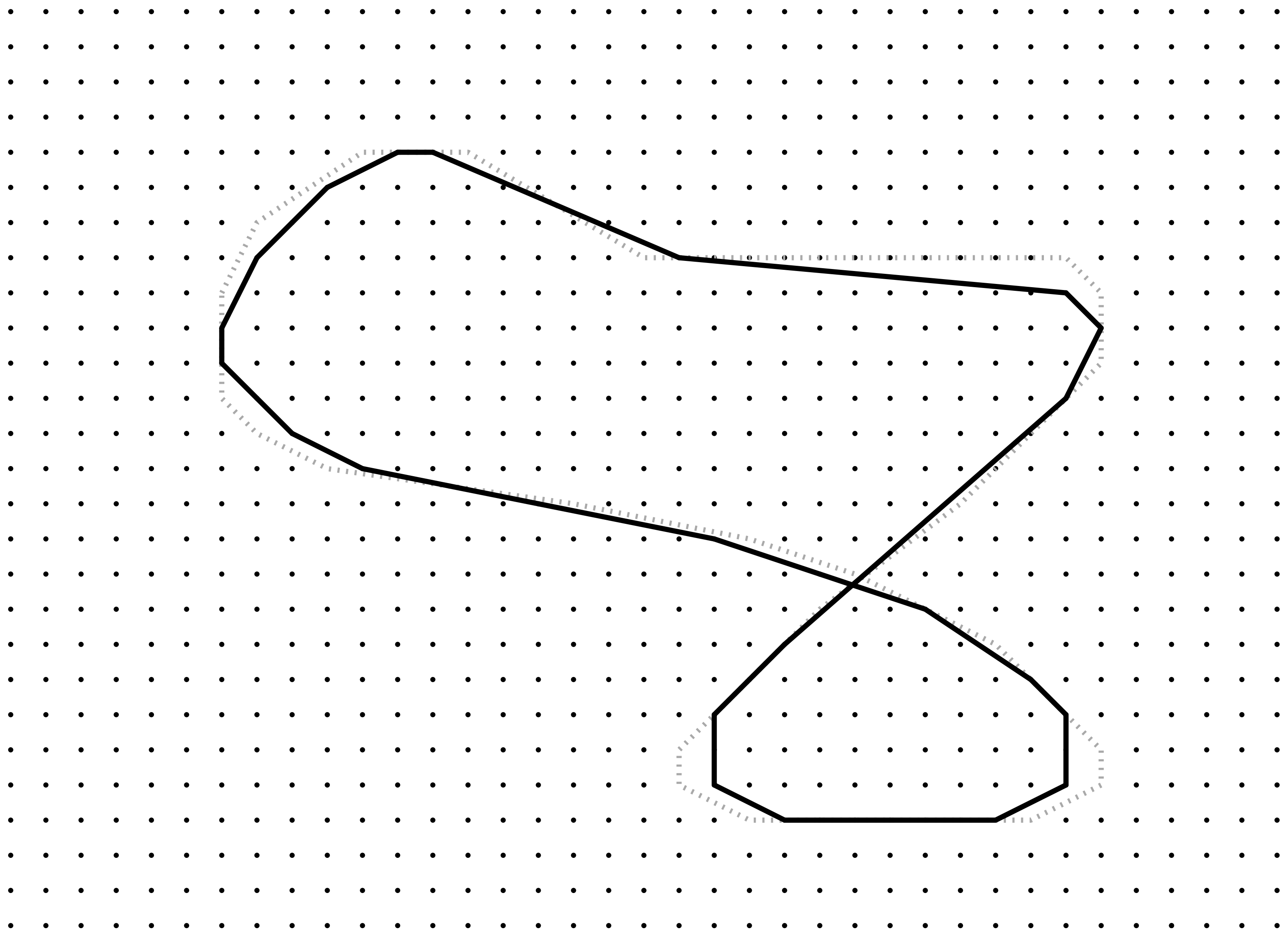

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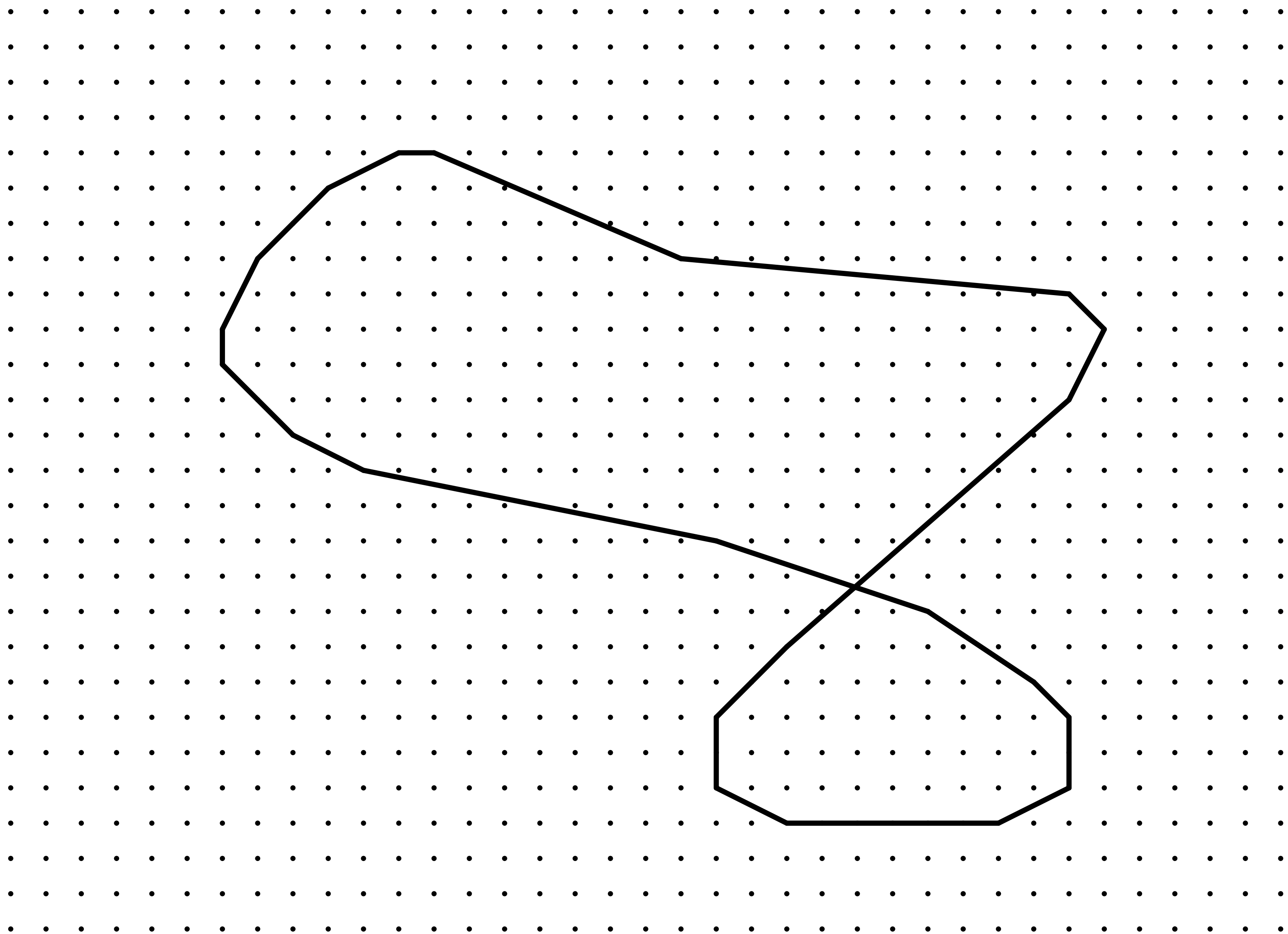

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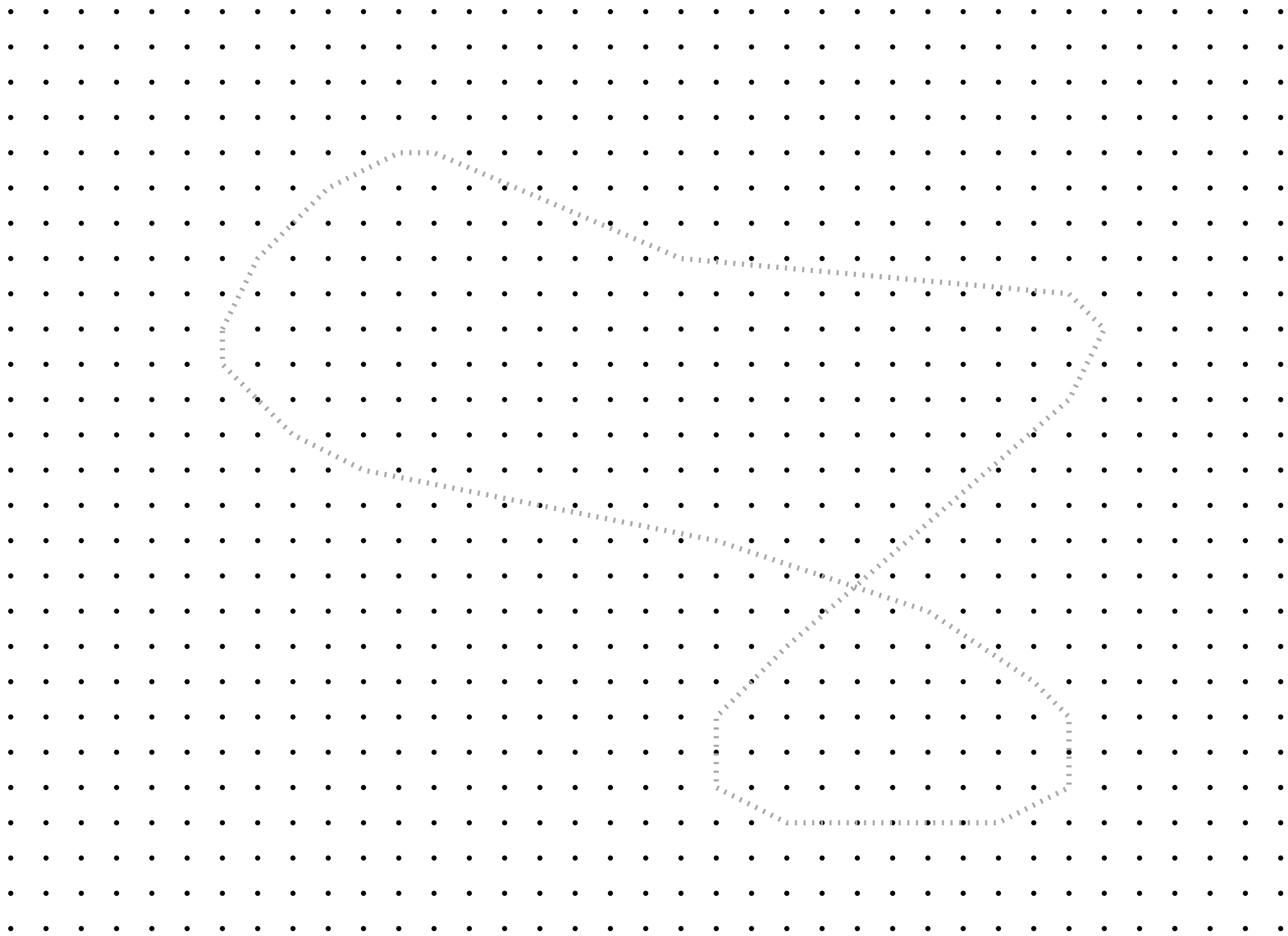

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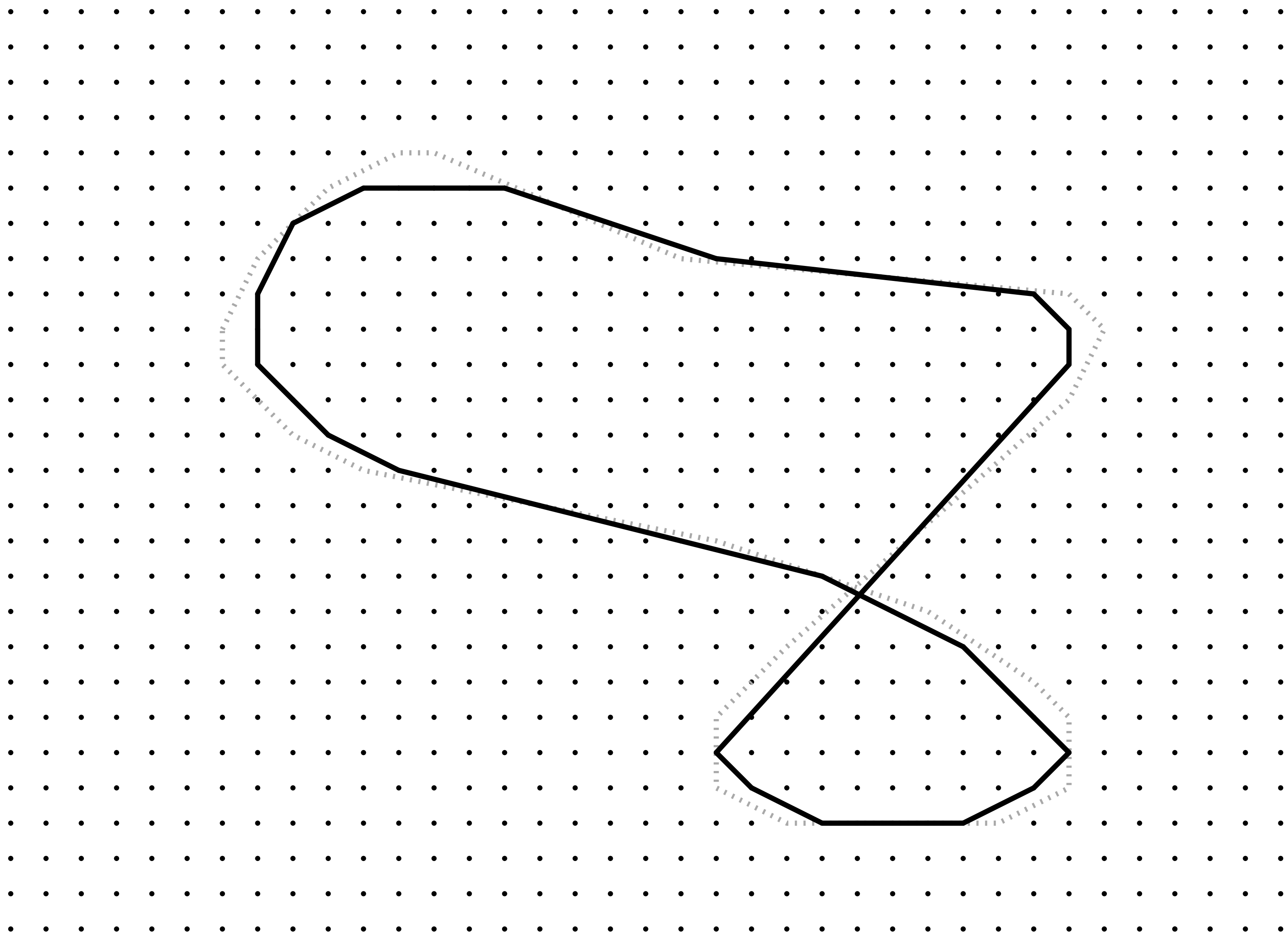

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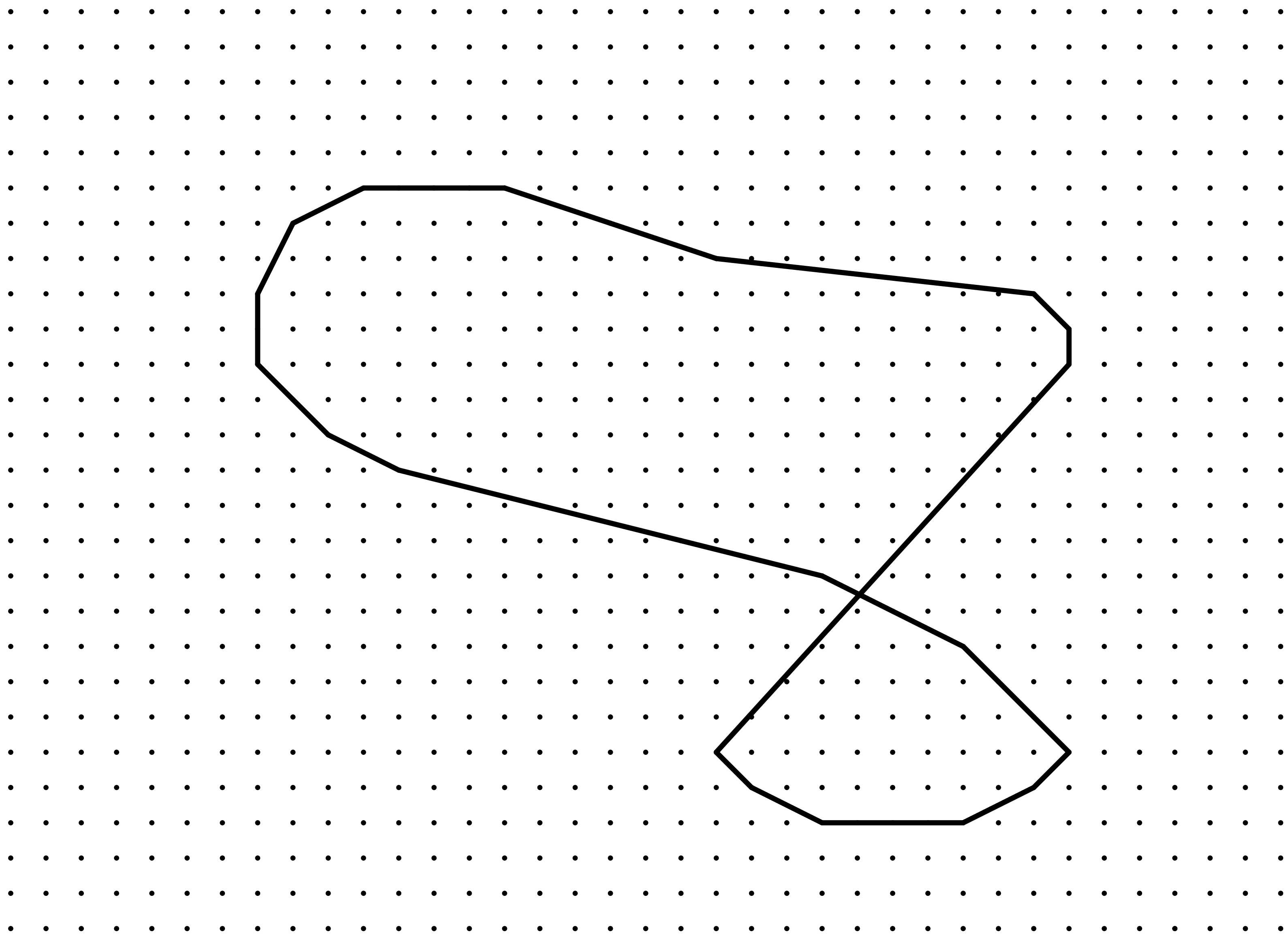

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Freie Universität Berlin

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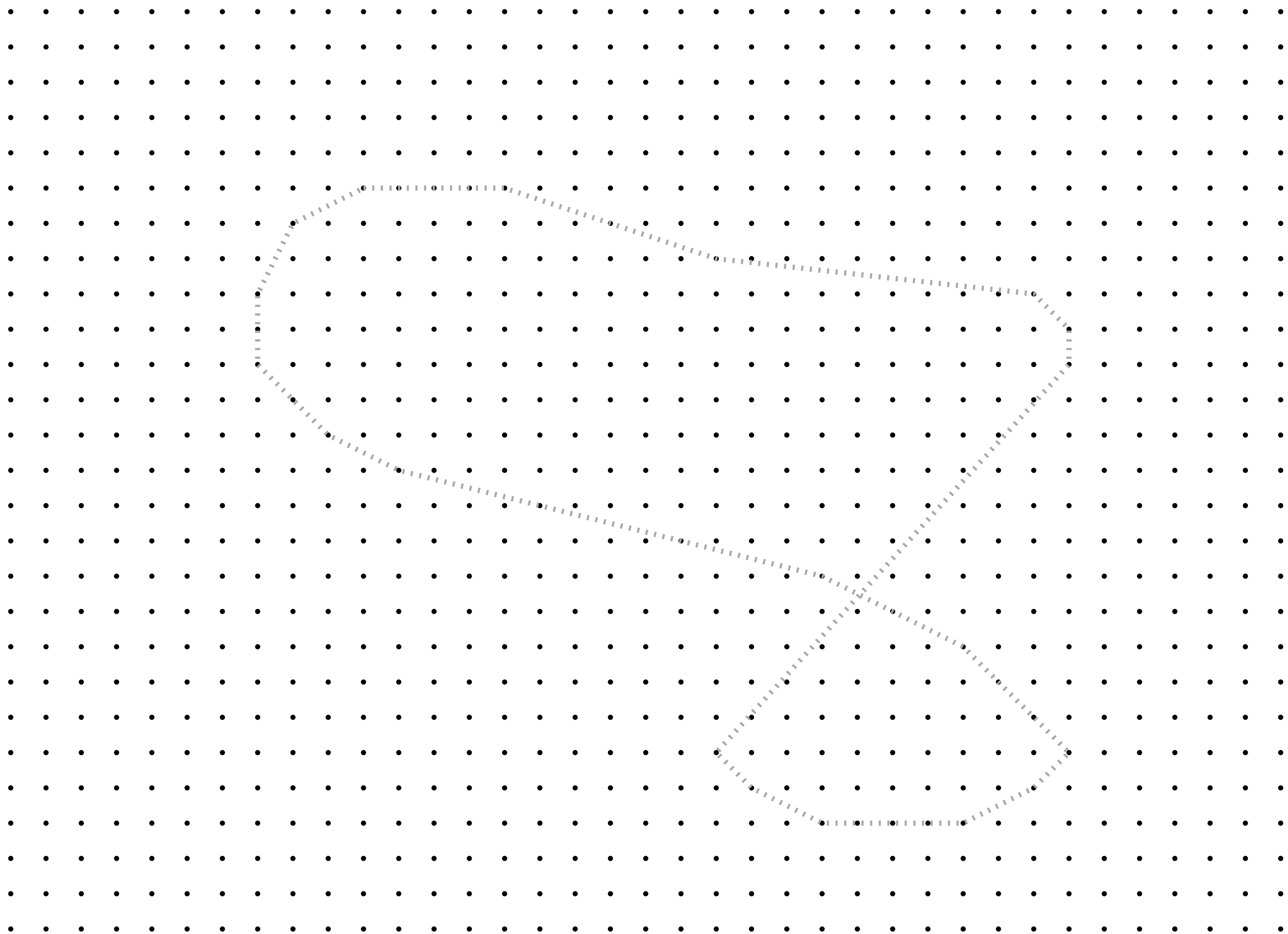

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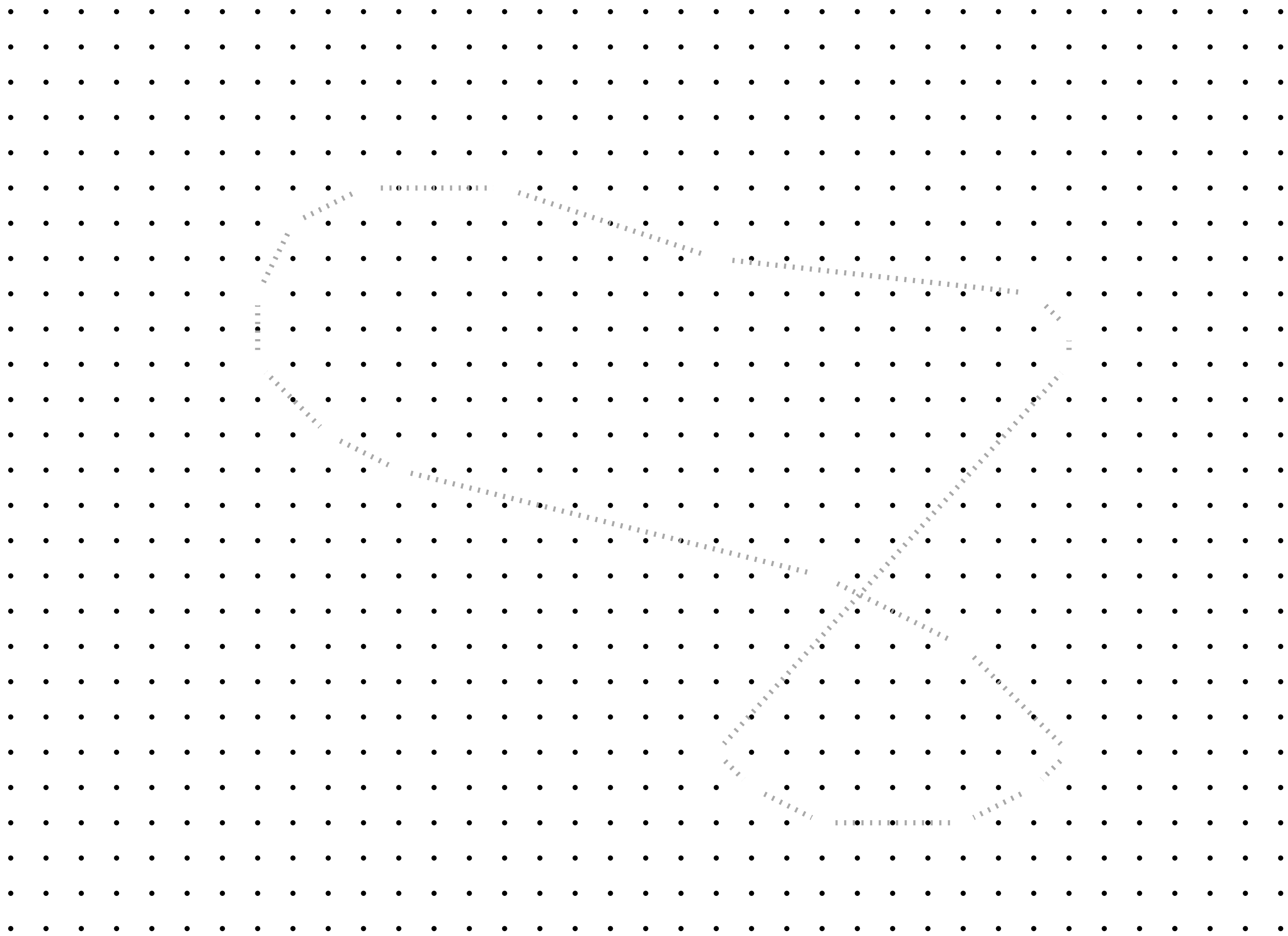

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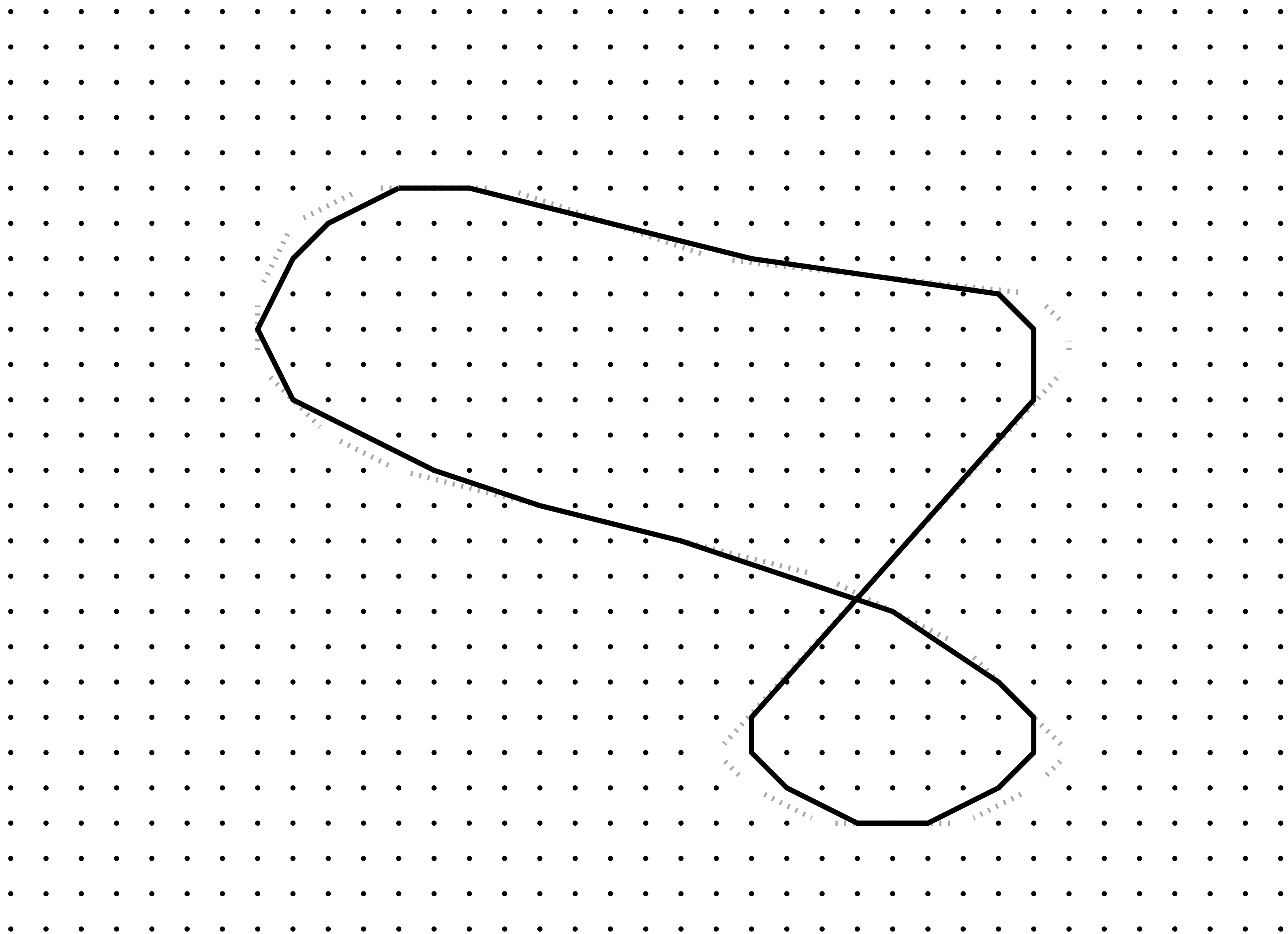

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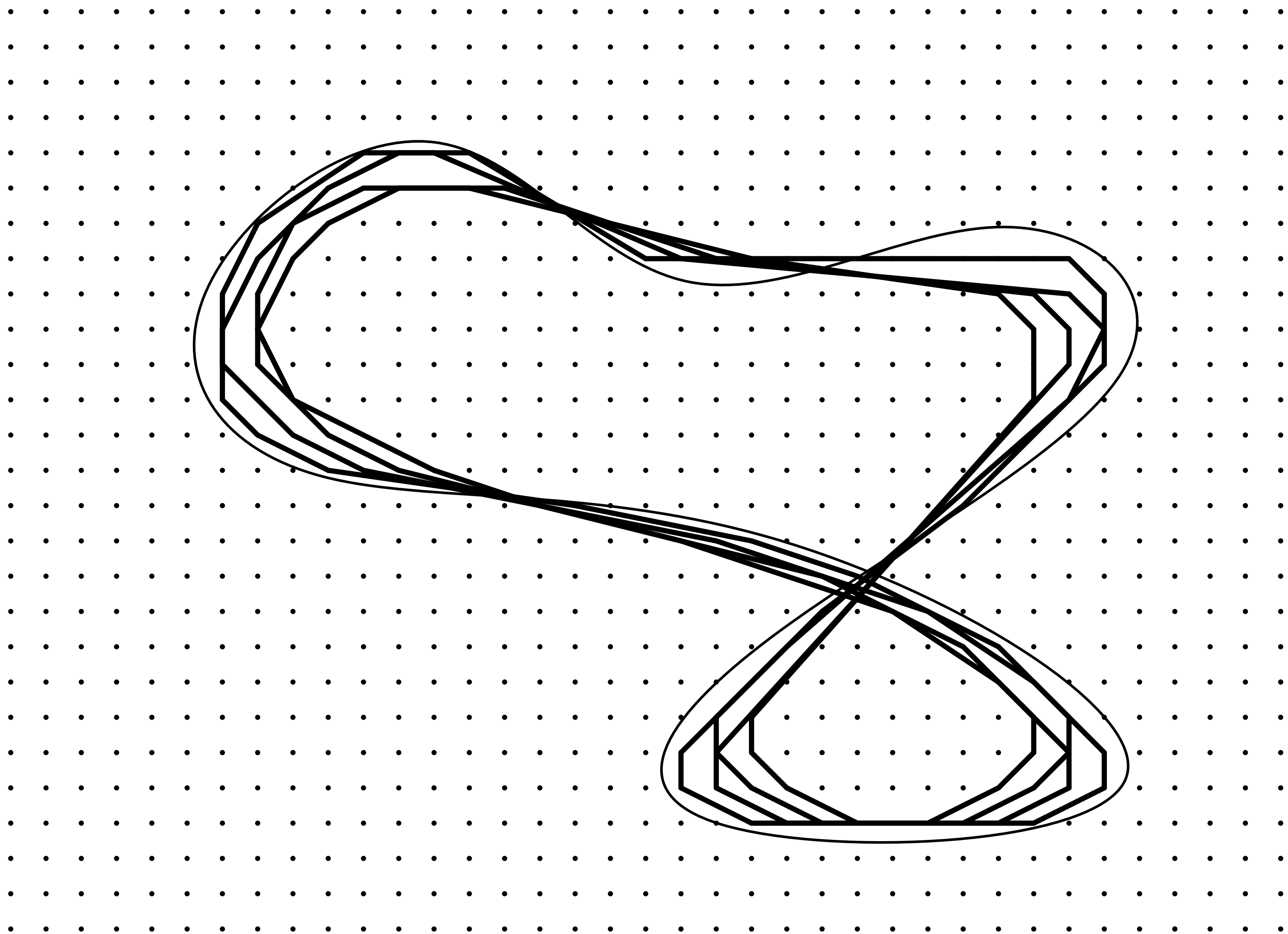
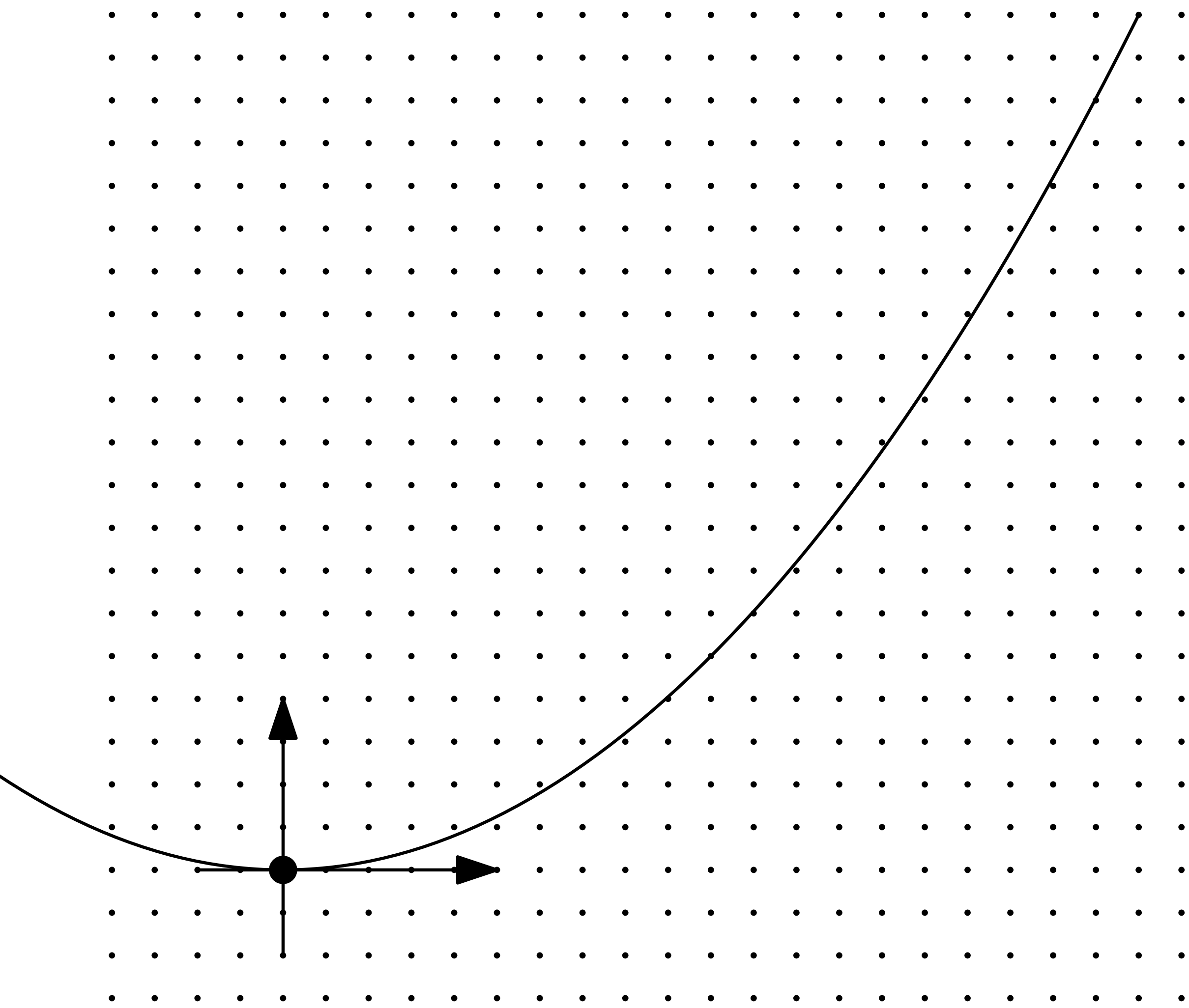


# The parabola!



$$y = \frac{1}{20}x^2$$

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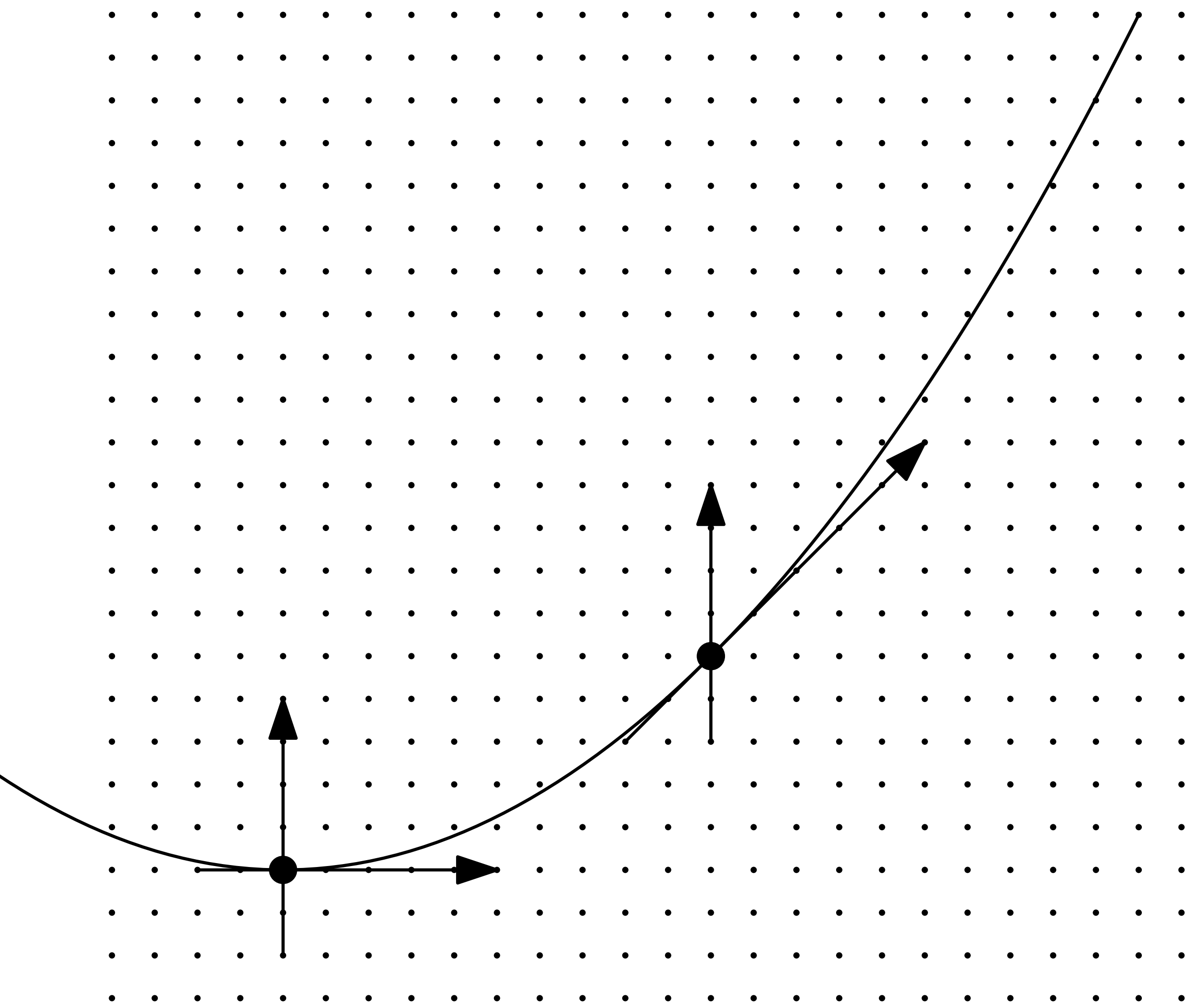

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affine lattice-preserving
shearing transformations

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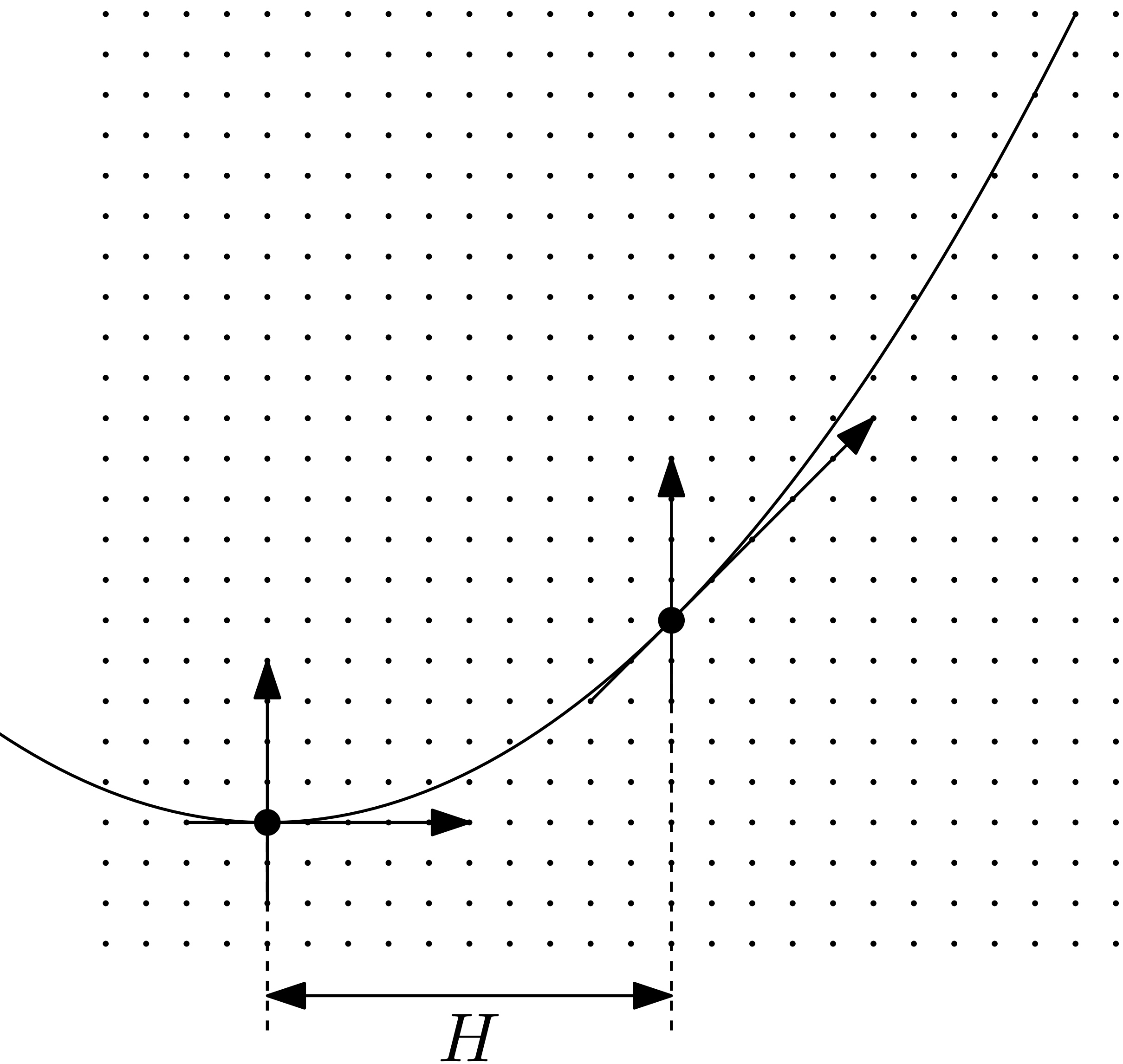


$y = \frac{1}{20}x^2$

affine lattice-preserving
shearing transformations

$y = \frac{a_N}{a_D}x^2 + \frac{b_N}{b_D}x + c$

Lemma:
Horizontal period $H = \mathrm{lcm}(a_D, b_D)$ or $H = \mathrm{lcm}(a_D, b_D)/2$

# Conics

Conics maintain their shape under ACSF.

- Ellipses (and circles) *shrink* (and collapse to the center).
- Parabolas are *translated*.
- Hyperbolas *expand*.

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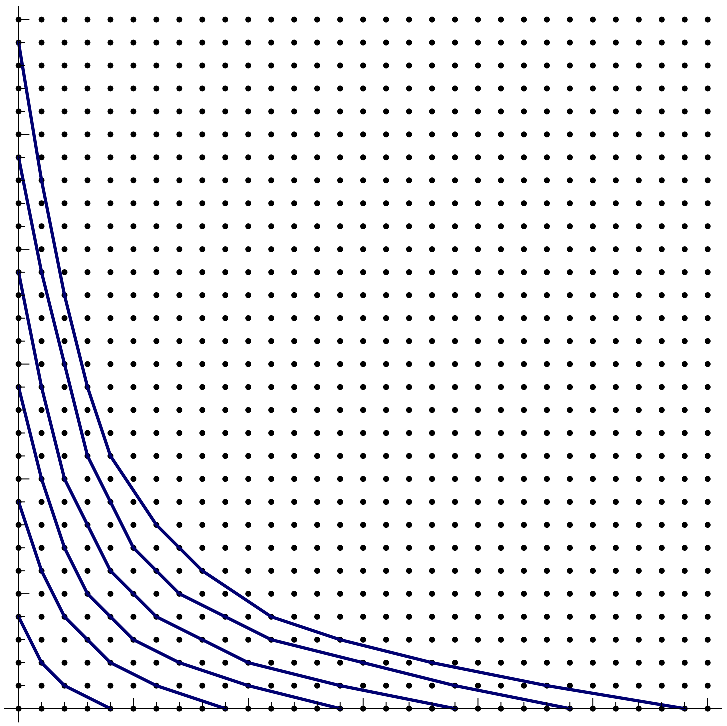

layers 5, 10, . . . , 30 of $\mathbb{N}^2$

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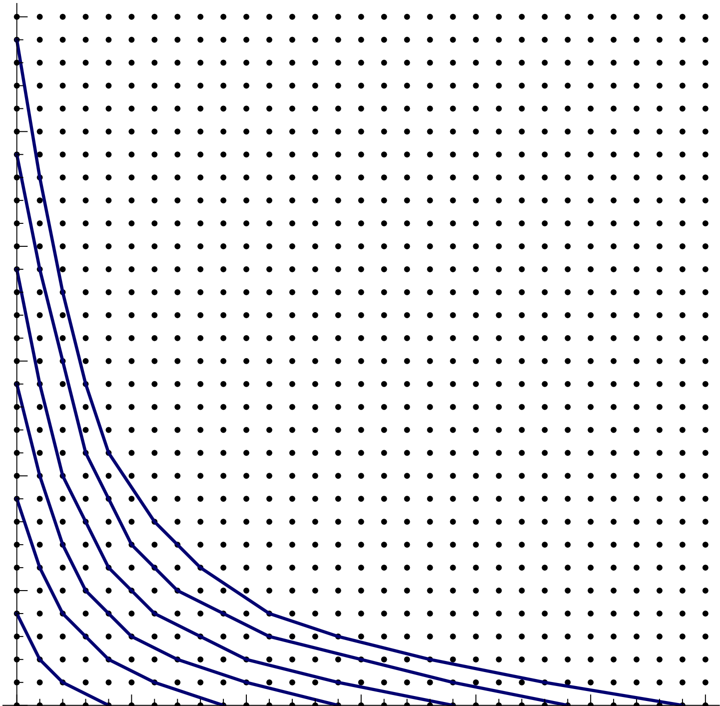


layers 5, 10, ..., 30 of $\mathbb{N}^2$

THEOREM:
The $n$-th layer of $\mathbb{N}^2$ is sandwiched between two hyperbolas:

$$c_1 n^{3/2} \leq xy \leq c_2 n^{3/2}$$

(except within $\sqrt{n}\log^2 n$ of the axes)

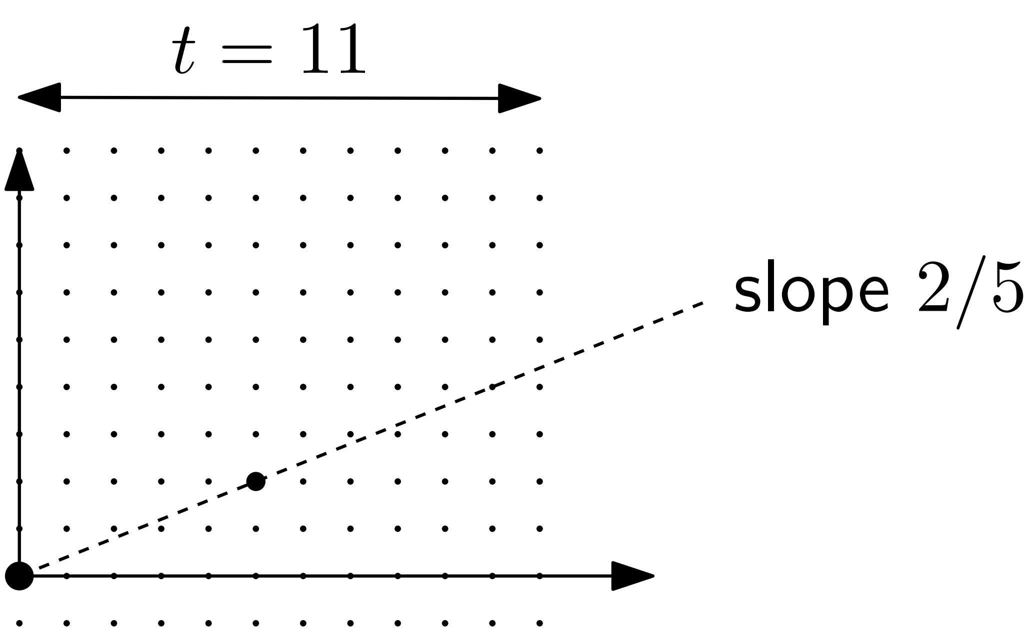
# "The grid parabola"

- integer parameter $t \geq 1$
- $S_t :=$ { all slopes $a/b$ with $0 < b \leq t$ }
- for each slope $a/b \in S_t$, take the longest integer vector

$$\binom{x}{y} = k\binom{b}{a} \quad (k \in \mathbb{Z})$$

with $0 < x \leq t$

Example



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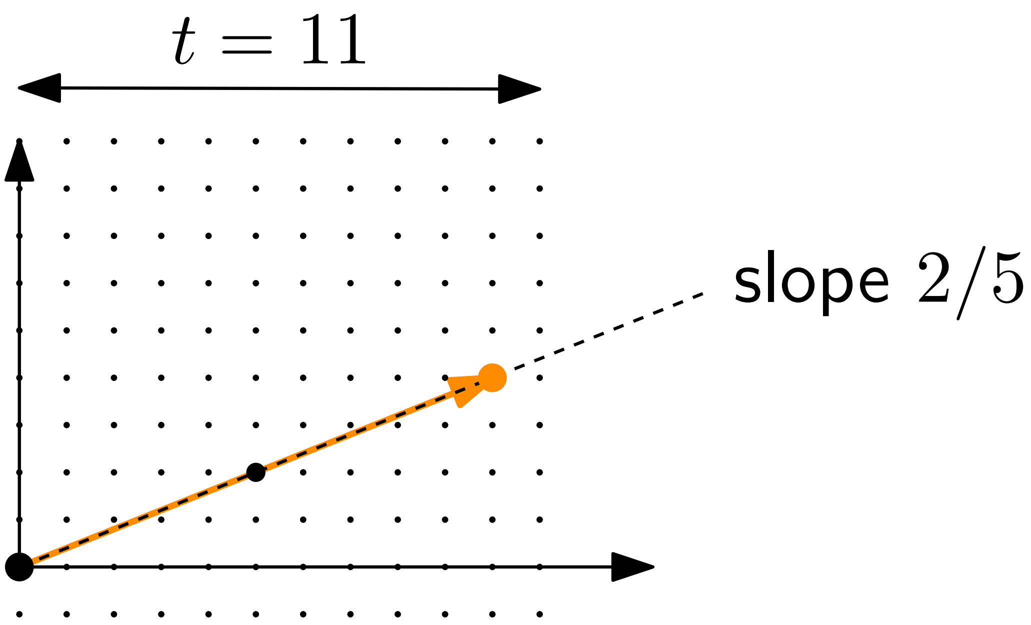

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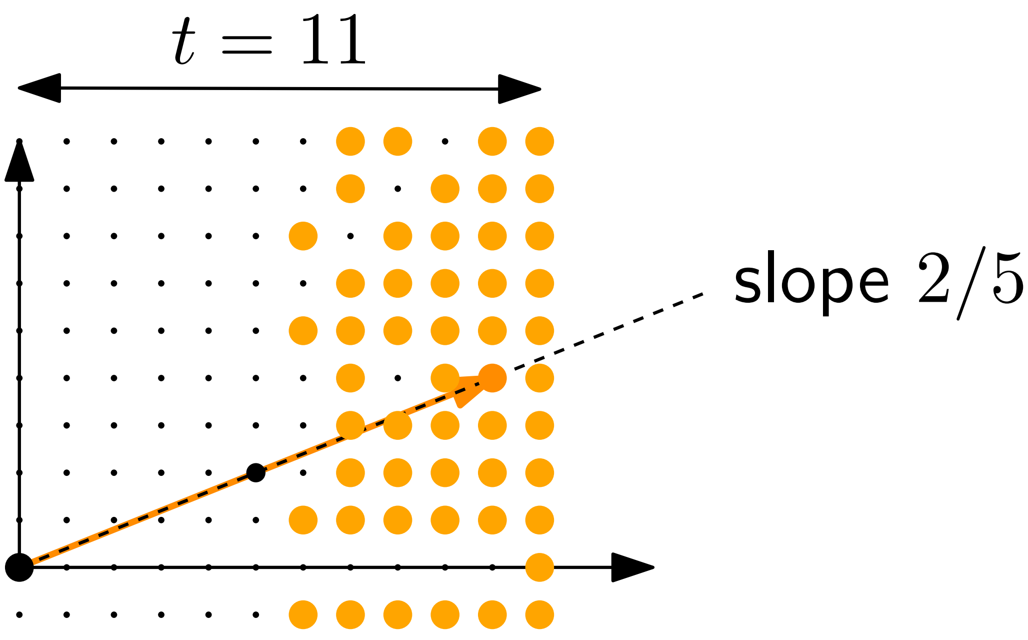

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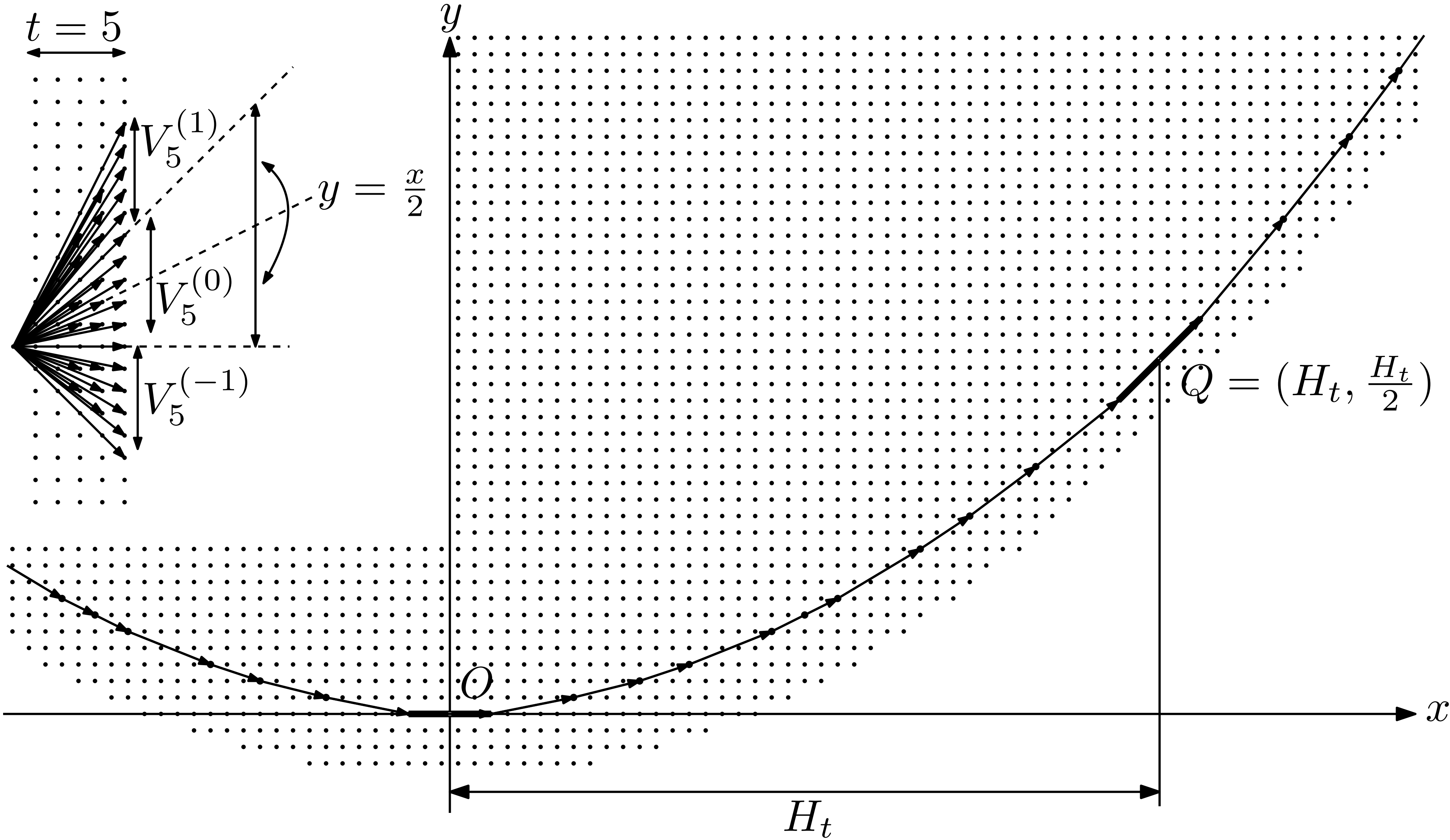

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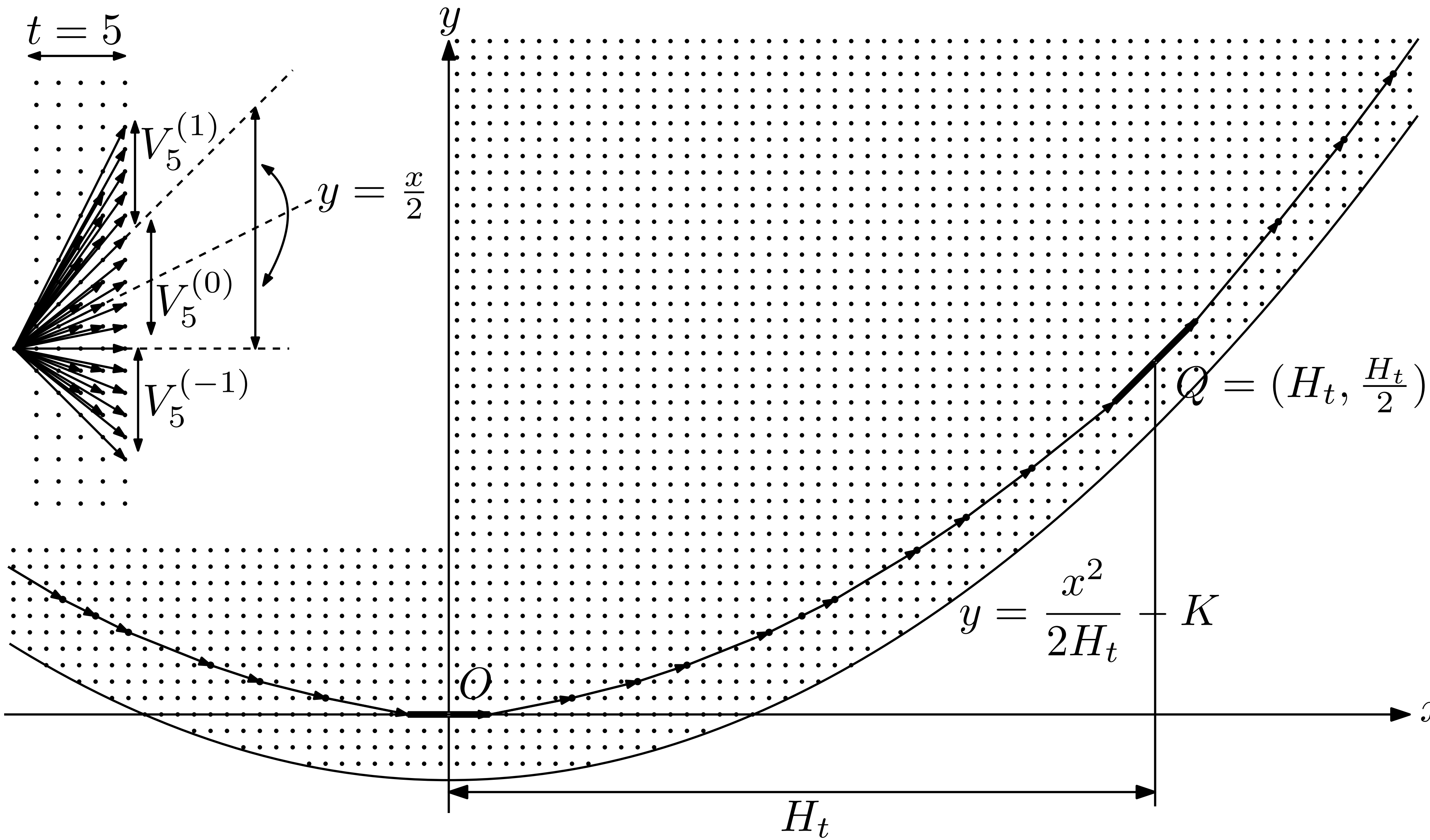

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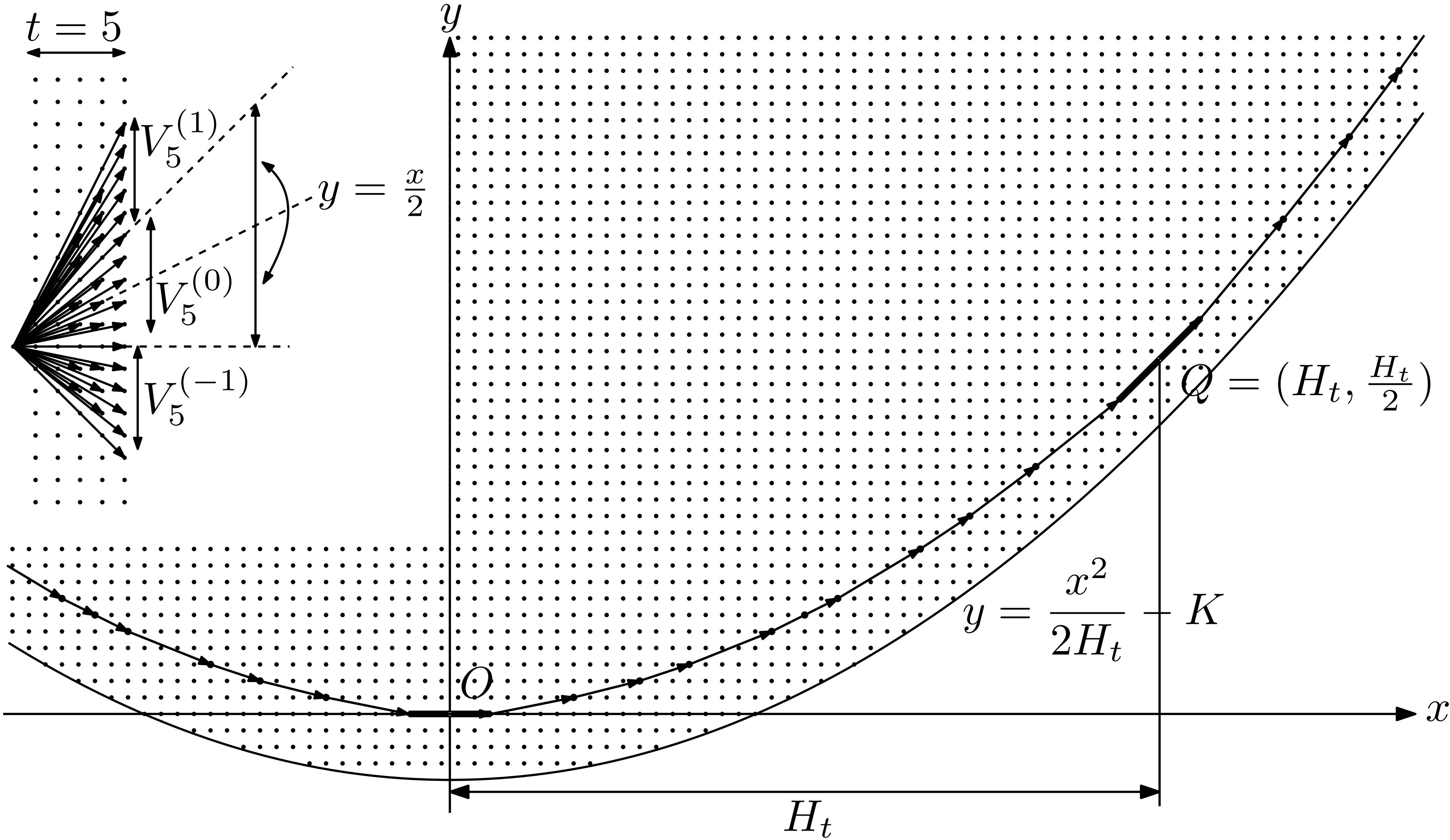
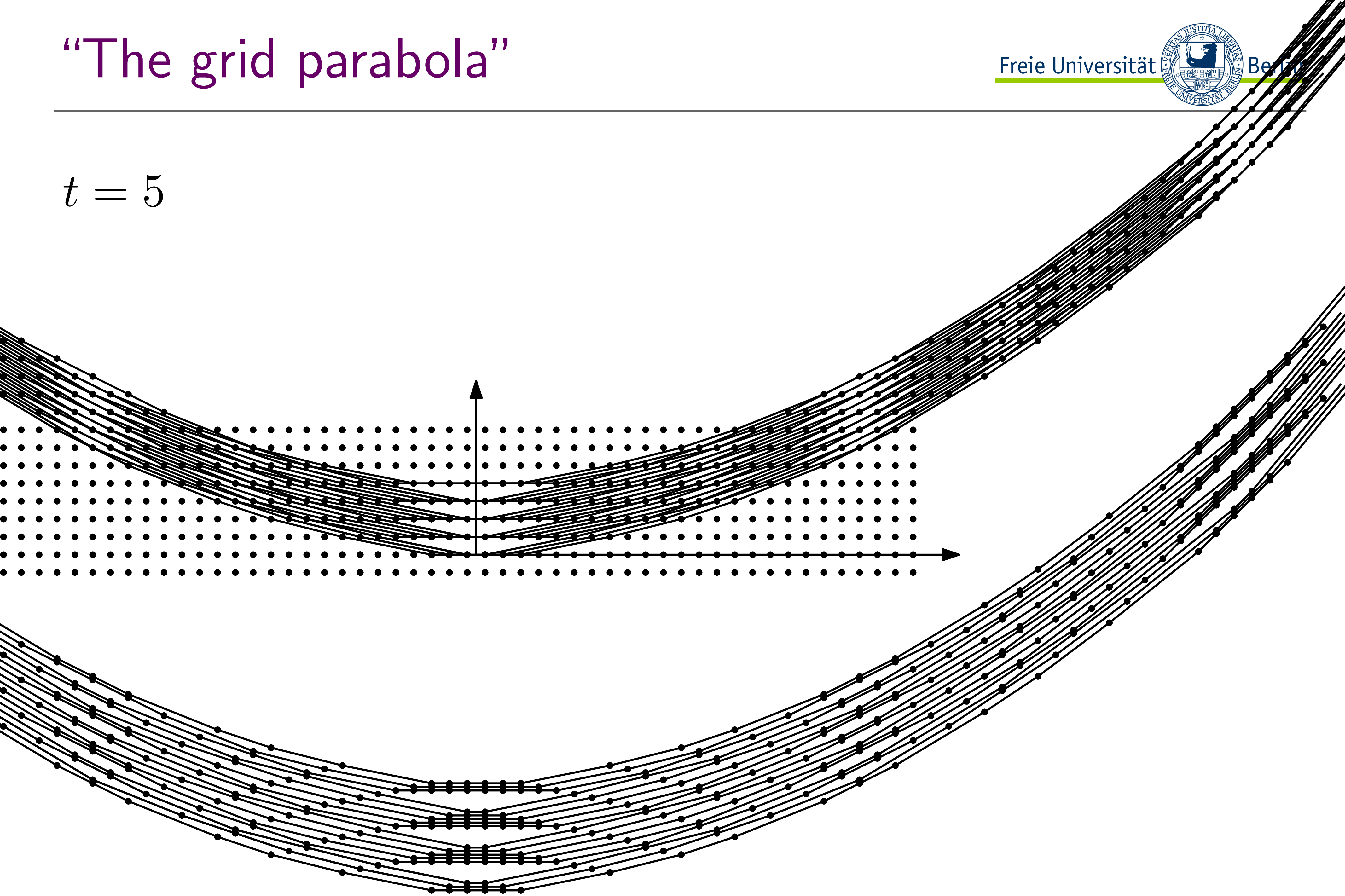



$$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$$

# “The grid parabola”

$t = 5$



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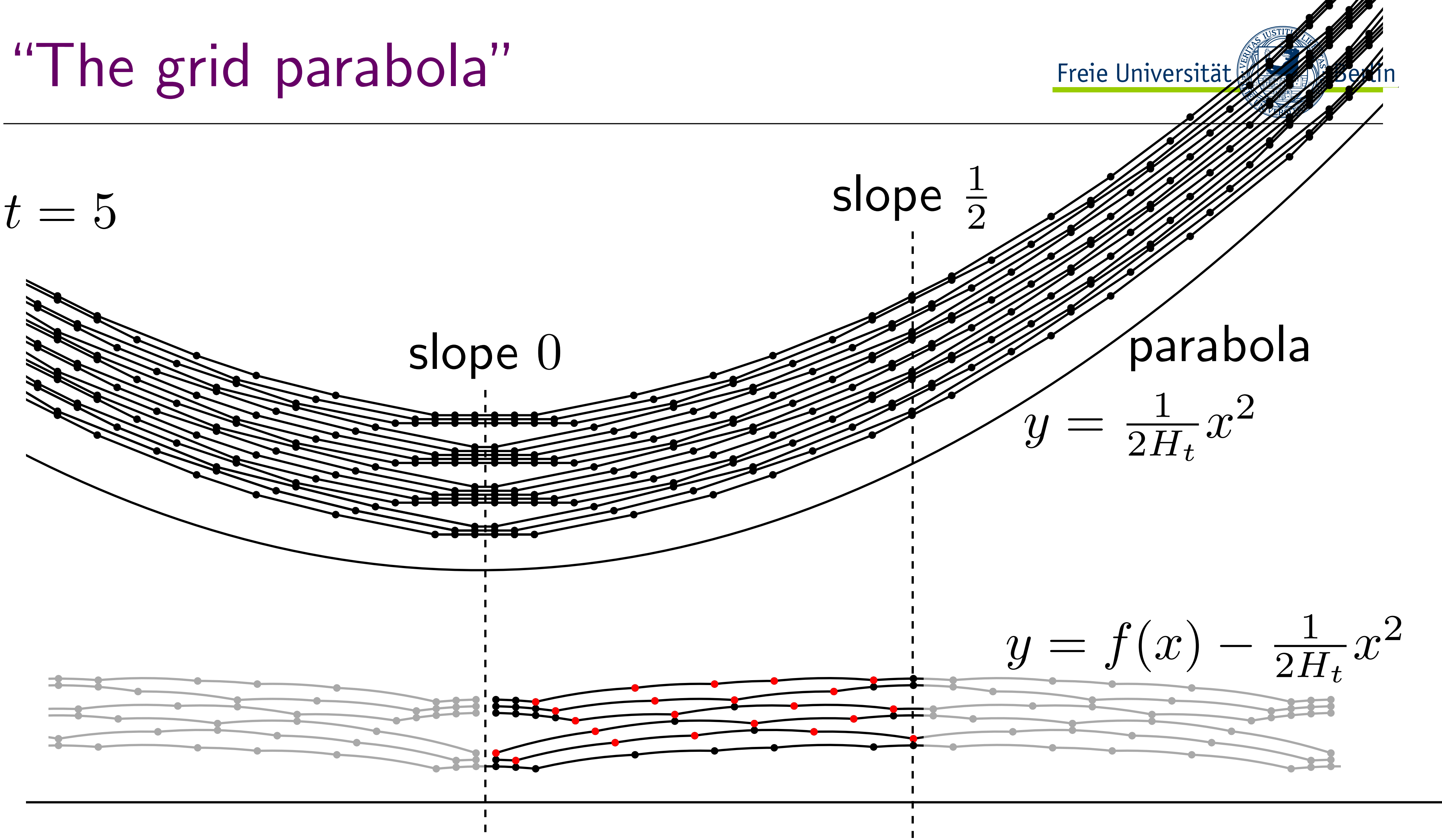
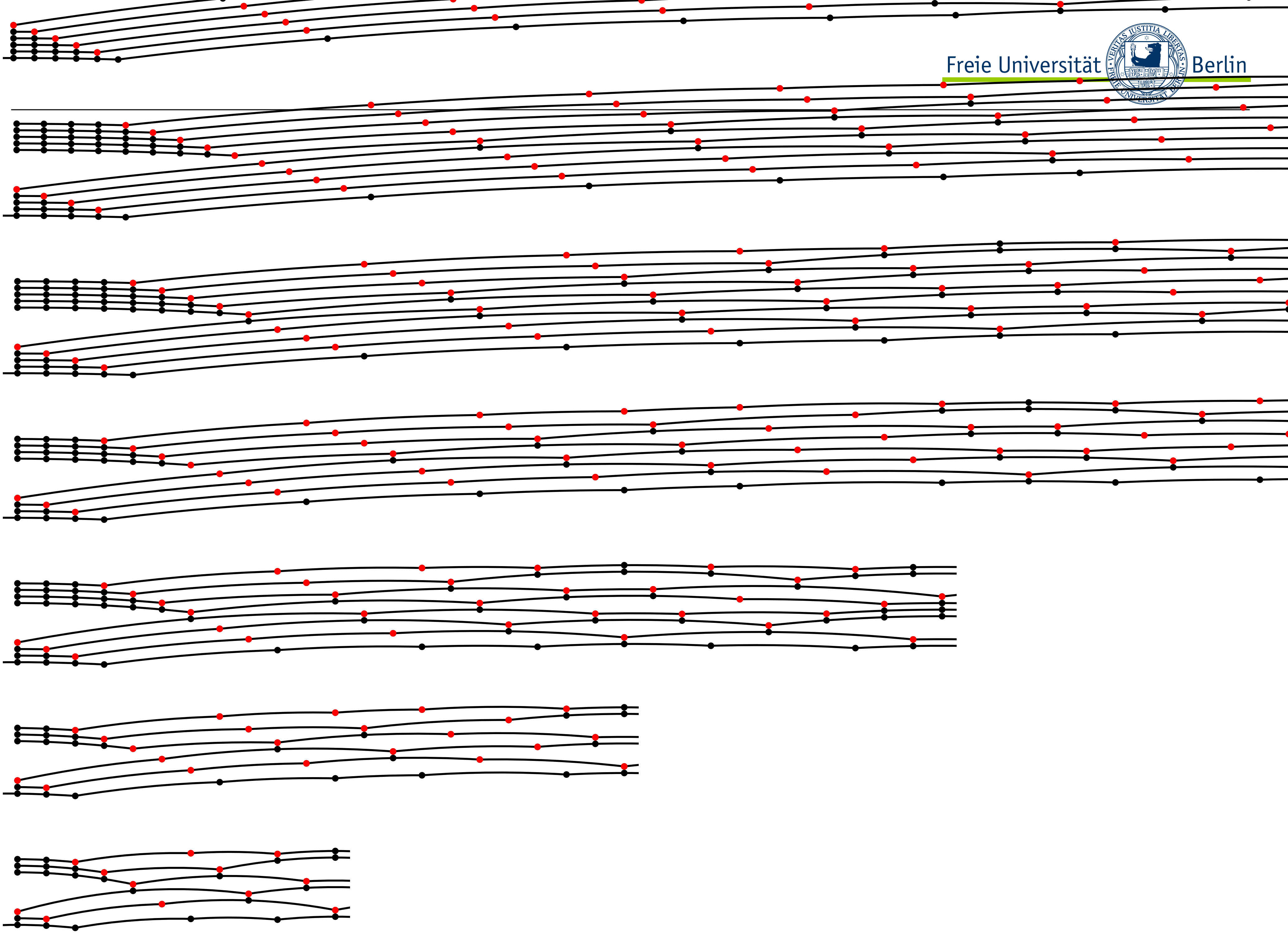



Theorem:
For odd $t$, the polygon repeats after $t$ steps, one level higher.
(For even $t$: after $t+1$ steps.)



# $t = 4, 5, 6, \ldots$

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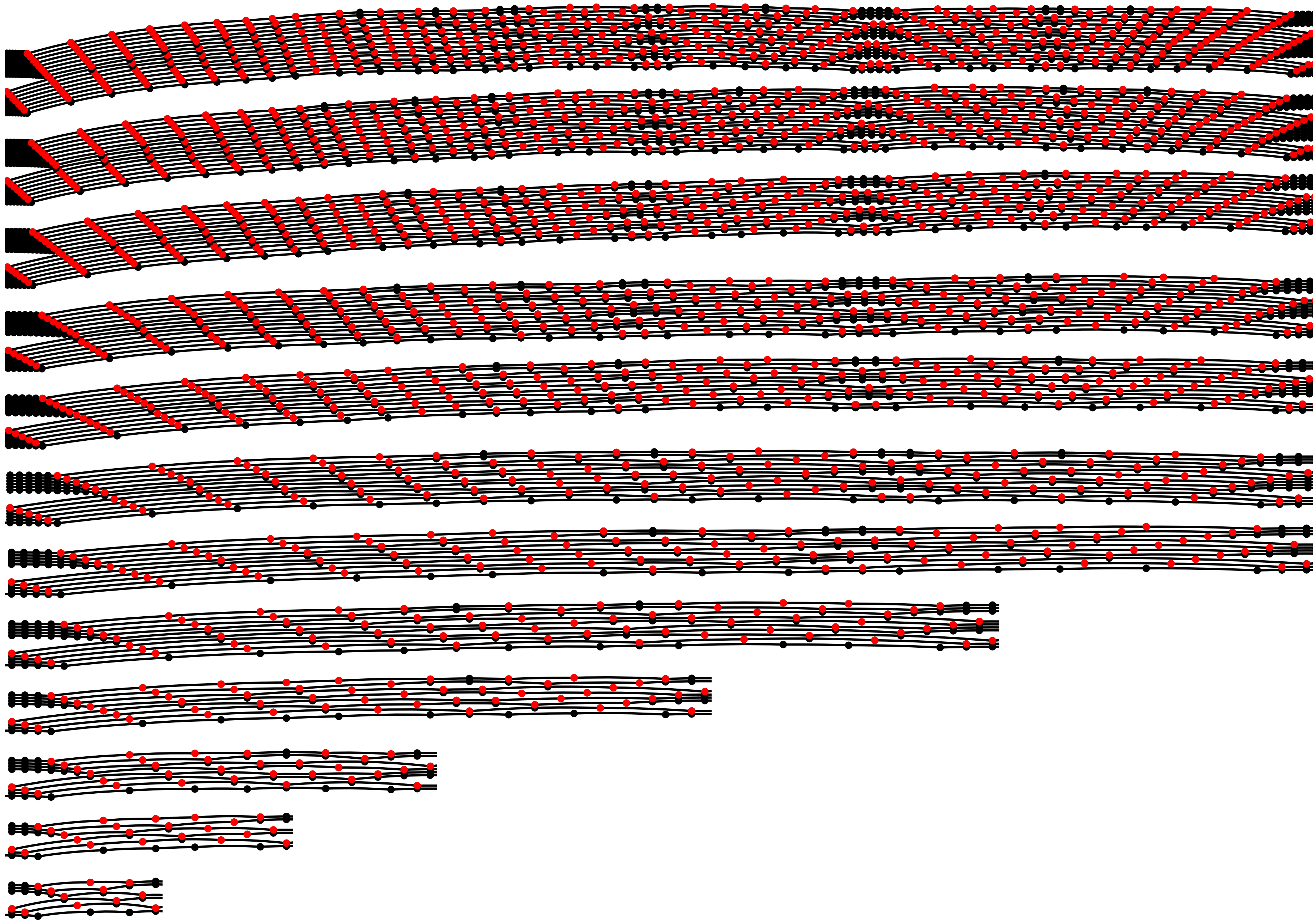

# Asymptotic period

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$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$

[ OEIS A174405 ]

$$H_t := \sum_{\substack{0<y\le x\le t\\ \gcd(x,y)=1}} \left\lfloor \frac{t}{x} \right\rfloor x = \sum_{1\le i\le t} \sum_{d|i} d\varphi(d)$$

$$H_t = \frac{2\zeta(3)}{\pi^2} t^3 + O(t^2 \log t)$$

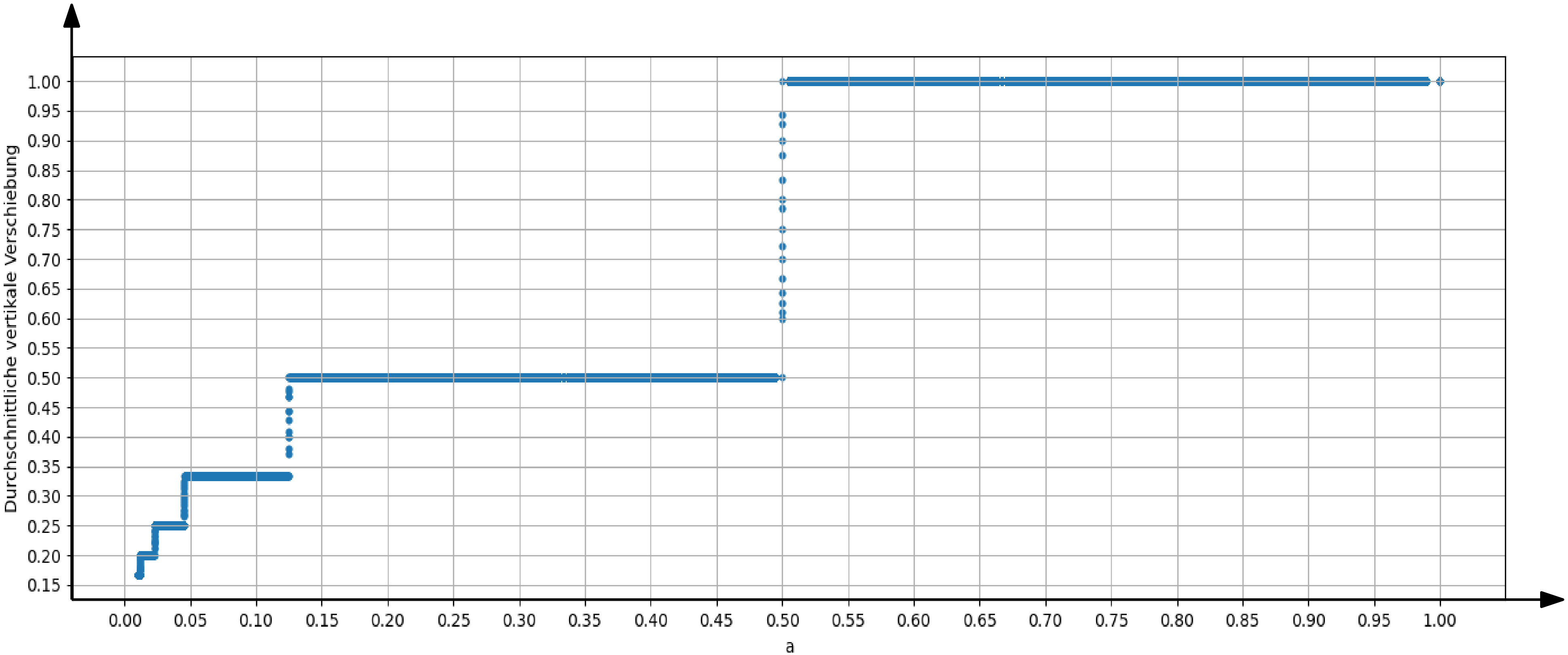
with $\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.2020569$

[ Sándor and Kramer 1999 ]

# Time period for various parabolas

$$y = ax^2 + bx$$

average vertical speed depending on $a$ (various values of $b$)



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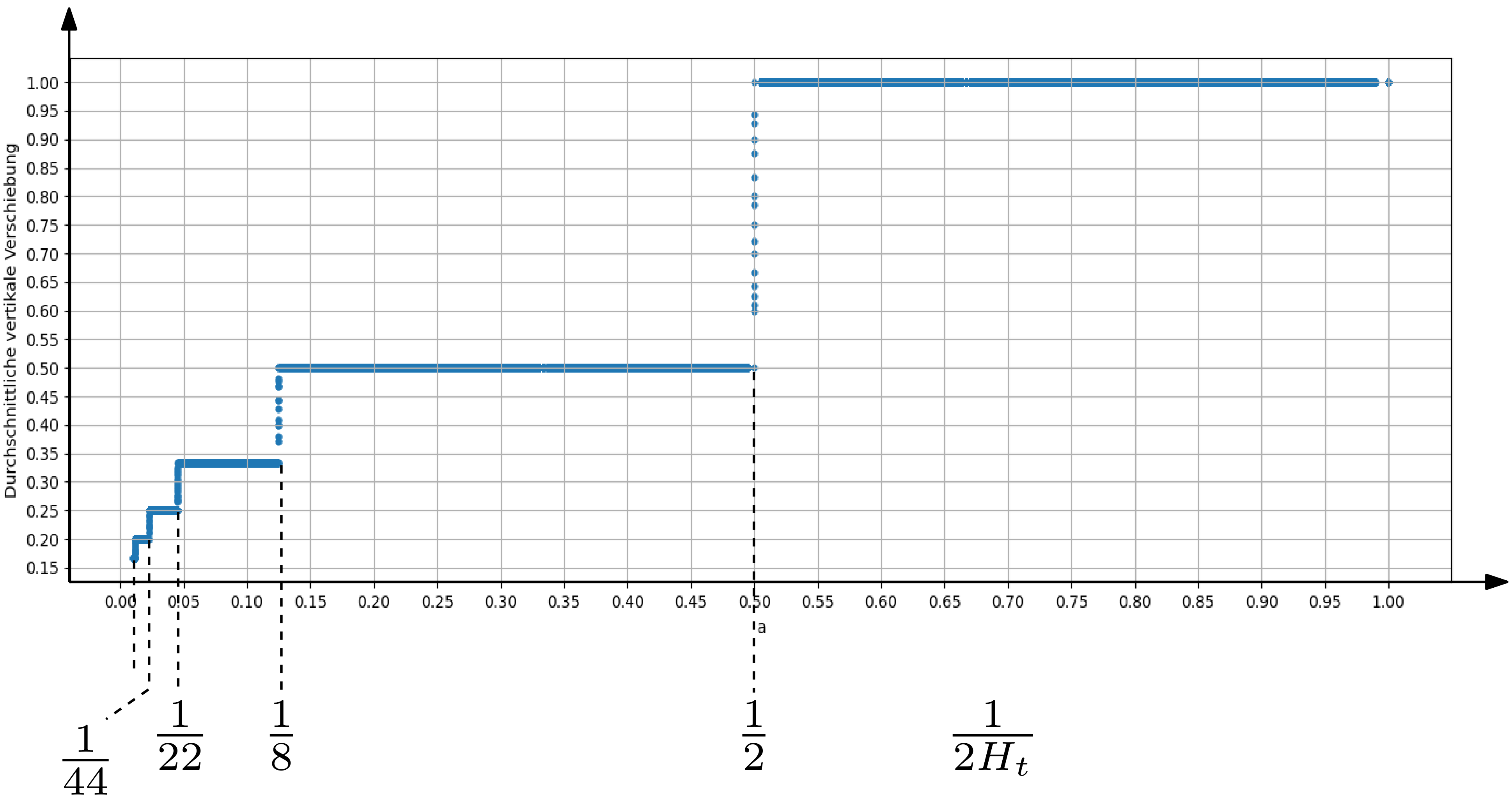

# Time period for various parabolas

vertical speed

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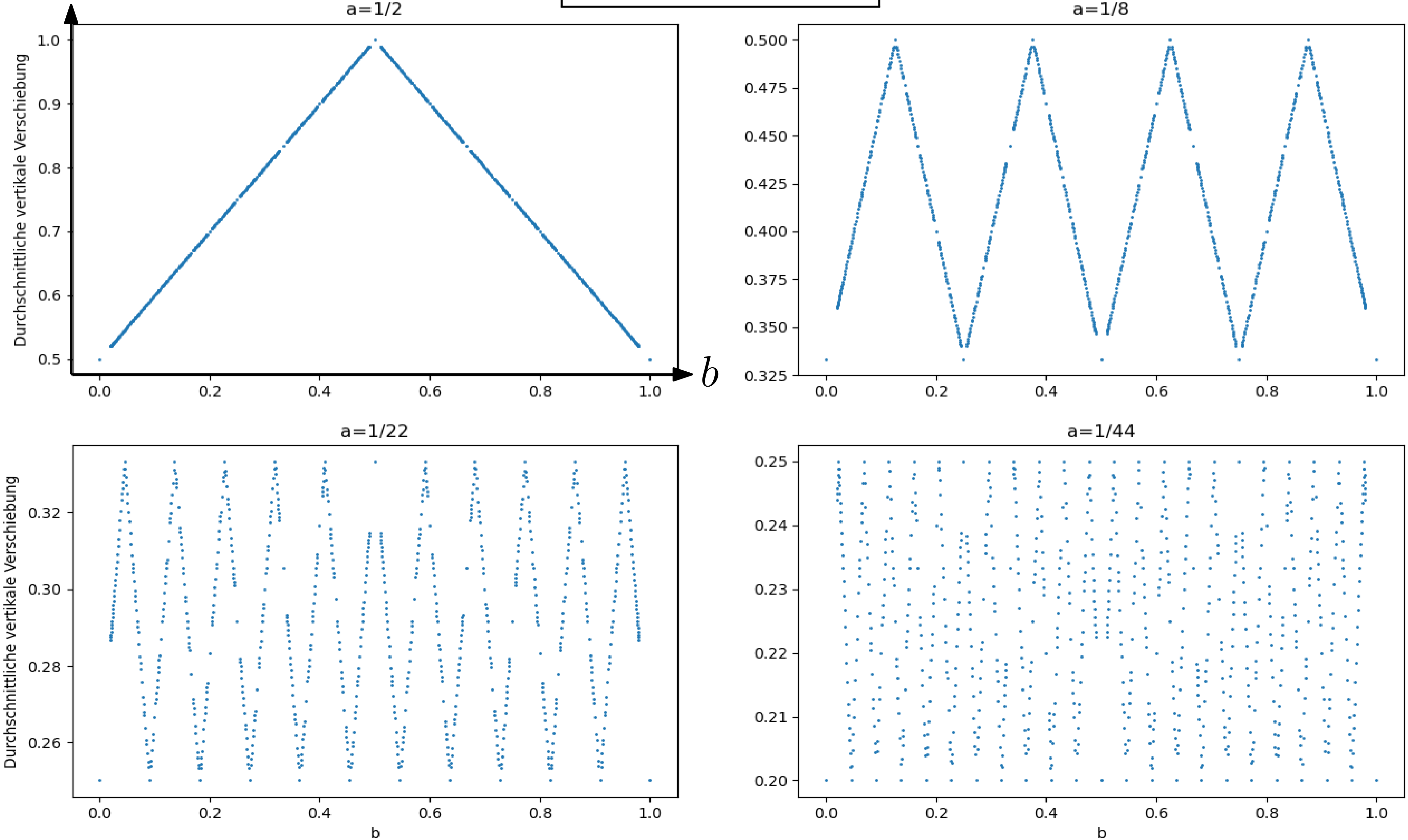

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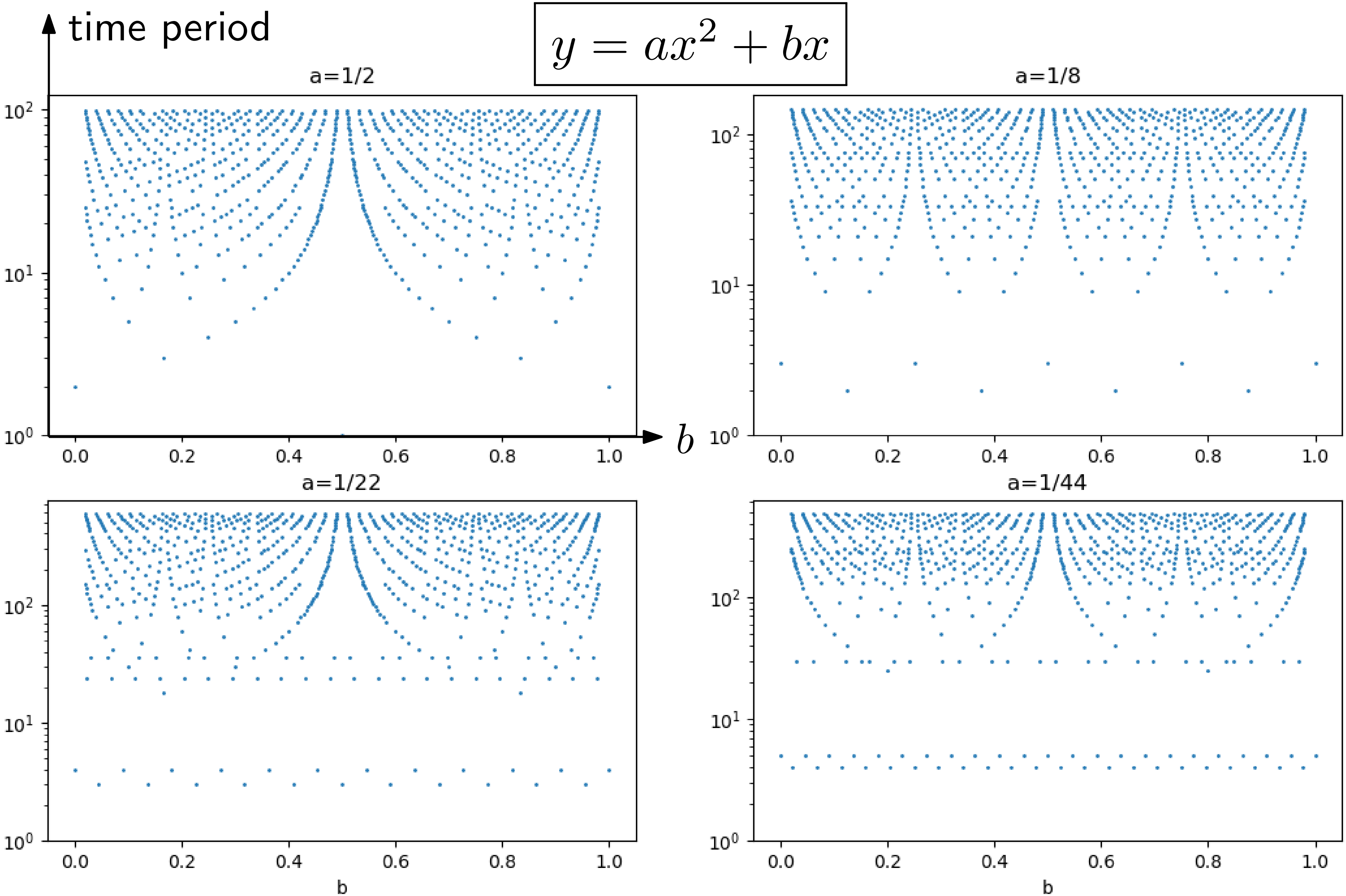

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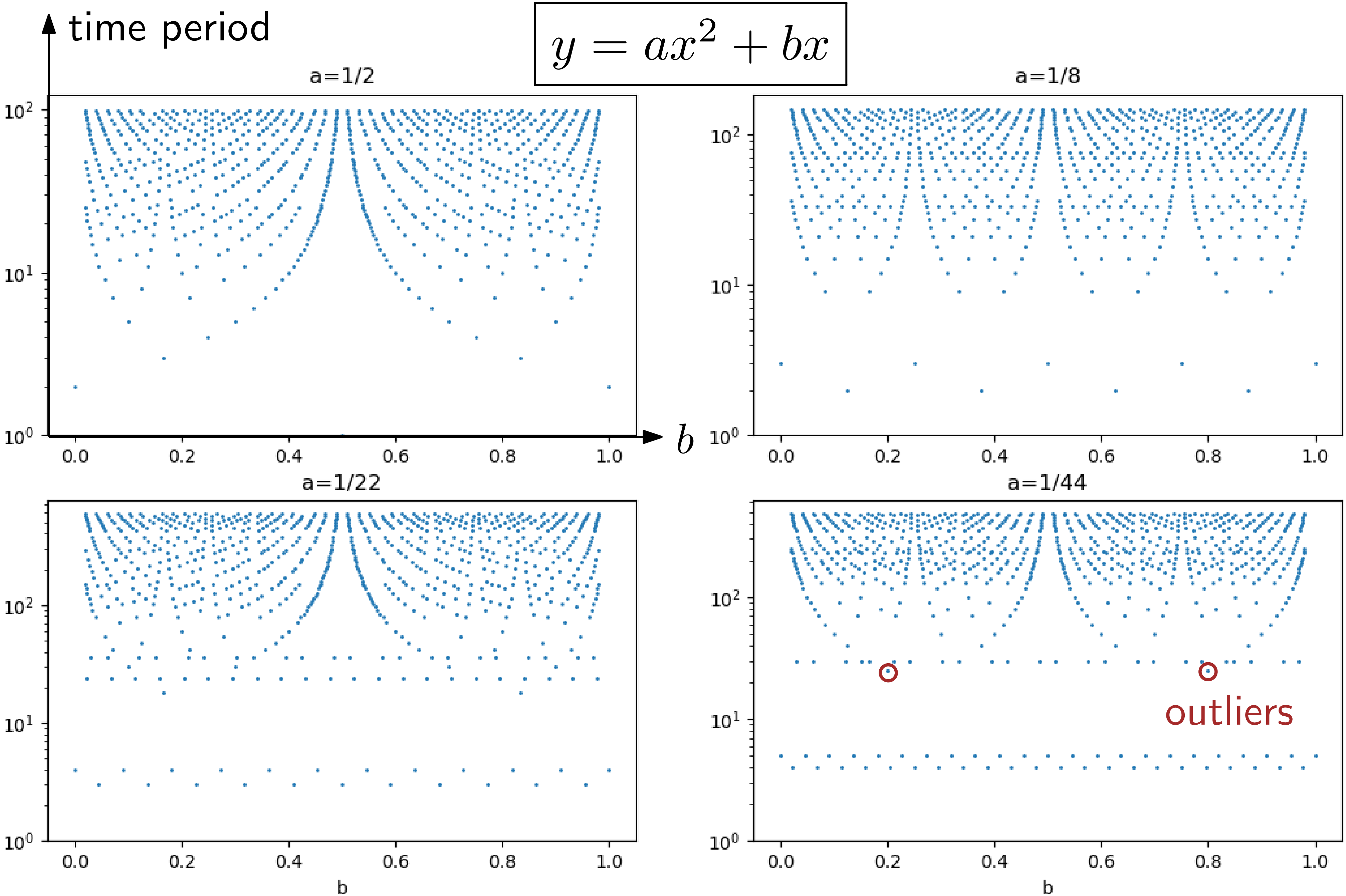
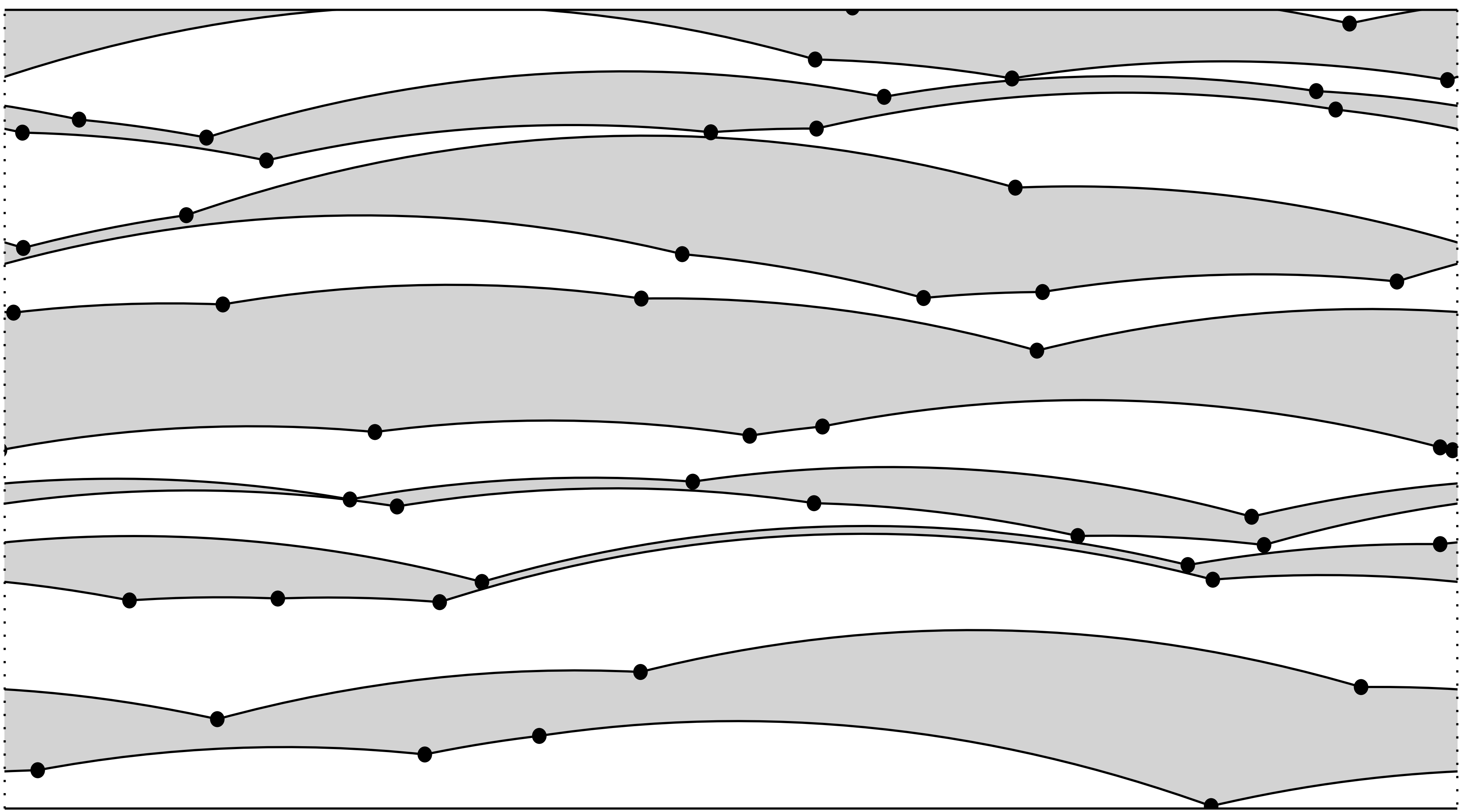


# Random-set peeling

Freie Universität Berlin

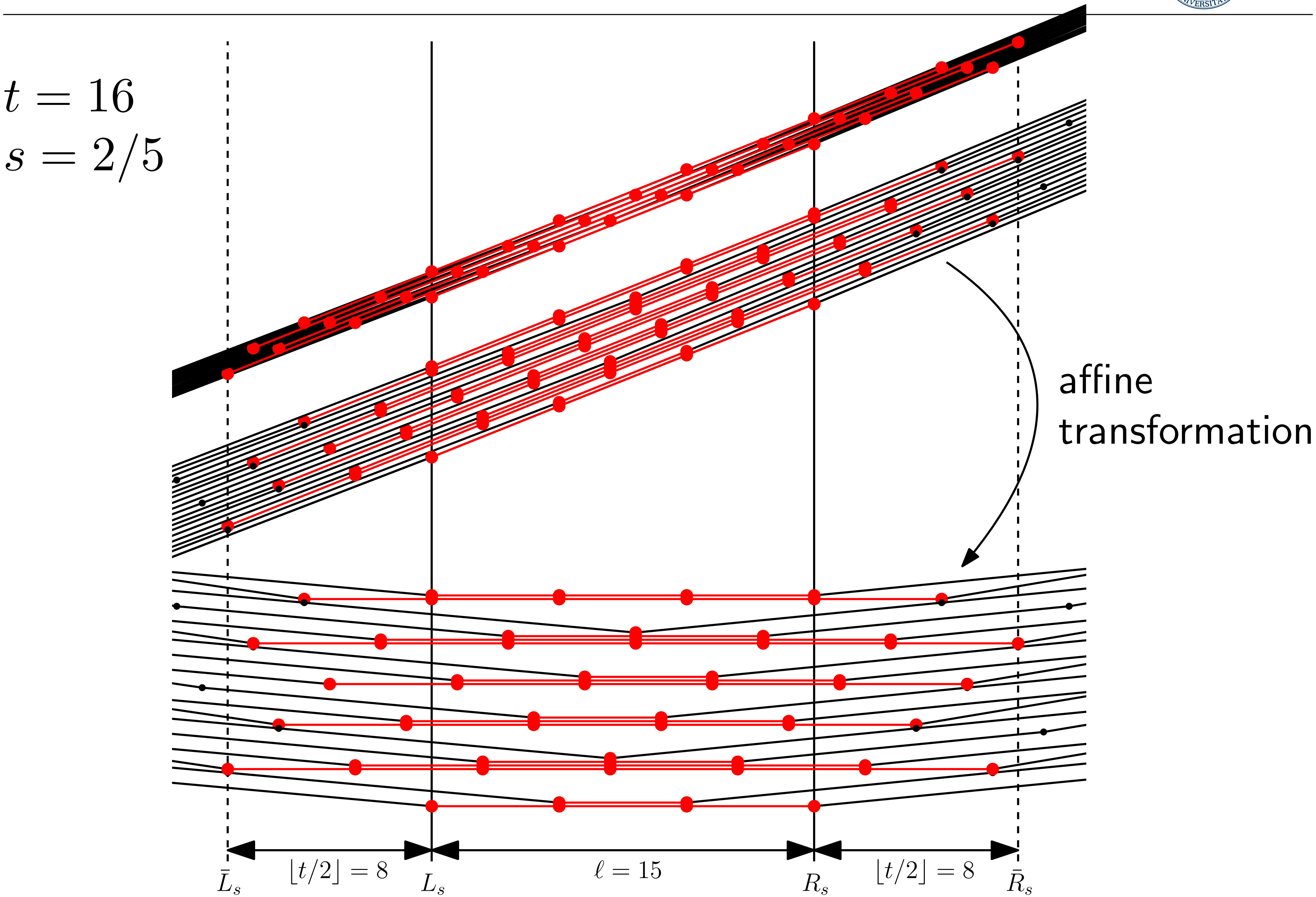
Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020



semiconvex peeling, on a cylinder

# Focus on one slope



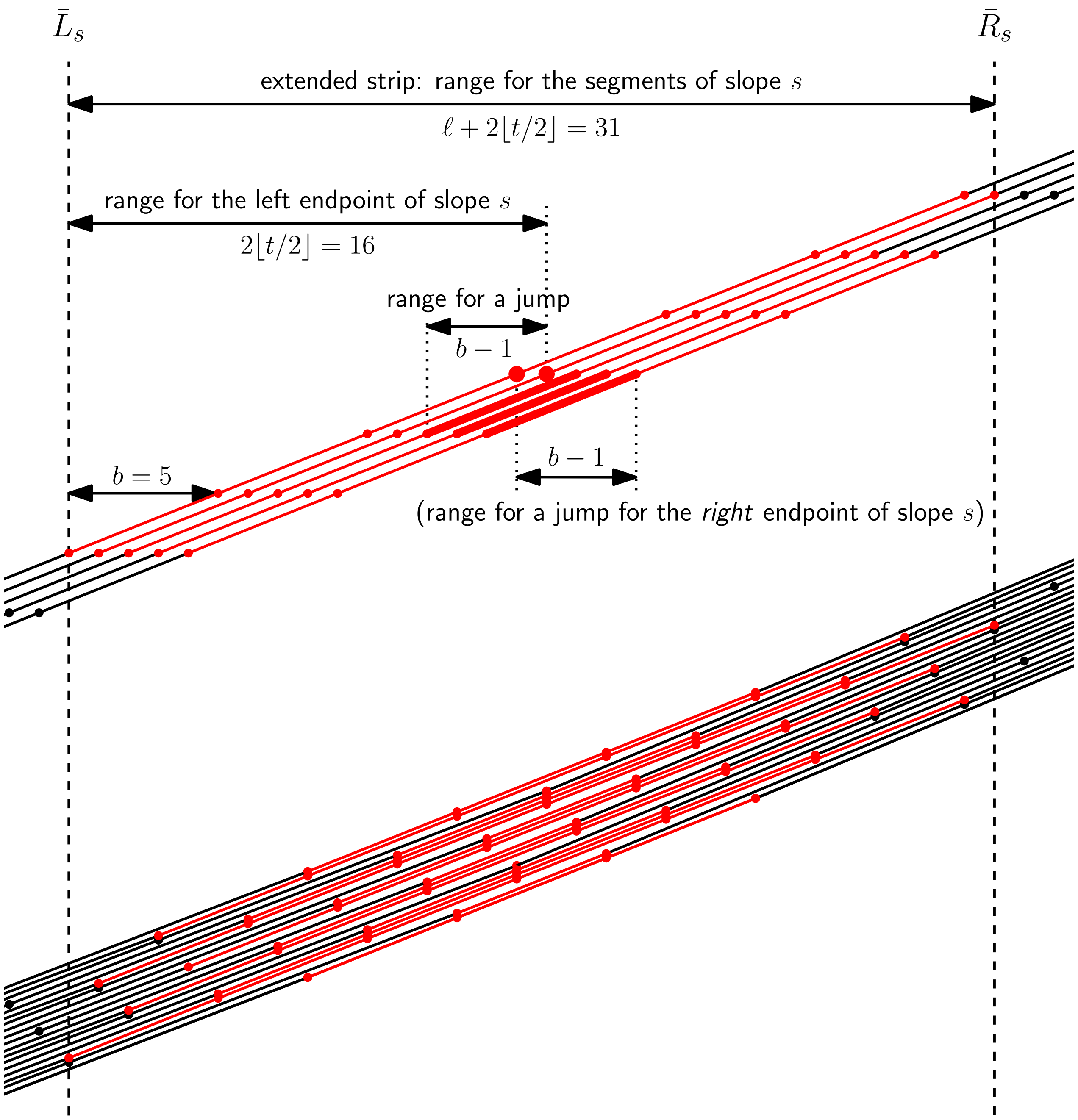


# What happens at a jump?

JUMP RULES:

- jump to the *next* grid line of slope $s$
- fill the extended strip $[\bar{L}_s, \bar{R}_s]$ as much as possible

# All possible grid lines of slope $s = 2/5$



# Two adjacent slopes $s, s'$