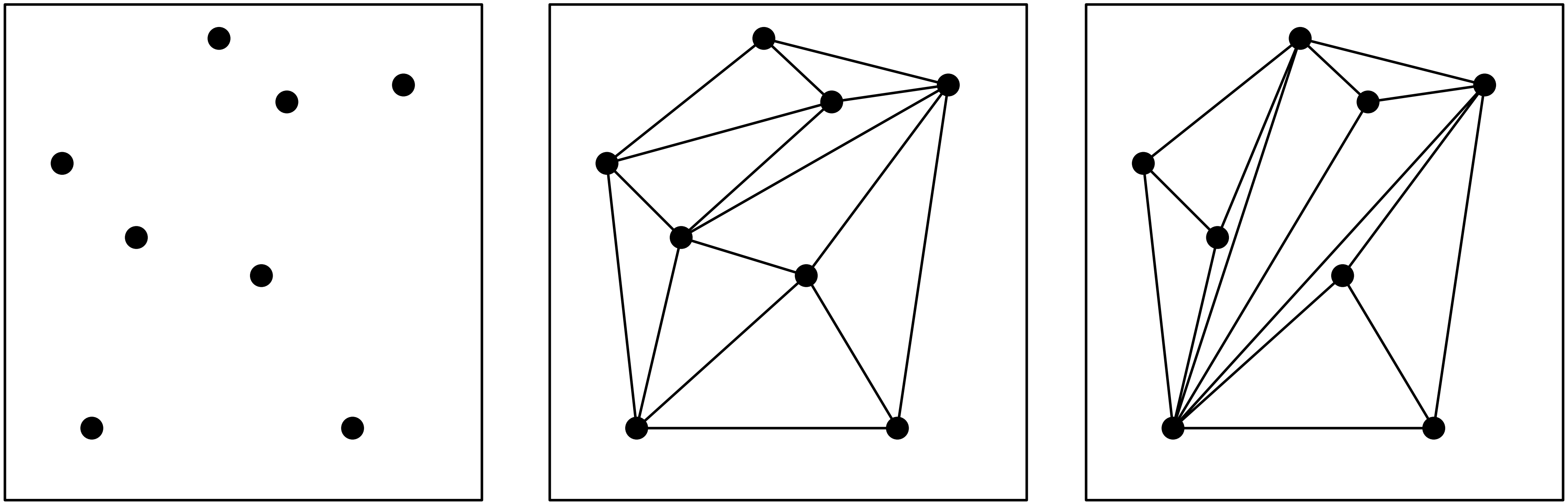
Counting and Enumeration in Combinatorial Geometry

Günter Rote
Freie Universität Berlin



two triangulations

General position: No three points on a line

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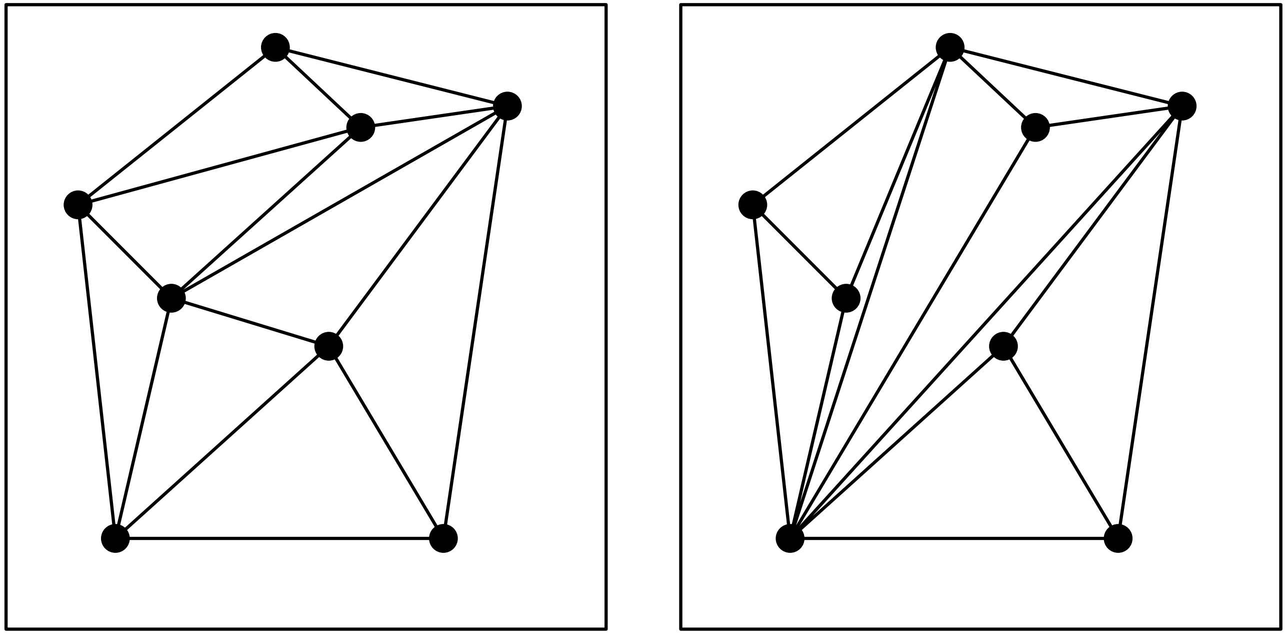
- enumeration
- counting and sampling
- bounds
- optimization
- . . .



two triangulations

General position: No three points on a line

Background

Given a set of n points in the plane in general position, how many

- triangulations
- non-crossing spanning trees
- non-crossing Hamiltonian cycles
- non-crossing matchings
- non-crossing perfect matchings
- ...
- *[your favorite straight-line geometric graph structure]*

can it have?

We first consider the more popular variants – those with new works studying them every several years.

GRAPH TYPE	LOWER BOUND	REFERENCE	UPPER BOUND	REFERENCE
Plane Graphs	$\Omega(41.18^N)$	[AHHHKV]	$O(187.53^N)$	[SS12]
Triangulations	$\Omega(8.65^N)$	[DSST11]	30^N	[SS11]
Spanning Cycles	$\Omega(4.64^N)$	[GNT00]	$O(54.55^N)$	[SSW13]
Perfect Matchings	$\Omega(3.09^N)$	[AR15]	$O(10.05^N)$	[SW06]
Spanning Trees	$\Omega(12.52^N)$	[HM13]	$O(141.07^N)$	[HSSTW11; SS11]
Cycle-Free Graphs	$\Omega(13.61^N)$	[HM13]	$O(160.55^N)$	[HSSTW11; SS11]

Some less common variants:

Polynomial Method Lectu Notes #2
Polynomial Method Lecture Notes

Recent Comments
Incidences: l Bo... on Incidences: l Bounds (par
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adamsheffer The Two Formulation the Sz...
Brendan Mu on The Two Formulation the Sz...

Numbers of Plane Gra...

Anwendungen Orte System rote

Numbers of Plane Graphs | Some Plane Truths - Mozilla Firefox

Datei Bearbeiten Ansicht Chronik Lesezeichen Extras Hilfe

Numbers of Plane Graphs | Som…

https://adamsheffer.wordpress.com/numbers-of-plane-graphs/

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Min #Triangulations:	$\Omega(2.43^n)$		$O(3.455^n)$	
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Previous Results on Perfect Matchings

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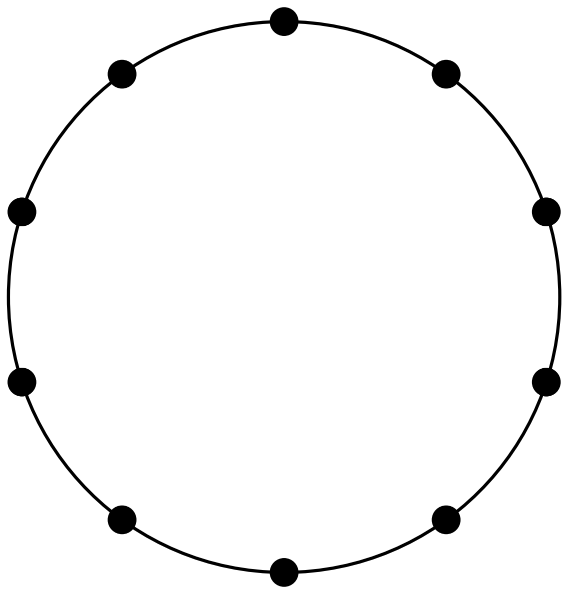
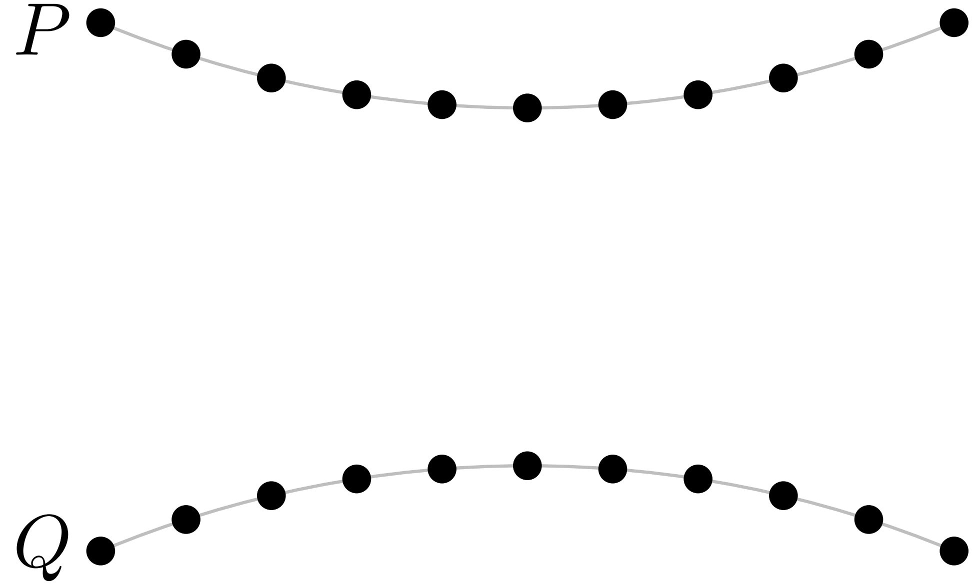


convex position

smallest possible number of perfect matchings: $\Theta^*(2^n)$



double-chain

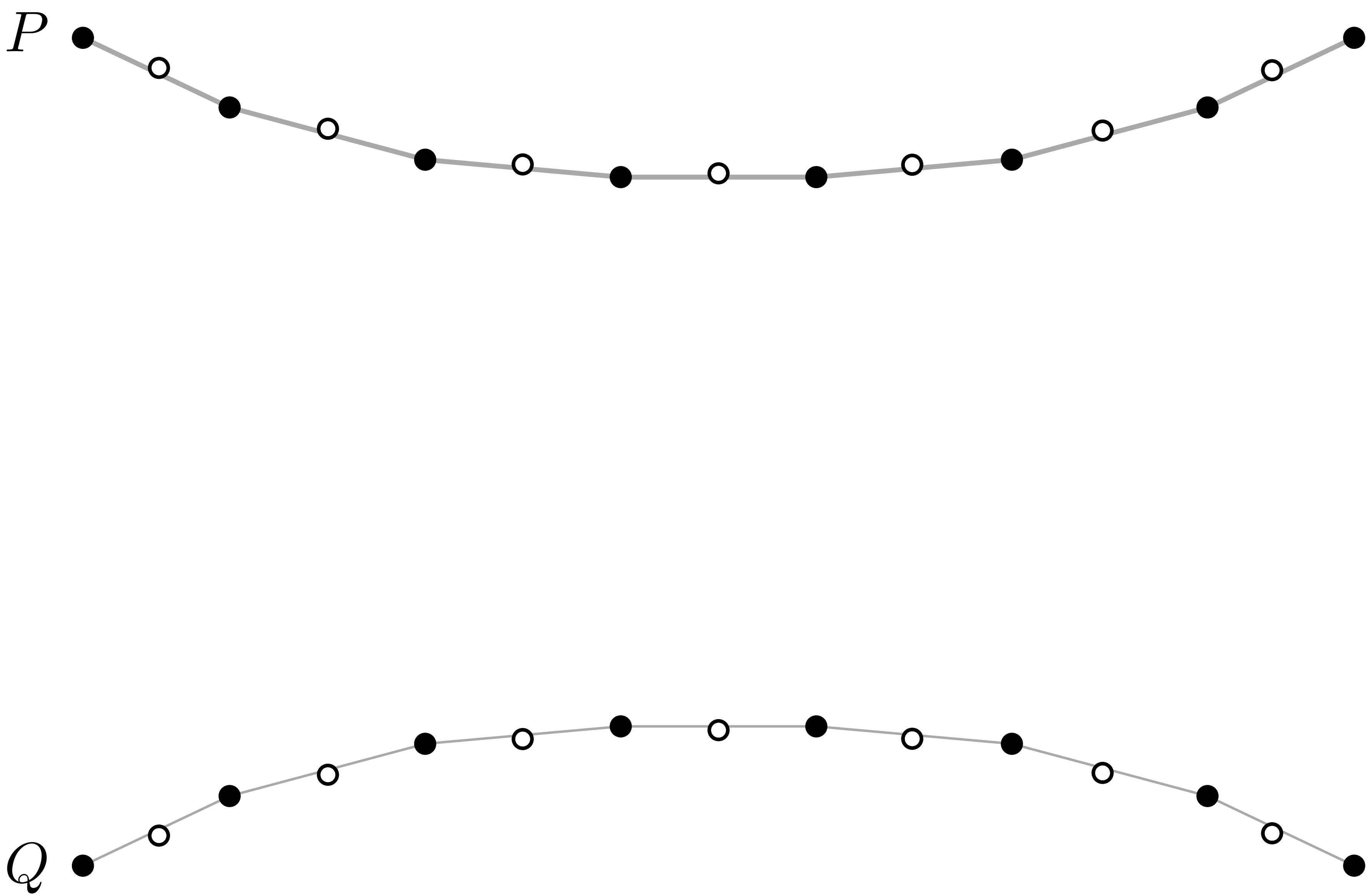
previous record: $\Theta^*(3^n)$

[García, Noy, Tejel 2000]

Upper bound: $O^*(10.06^n)$ [Sharir, Welzl 2006]

$* =$ up to a polynomial factor

The Double-Zigzag Chain



The Proof

$$C = \frac{2(1+x+x^3) - \sqrt{2(1+x+x^3)\left(1-2x-8x^2-3x^3+\cdots\right.}}{4x(1+x)(1+x+x\cdots}$$

smallest singularity: $1 - 9x - 3x^2 = 0$

$$x_0 = \frac{\sqrt{93}}{6} - \frac{3}{2}$$

$$1/\sqrt{x_0} = \sqrt{6/(\sqrt{93}-9)} \approx 3.0532$$

$$\#(\text{perfect matchings in } P \cup Q) = \Theta^*(3.0532^n)$$

Longer Arcs

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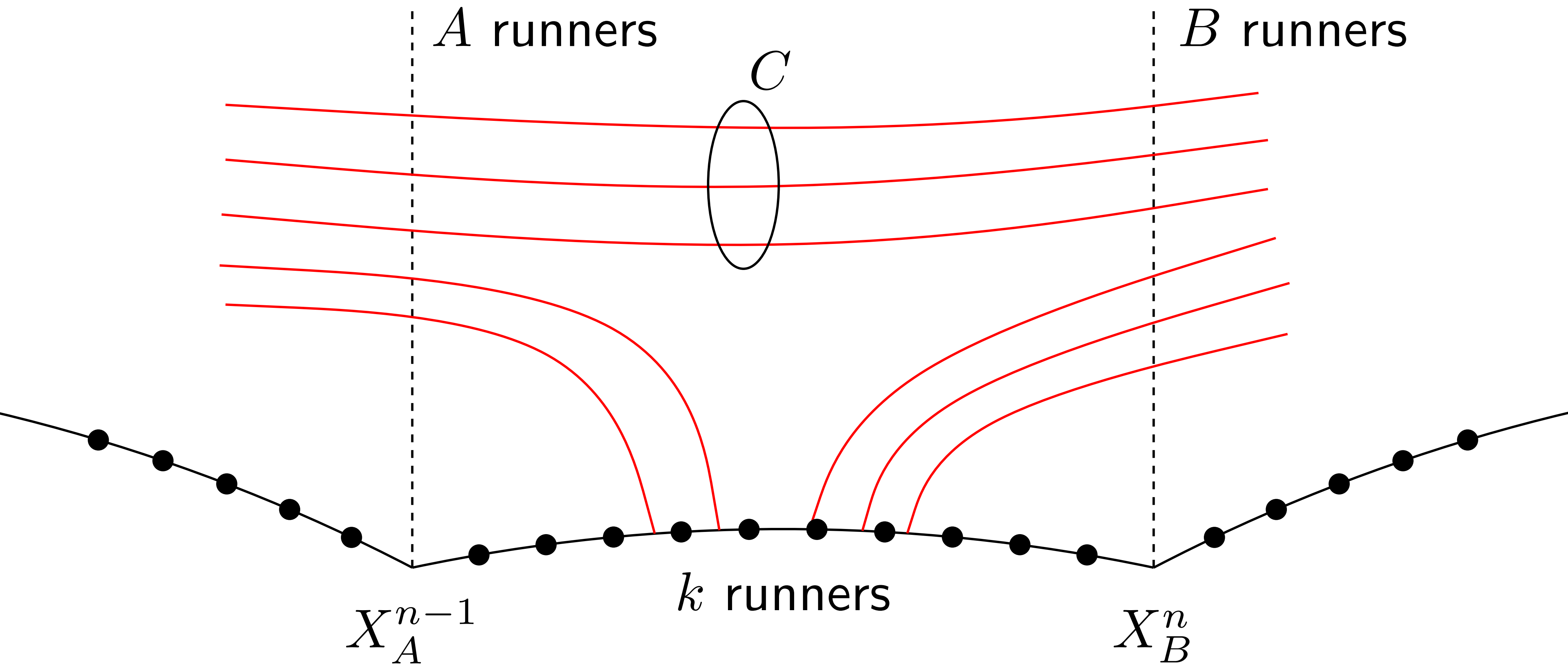
$|P| = nr + 1$

$r = 8$: $\Theta^*(3.0930^n)$ [joint work with Andrei Asinowski]

Dynamic Programming Recursion



$X_B^n = \#$ possibilities after n arcs with B crossing runners

Example: $r = 5$

matrix for transforming $(X_0^{n-1}, X_1^{n-1}, X_2^{n-1}, \ldots)$ into $(X_0^n, X_1^n, X_2^n, \ldots)$

$$\begin{pmatrix}
10 & 30 & 30 & 20 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
30 & 40 & 50 & 35 & 21 & 5 & 1 & 0 & 0 & 0 & 0 & \cdots \\
30 & 50 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & 0 & 0 & \cdots \\
20 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & 0 & \cdots \\
5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & \cdots \\
1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & \cdots \\
0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & \cdots \\
0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & \cdots \\
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0 & 0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

row sum $= 271 \implies$ vectors grow like $271^n/\mathrm{poly}(n)$

Counting Triangulations

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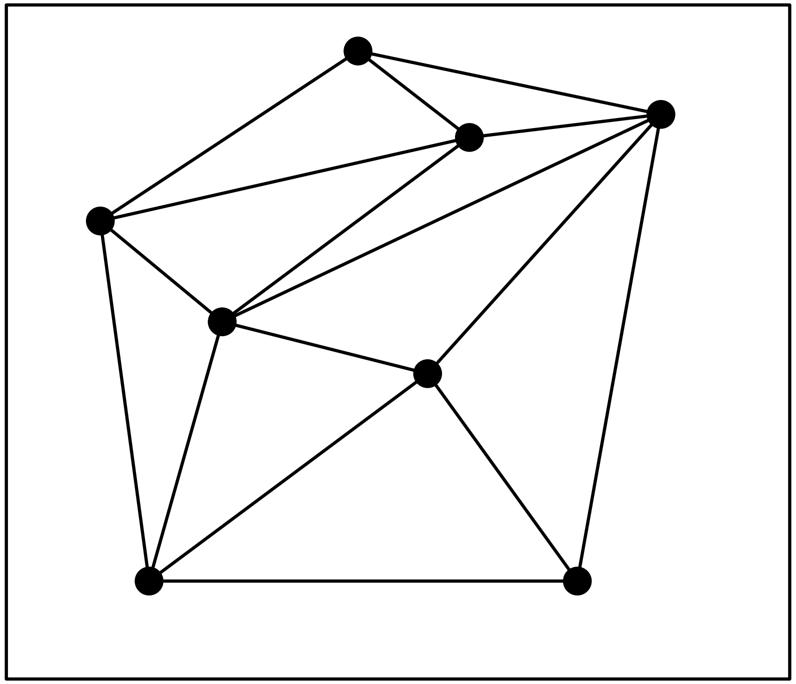
Counting, sampling, enumerating [V. Alvarez, R. Seidel 2013]

triangulation



Counting Triangulations

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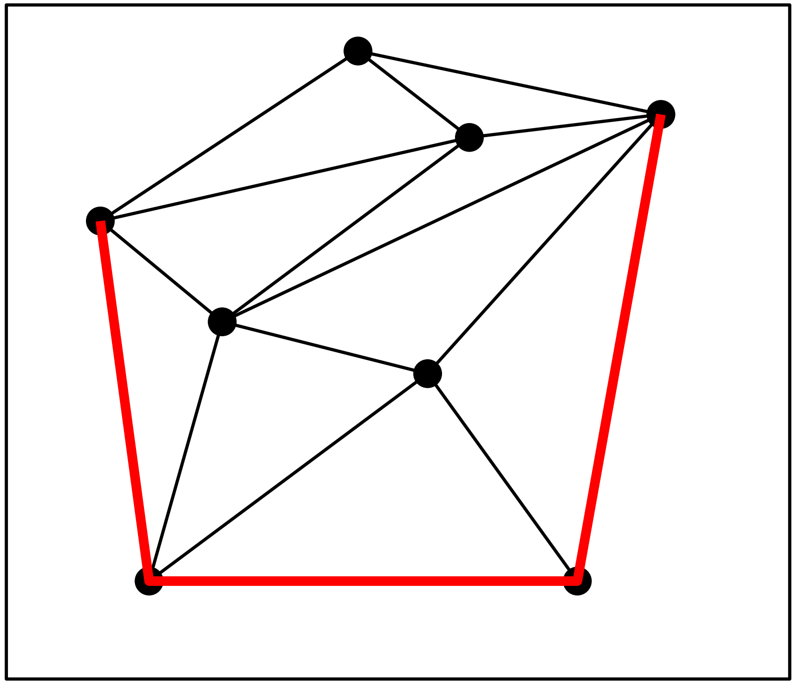
triangulation $\longrightarrow$ sequence of x-monotone paths



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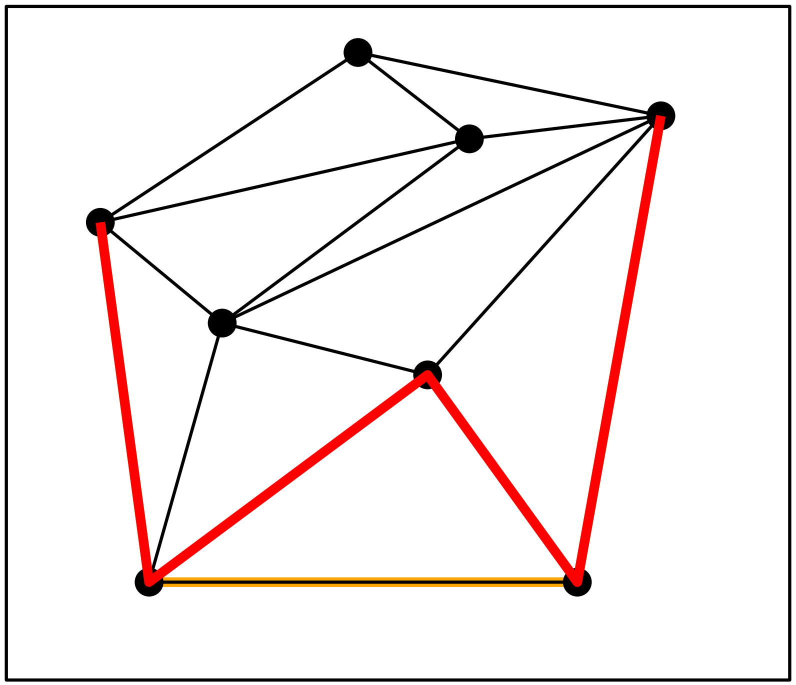

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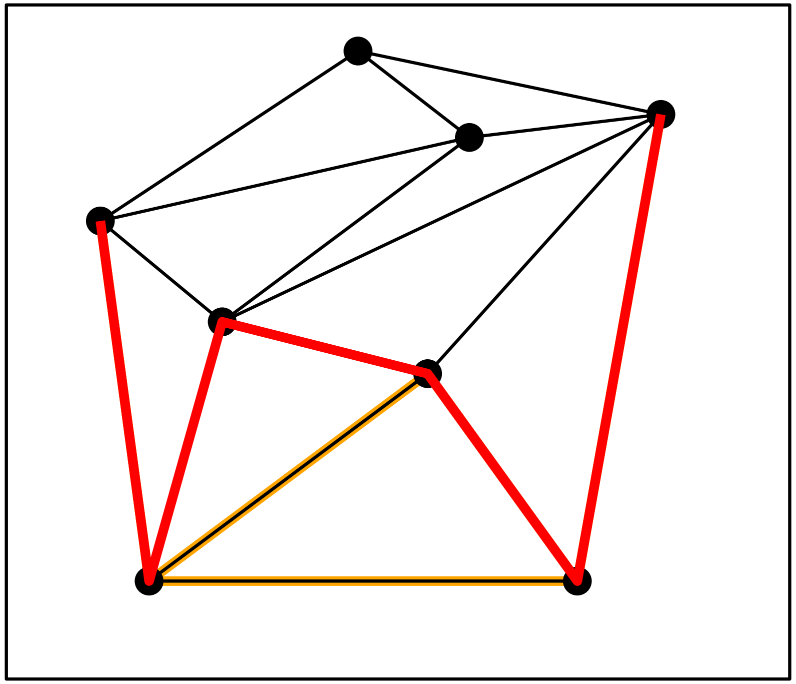

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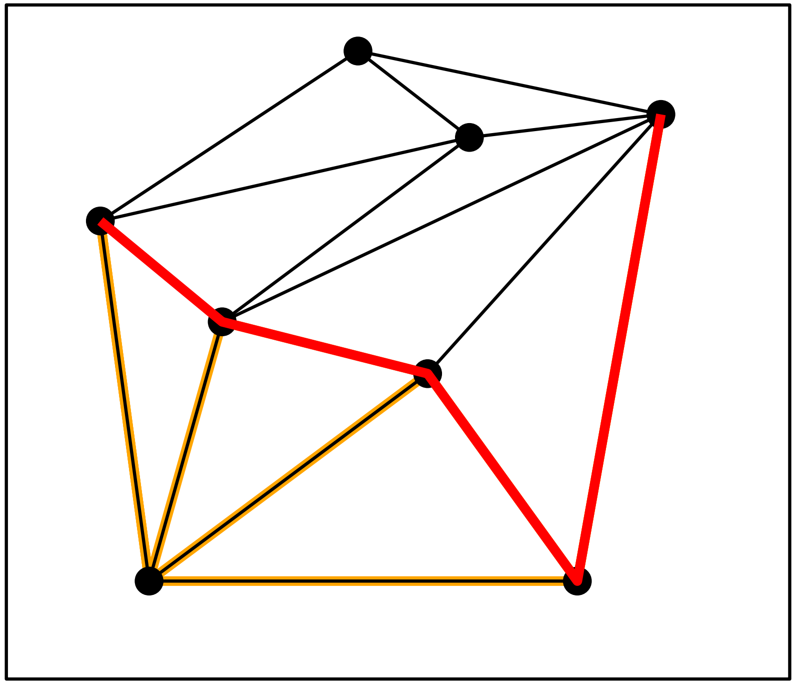

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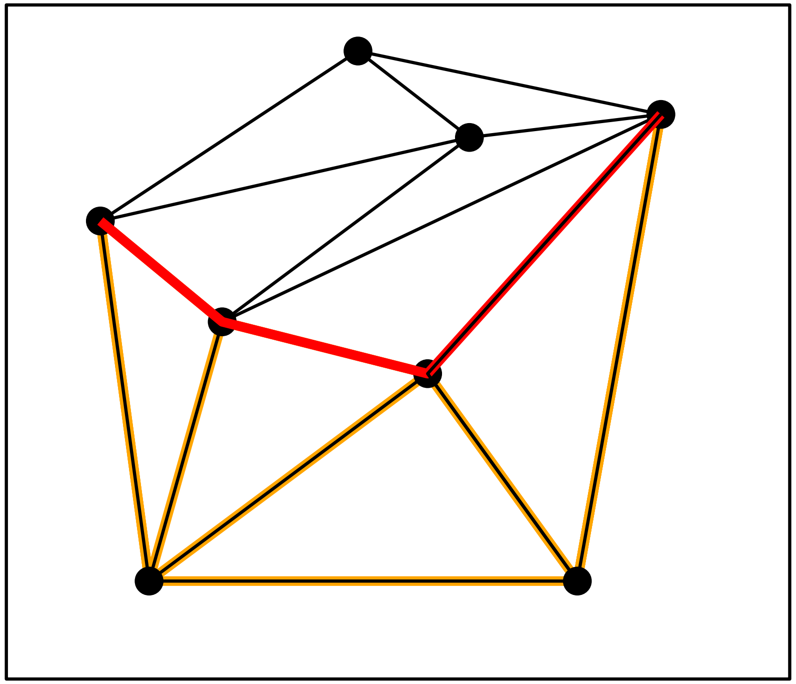

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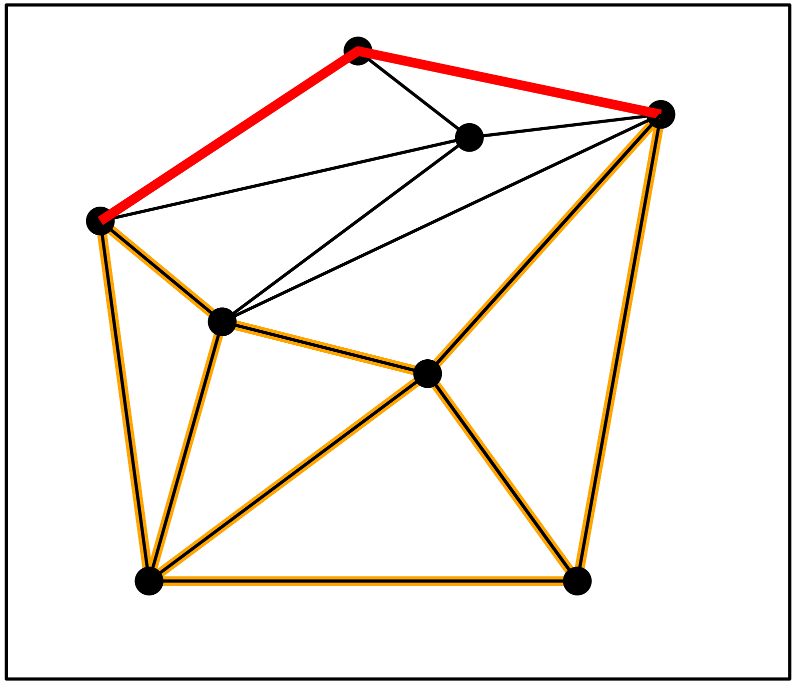

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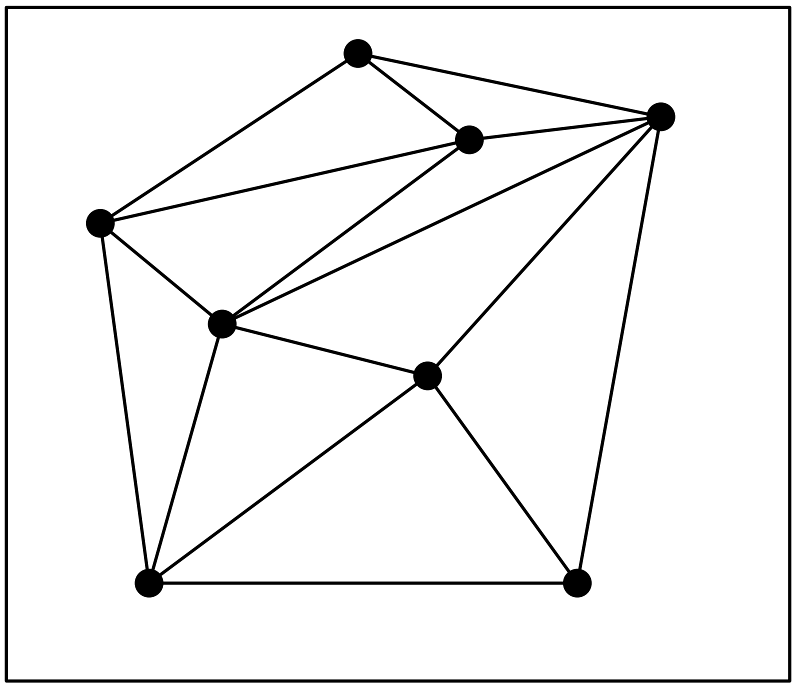
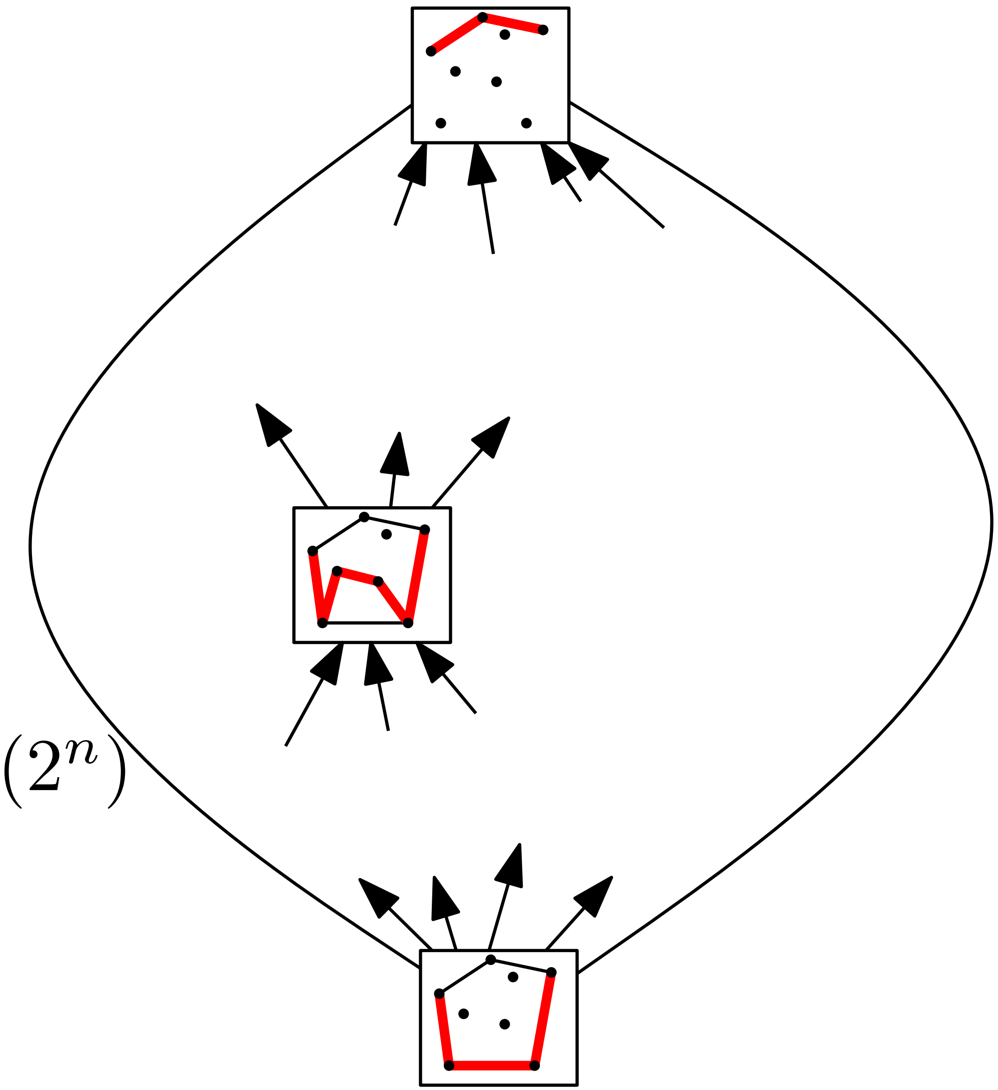


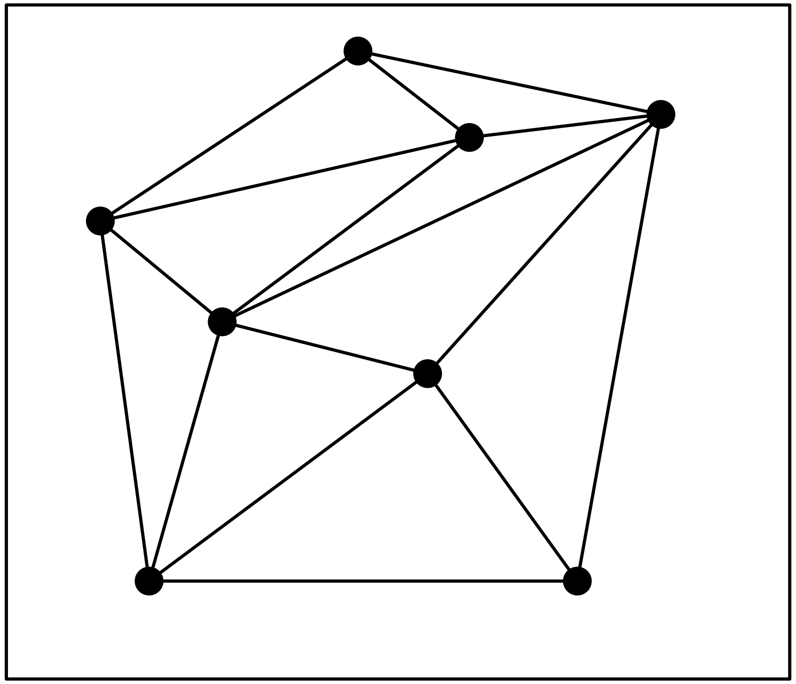


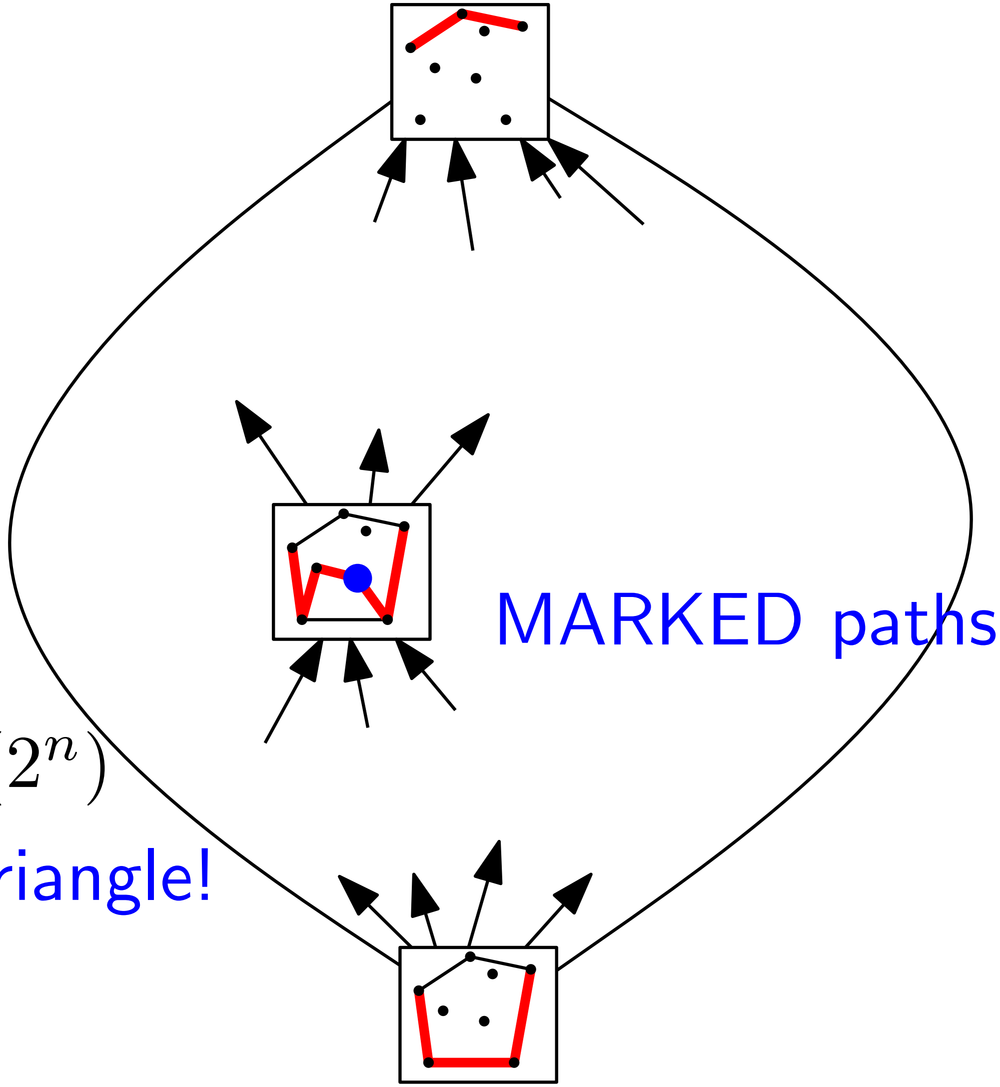
→ path in a DAG of size $O^*(2^n)$

Counting Triangulations

Counting, sampling, enumerating [V. Alvarez, R. Seidel 2013]

triangulation $\leftrightarrow$ sequence of x-monotone paths





MARKED paths

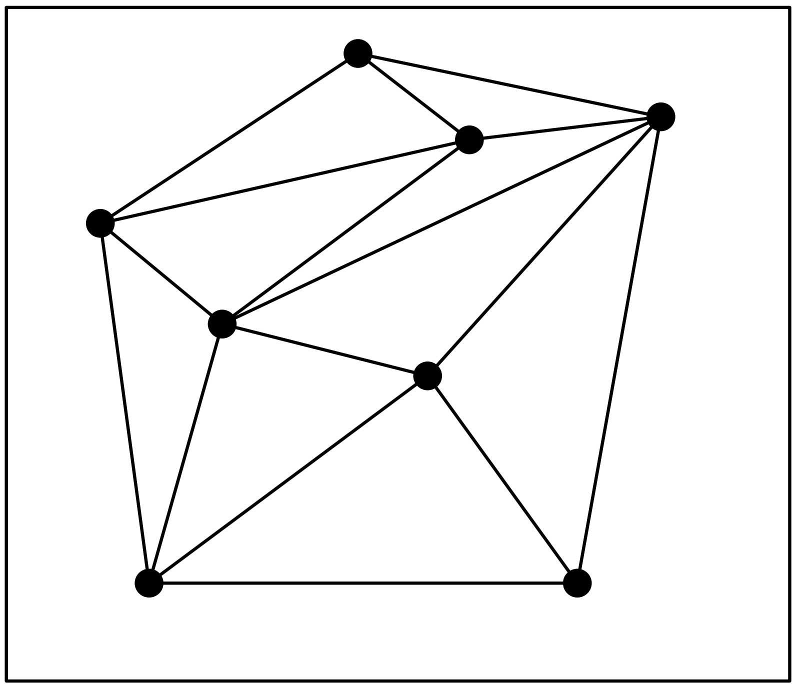
$\rightarrow$ path in a DAG of size $O^*(2^n)$

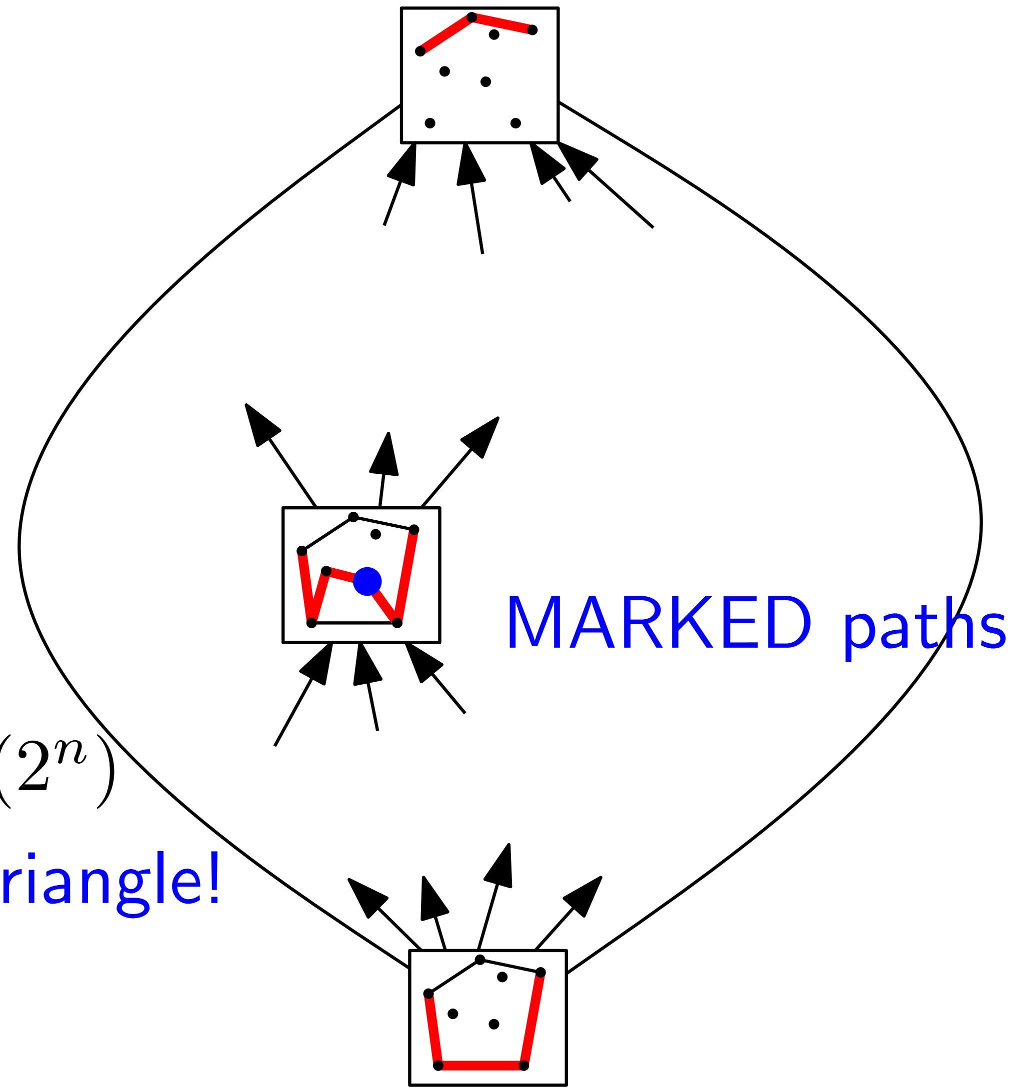
always choose the LEFTmost triangle!

Counting Triangulations

Counting, sampling, enumerating [V. Alvarez, R. Seidel 2013]

triangulation $\leftrightarrow$ sequence of x-monotone paths





$\rightarrow$ path in a DAG of size $O^*(2^n)$

always choose the LEFTmost triangle!

$O(1)$-delay enumeration,
with $O^*(2^n)$ preprocessing

Extension to Perfect Matchings

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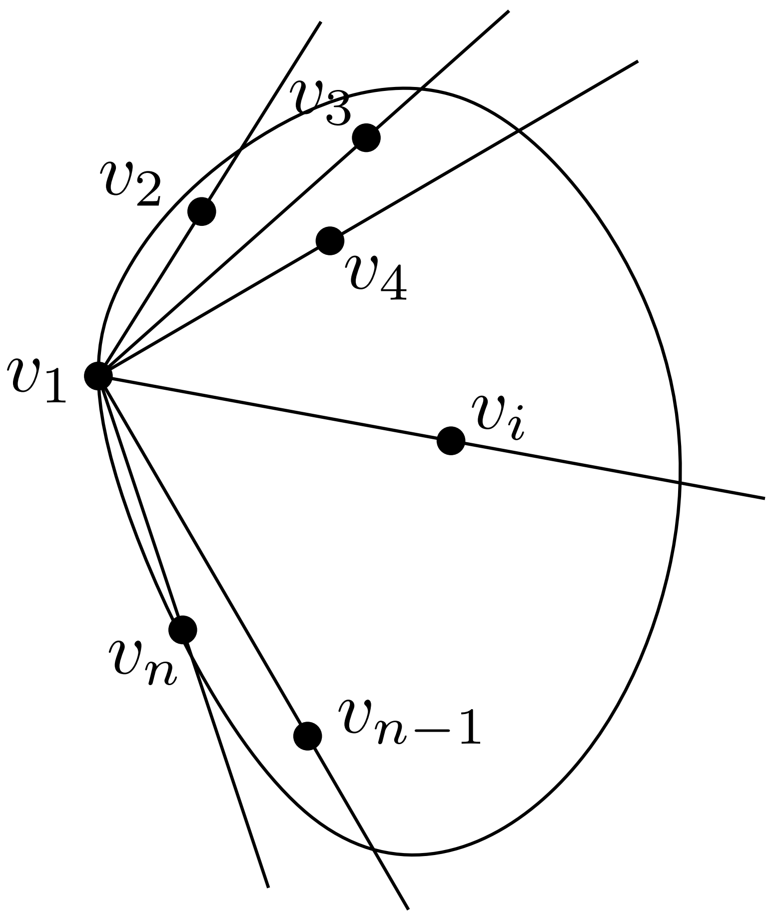
[Manuel Wettstein 2014]

Every point set has at least
$\text{Catalan}(n/2) \sim 2^n$ perfect non-crossing matchings.



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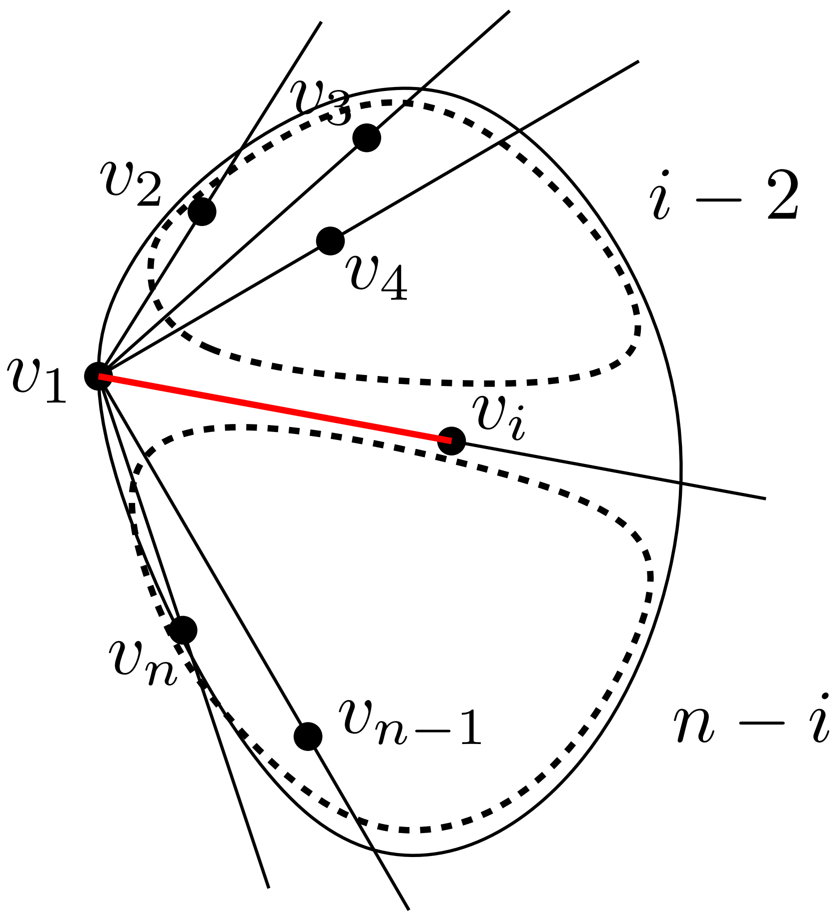

(tight (almost only) for point sets in convex position)

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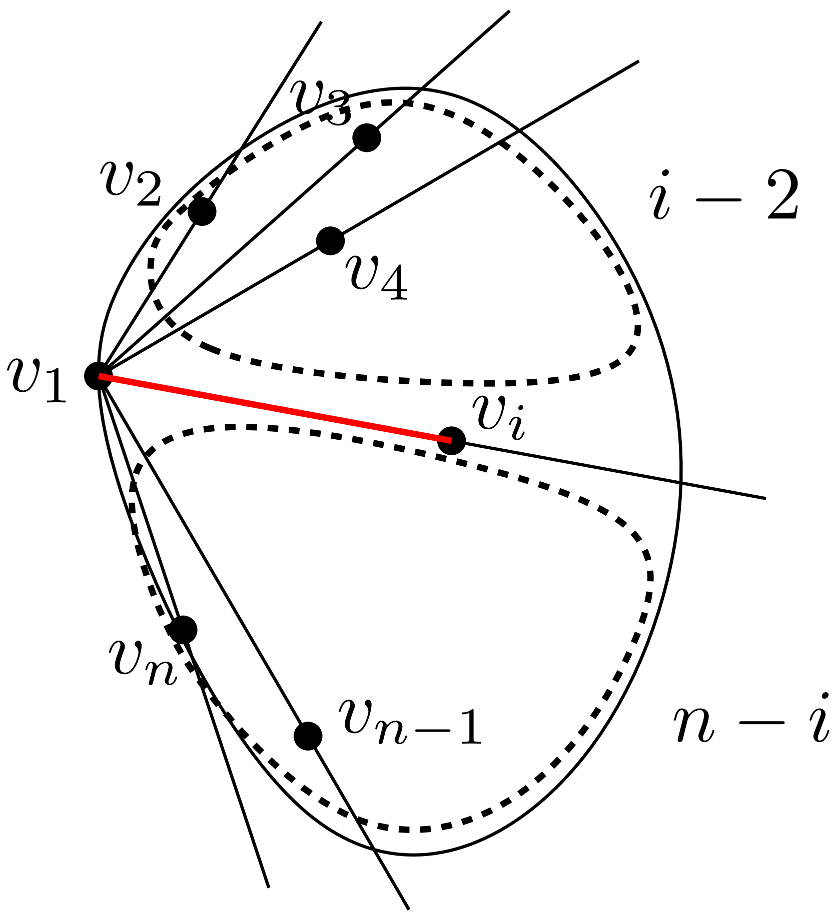

TRICK to achieve polynomial delay:

Output those “trivial” matchings while preparing the DAG.

(tight (almost only) for point sets in convex position)