Coloring Points for Bottomless Rectangles

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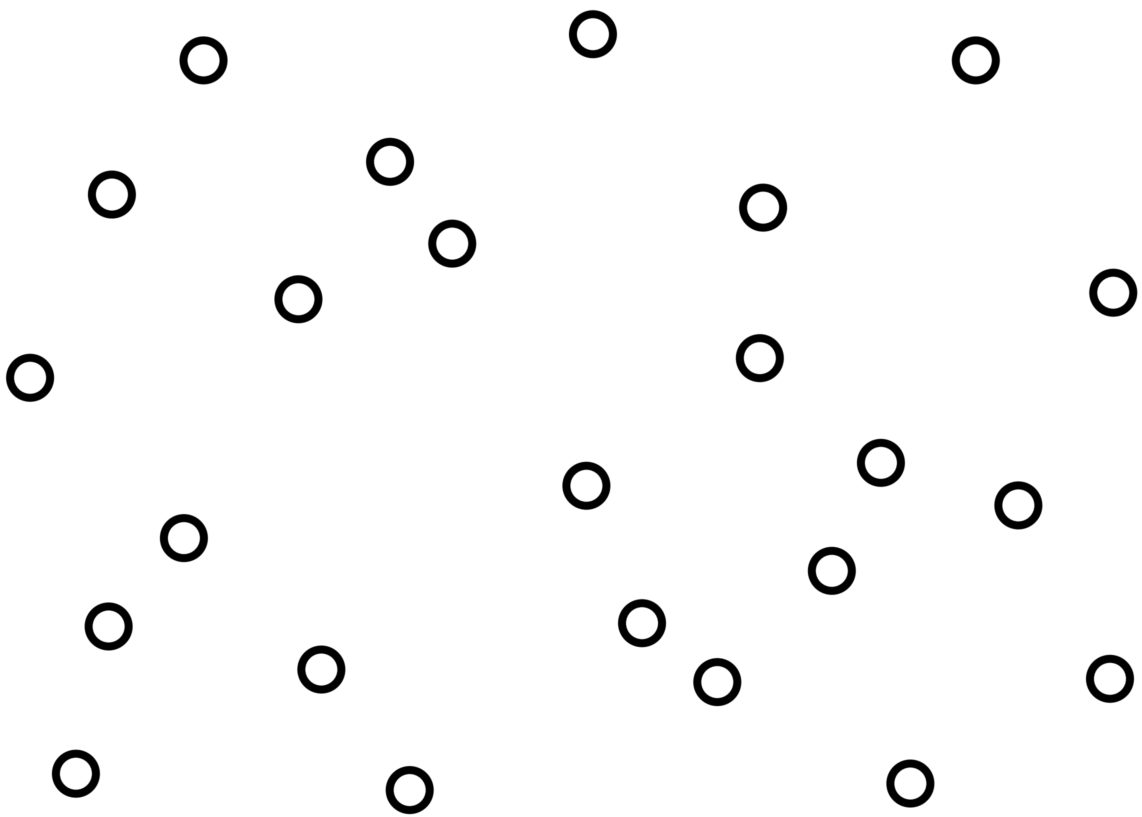
Berlin, Brussels, Graz, Kraków, Prague, Zürich

Problem Statement

GIVEN: point set, $k = 3$ colors

FIND a coloring such that every *bottomless rectangle* with at least $q = 7$ points contains all k colors



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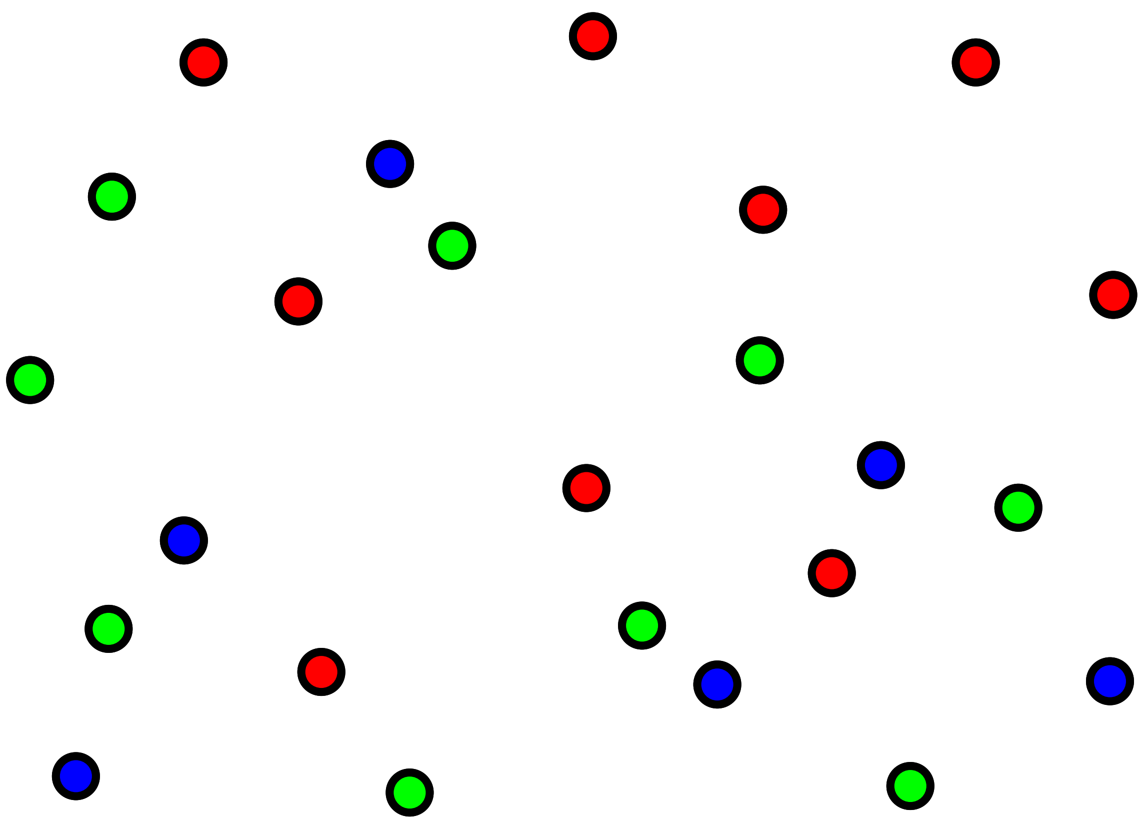

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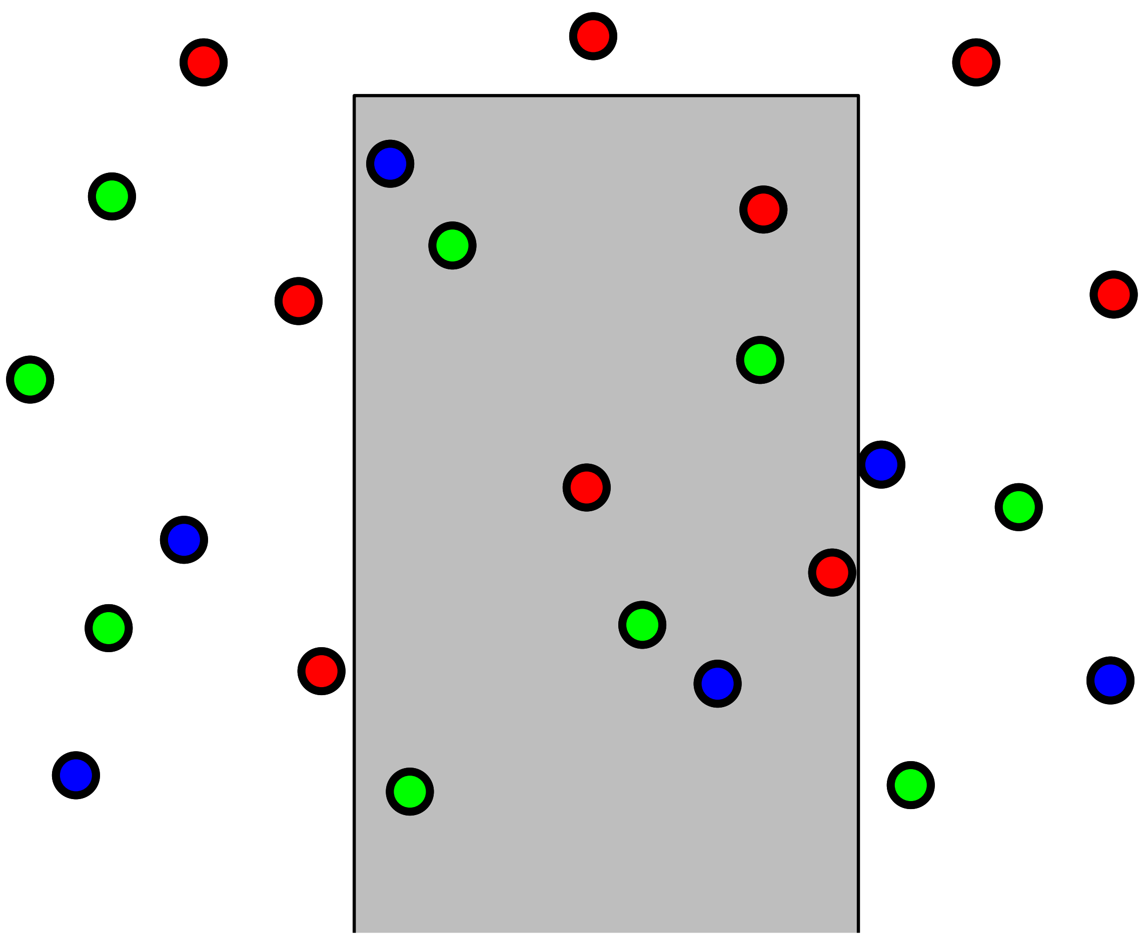

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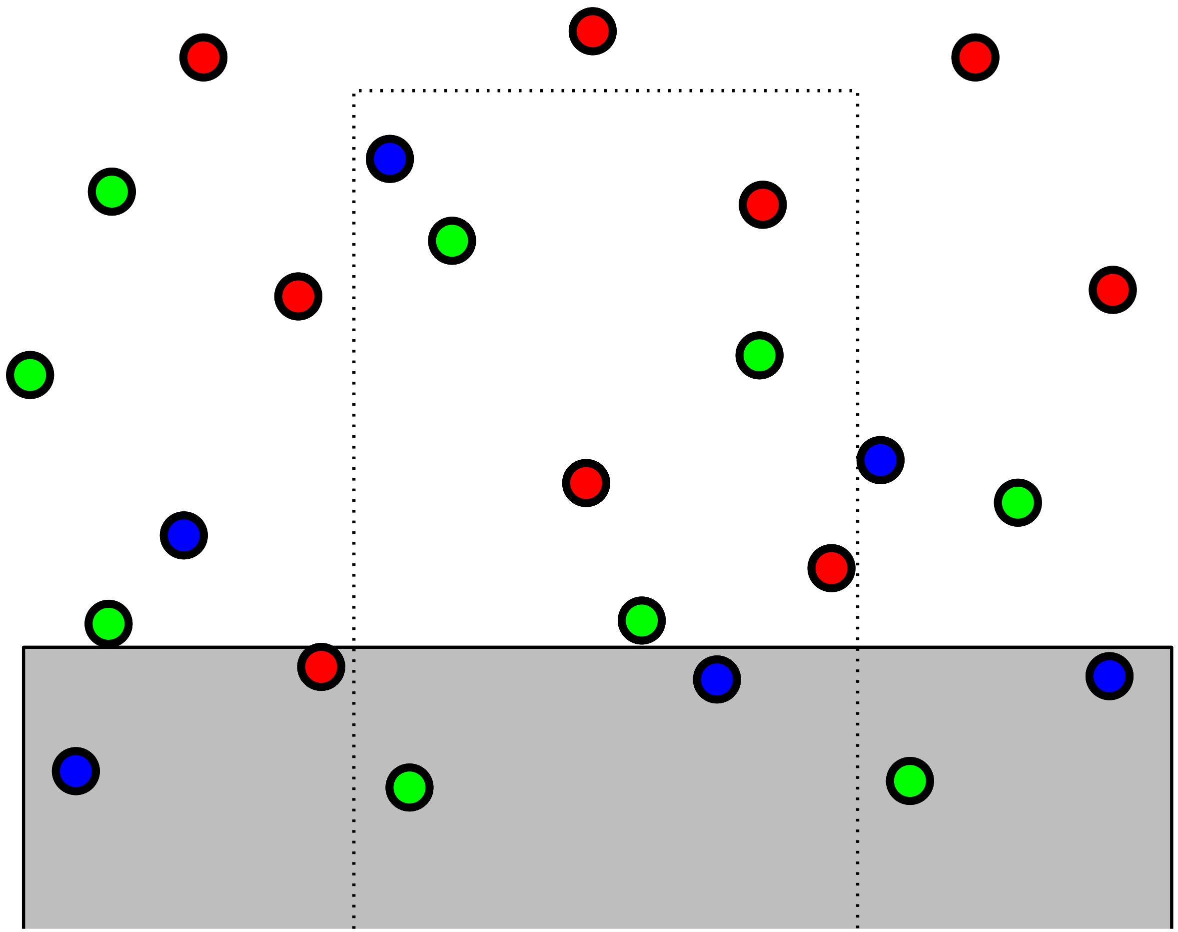

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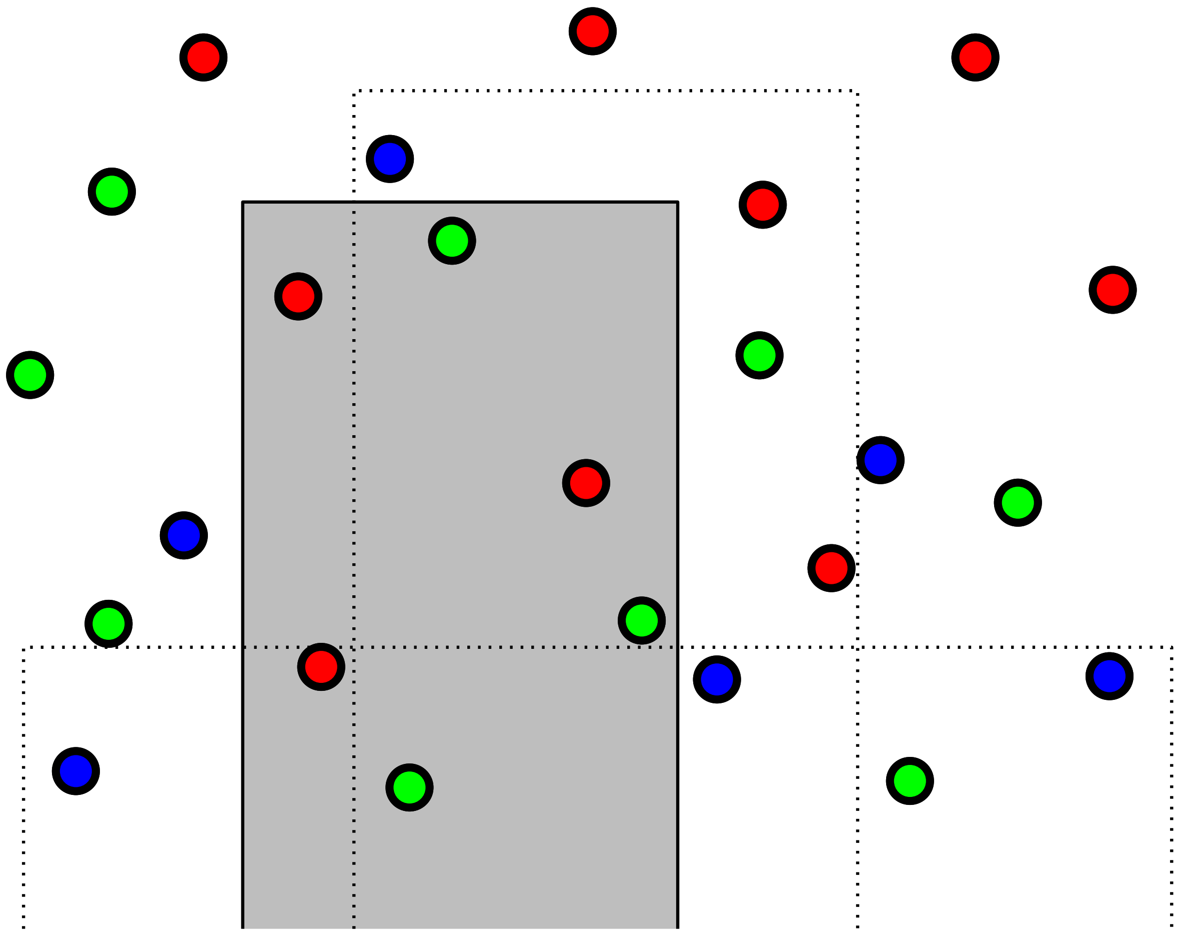

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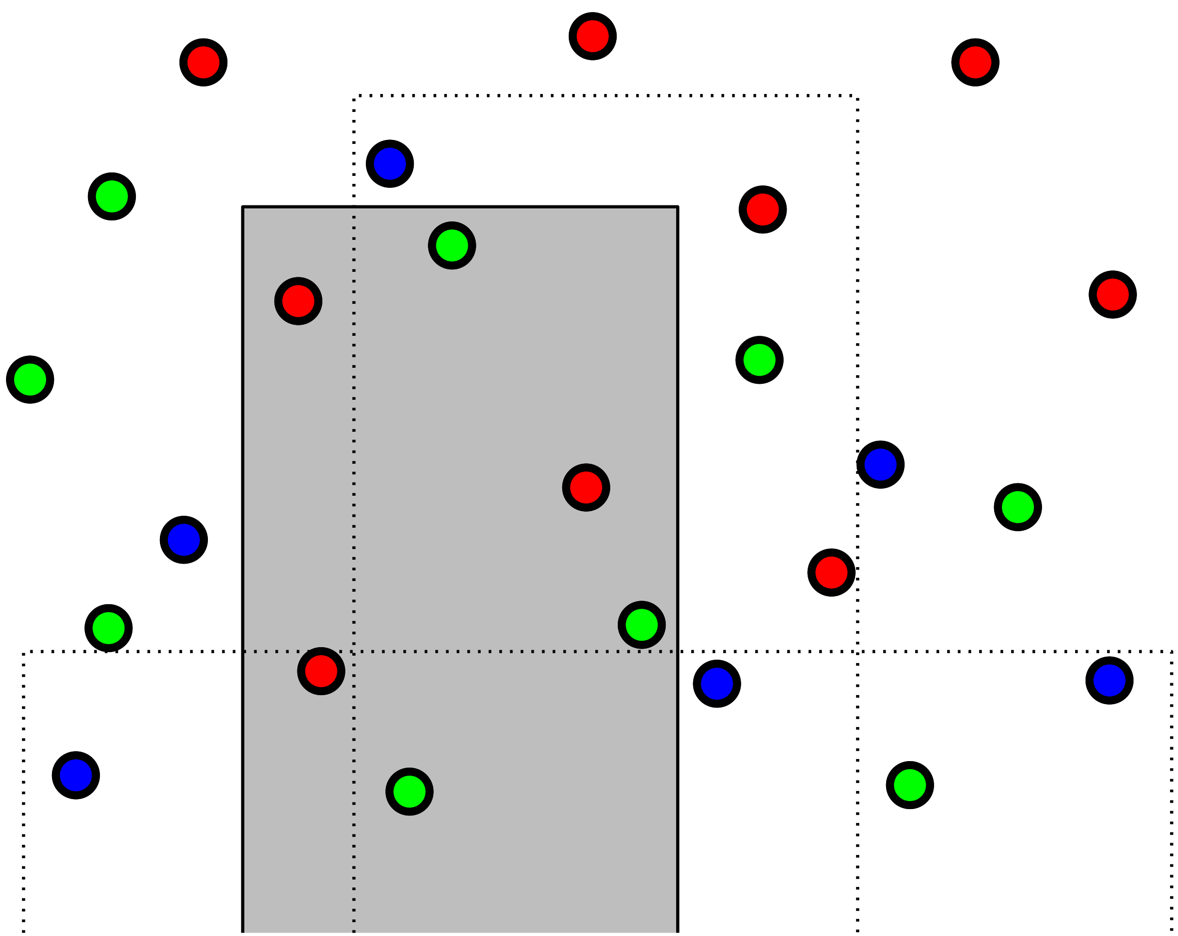

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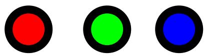
GIVEN: point set, $k = 3$ colors 

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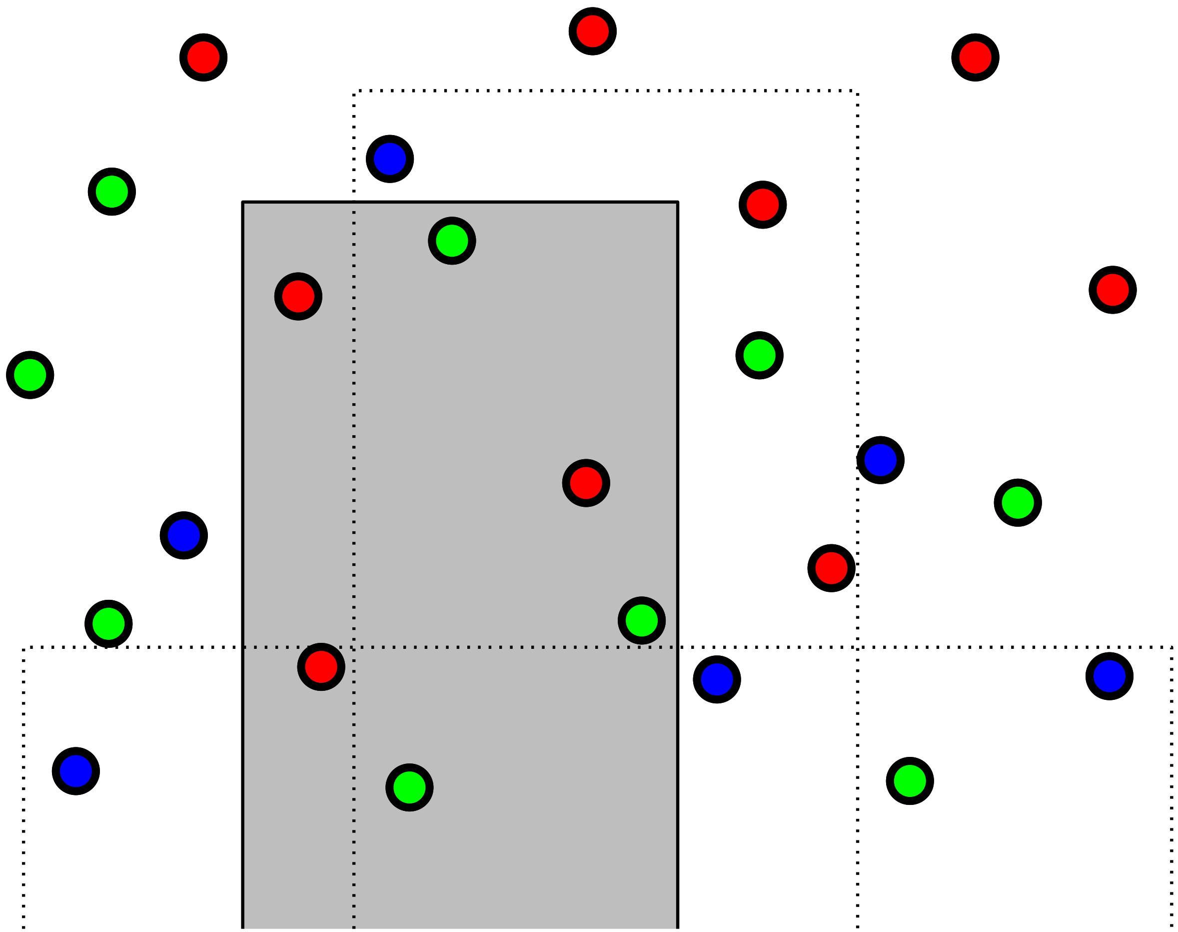


$f(k) :=$ the smallest q for which such a coloring always exists

RESULTS:
$1.63k \leq f(k) \leq 3k-2$

Other Ranges

Axis-aligned rectangles: $f(k) = \infty$, even for $k = 2$ colors

[Pach, Tardos 2010]

Aligned equilateral triangles: $f(2) \leq 12$

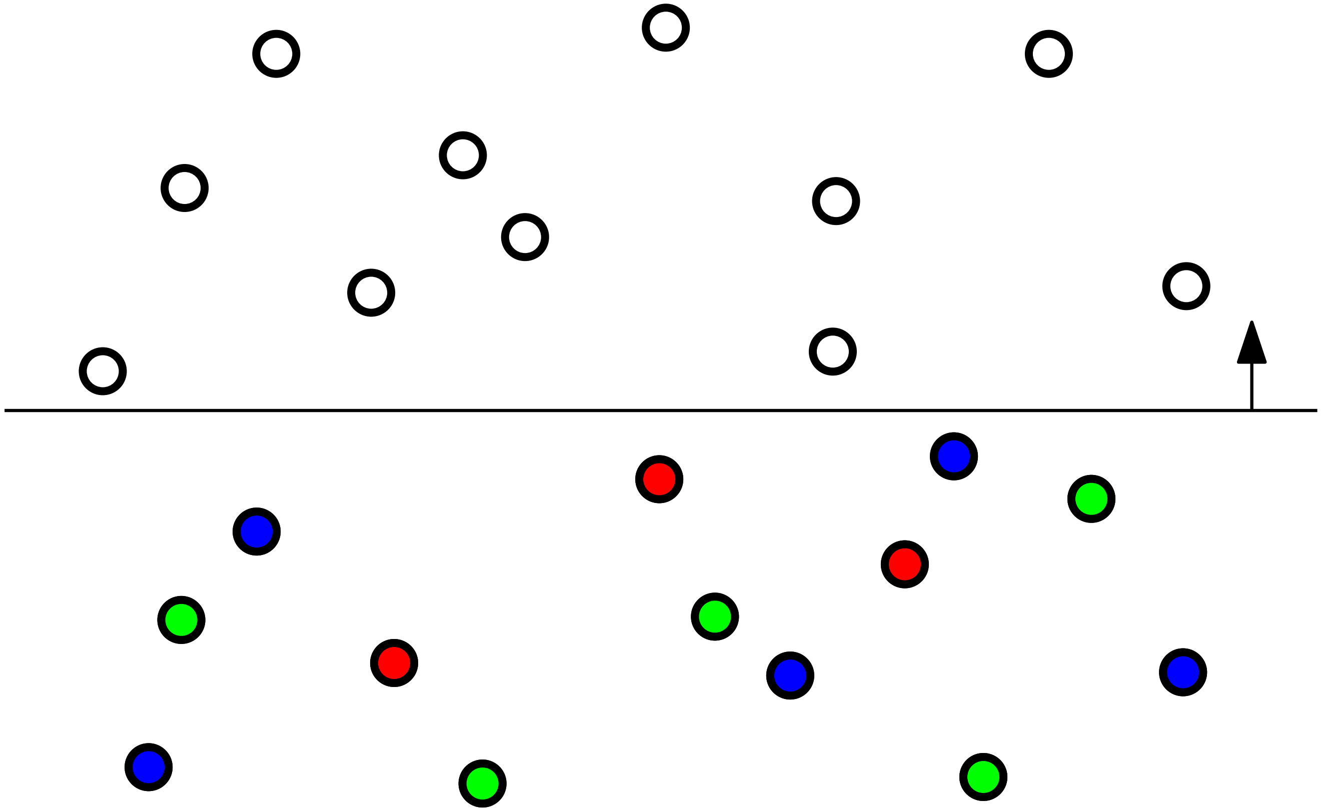
[Keszegh, Pálvölgyi 2011]

OPEN: $f(k) =$ finite or infinite?

related to cover-decomposability / dual cover-decomposability

Bottom-Up Sweep

IDEA: Color the points from bottom to top



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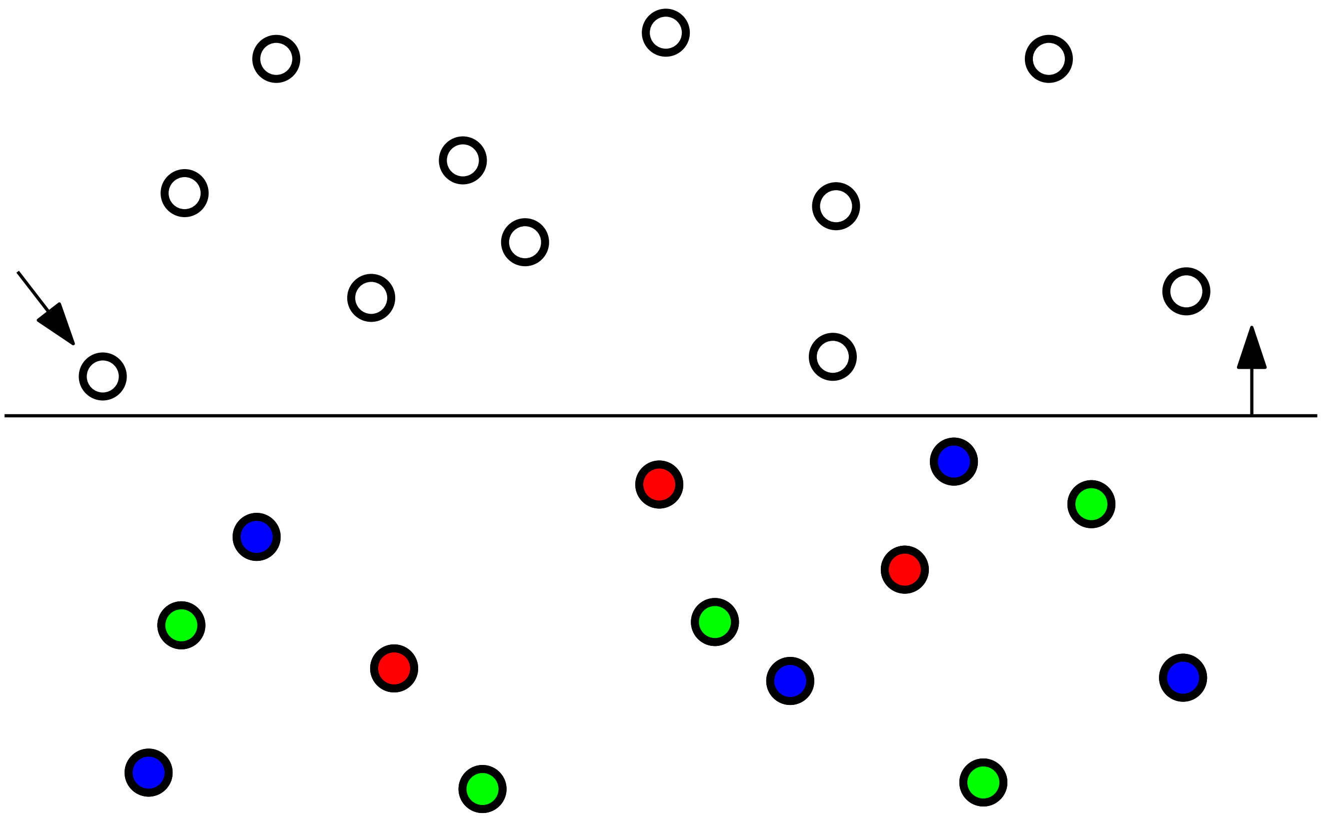

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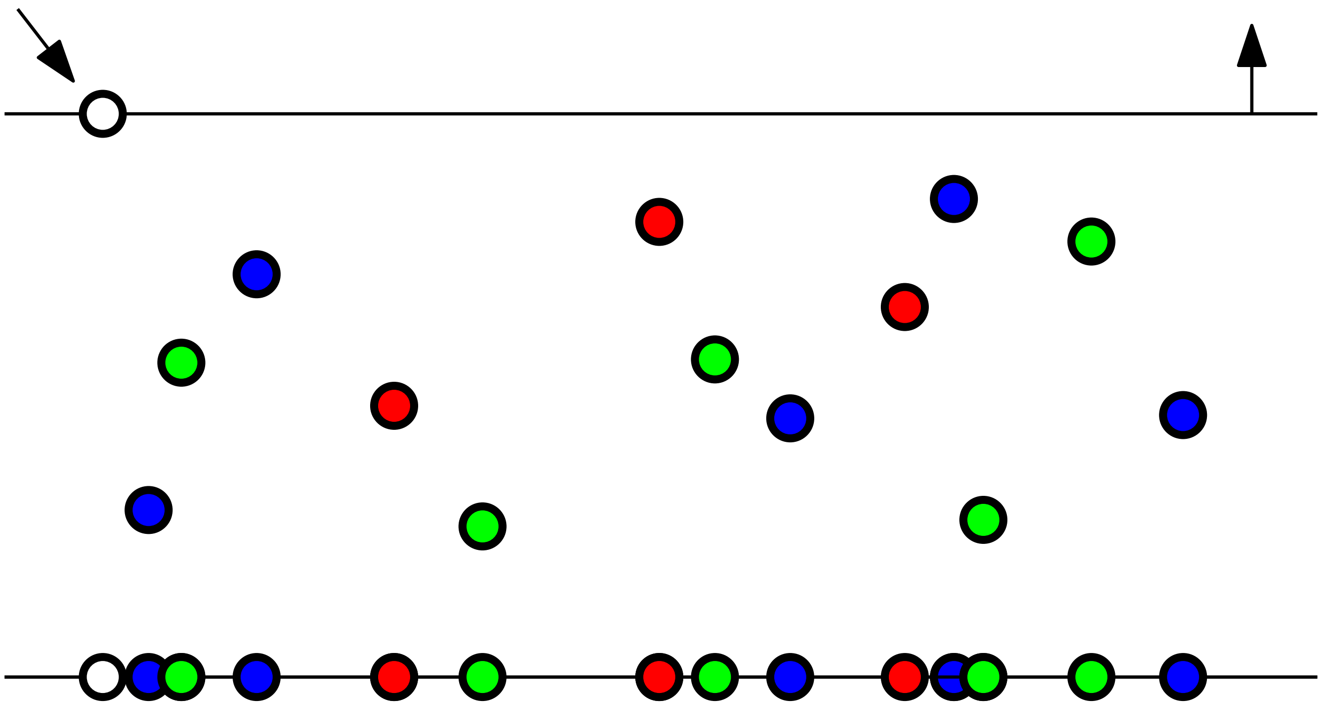

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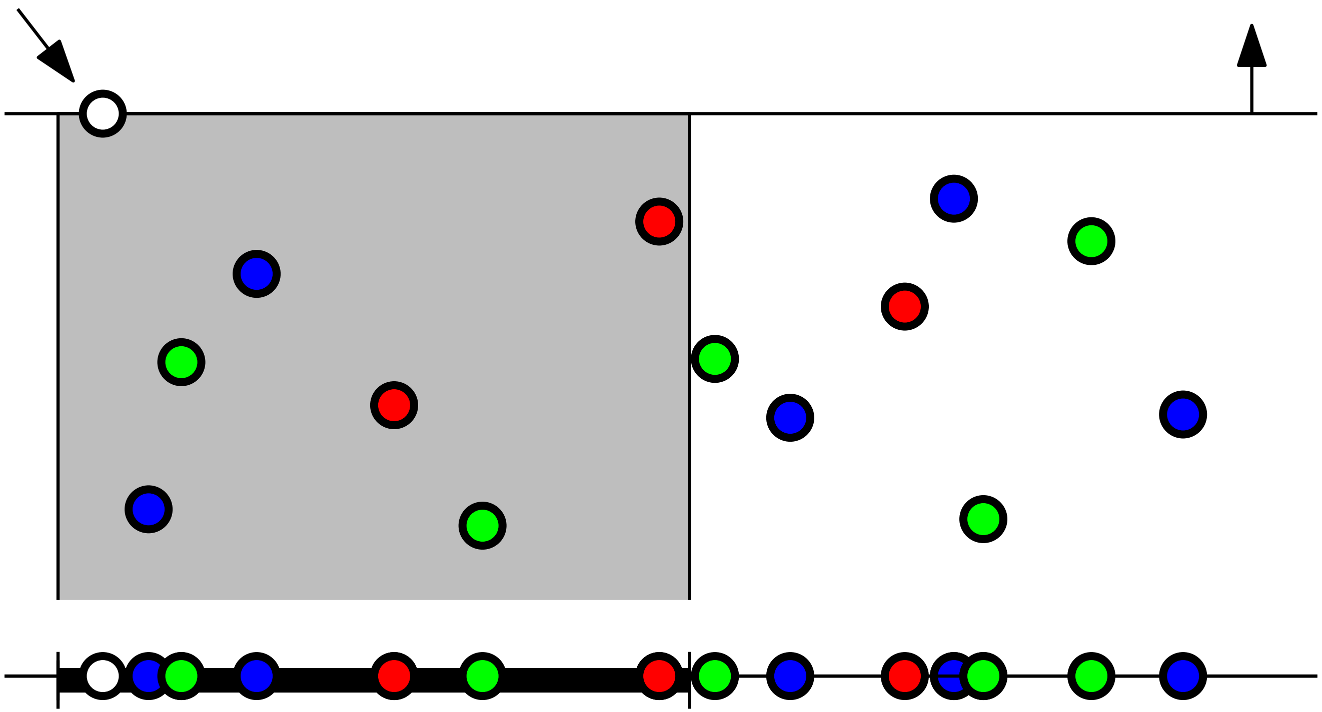

legal coloring:

Every *interval* of q consecutive points must contain all colors.

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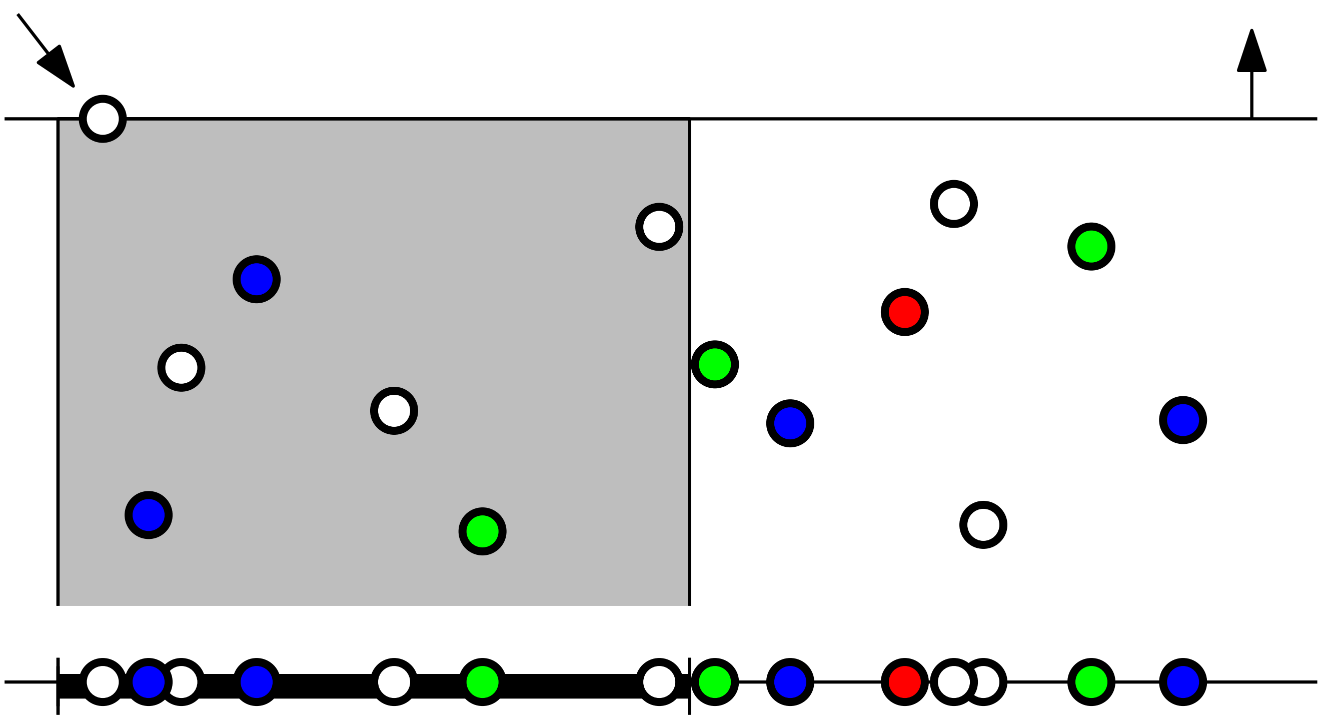
SEMI-ONLINE: Points need not be colored immediately.
Points can be colored *any time*, but then the color remains fixed.



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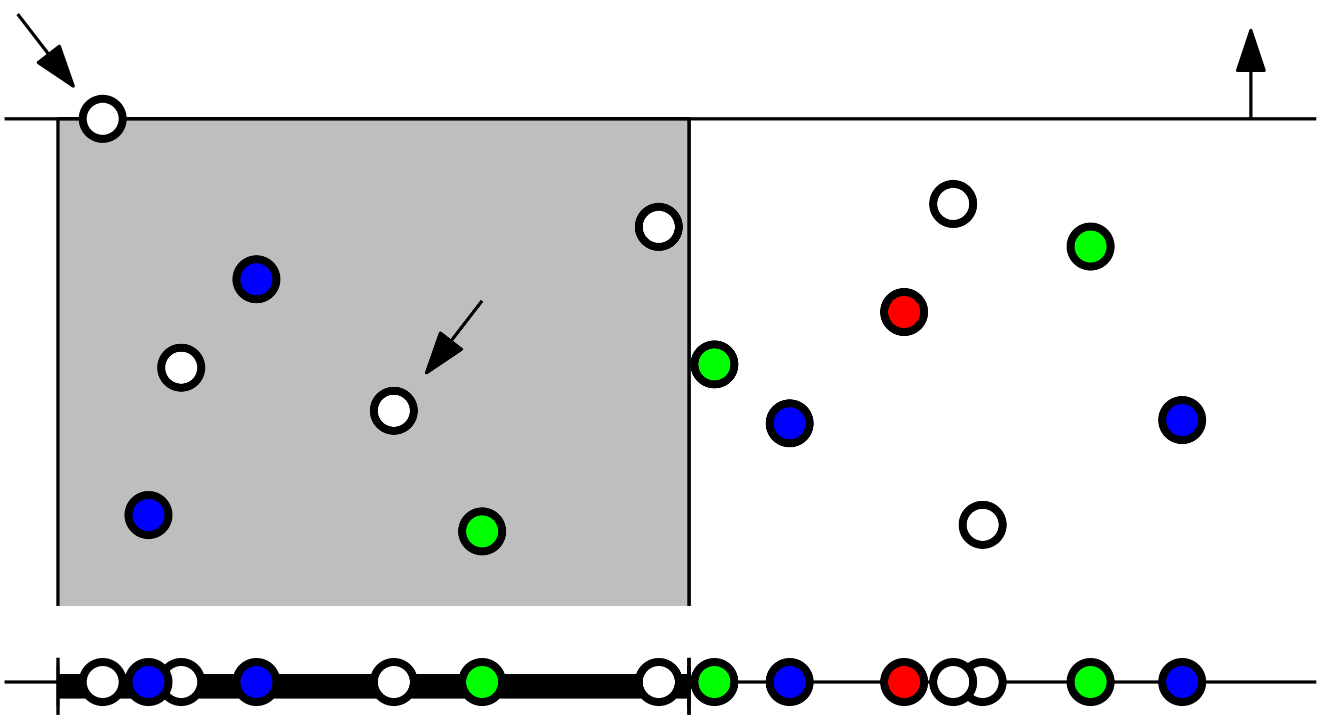

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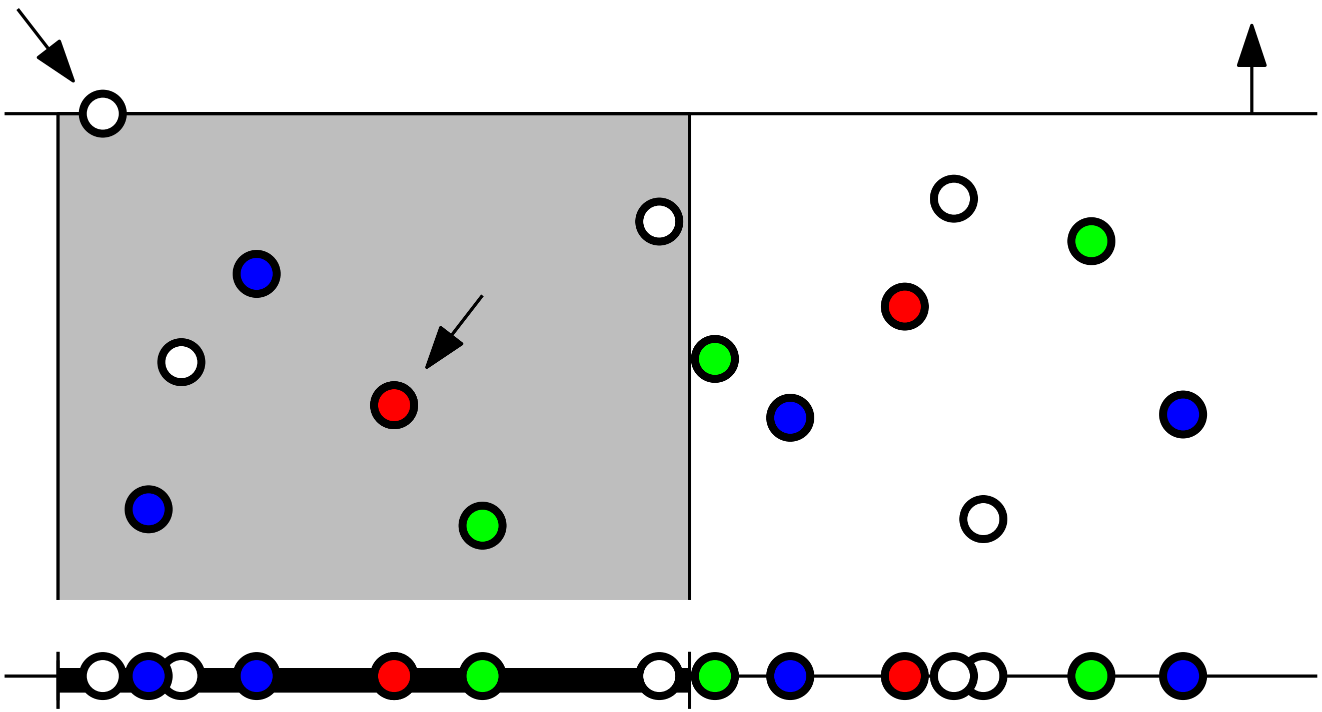

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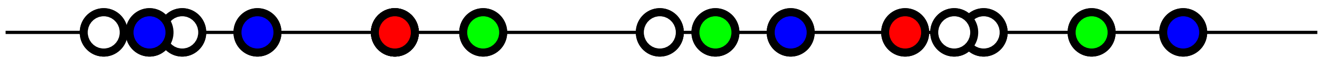
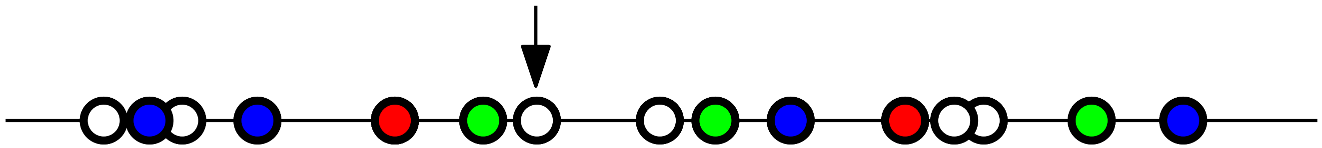
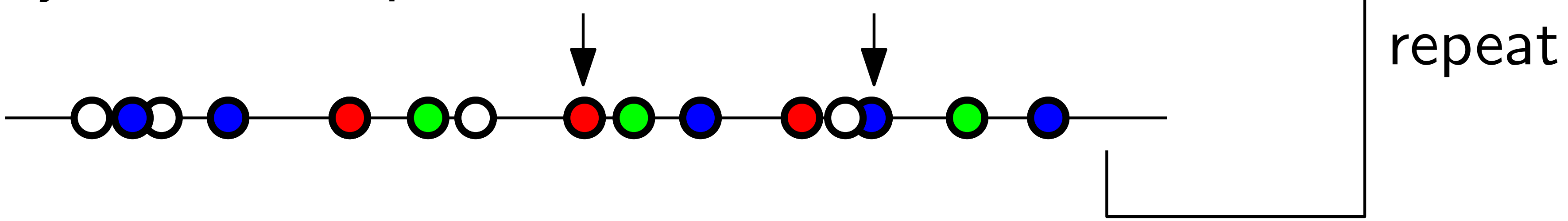
The Semi-Online Coloring Problem on the Line



A new uncolored point arrives:



Any uncolored points can be colored . . .



. . . to make the coloring *legal*:
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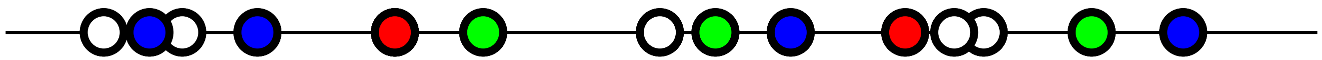
The Semi-Online Coloring Problem on the Line



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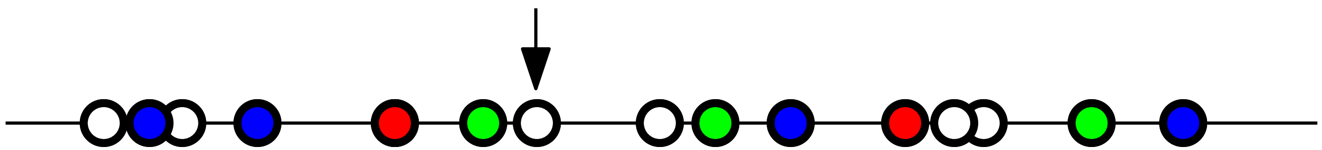

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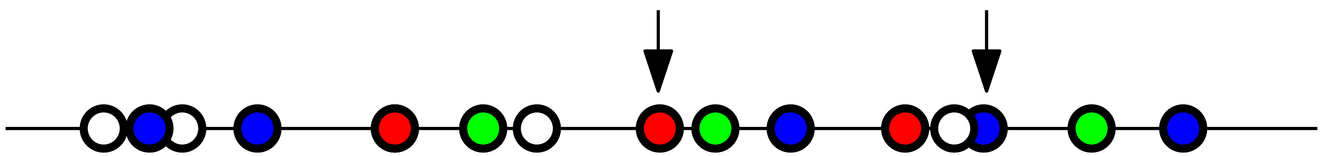


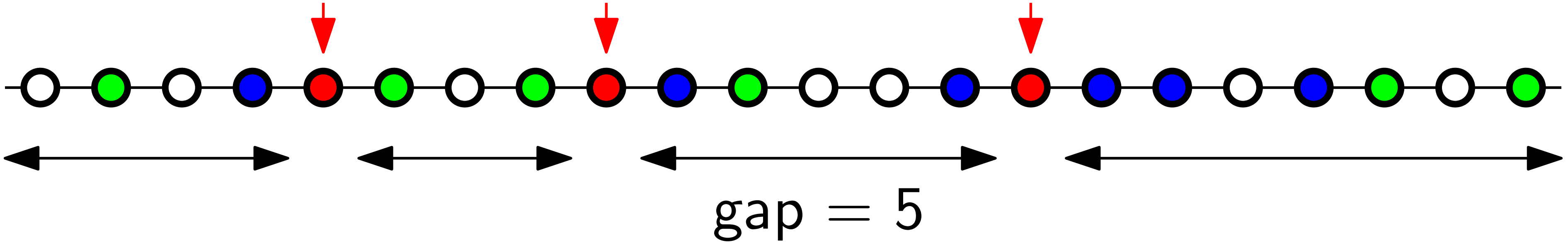
repeat

. . . to make the coloring *legal*:
Every interval of q consecutive points must contain all colors.

$f'(k) :=$ the smallest q for which there is a semi-online coloring algorithm

RESULTS: $f(k) \le f'(k) \le 3k - 2$
COMPUTER LOWER BOUNDS:
$f'(2) = 4$, $f'(3) = 7$, $9 \le f'(4) \le 10$

Upper Bound: $f'(k) \le 3k - 2$



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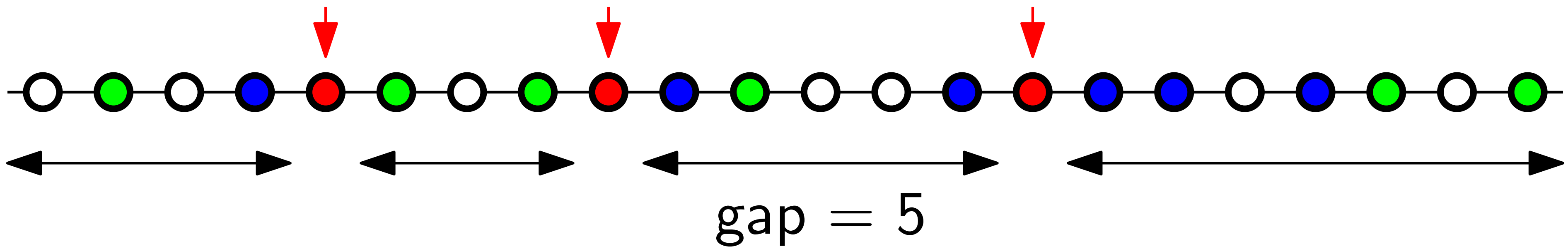
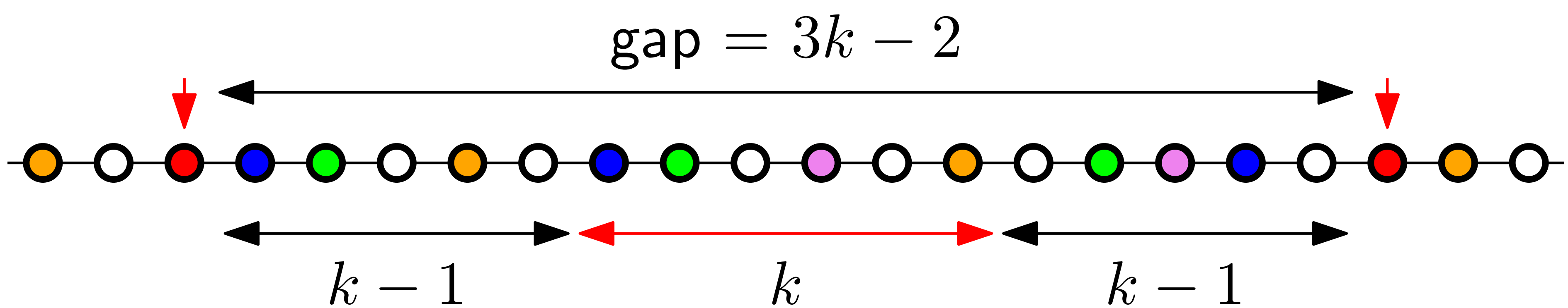
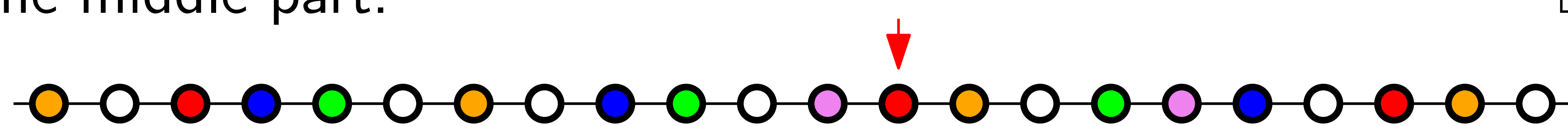


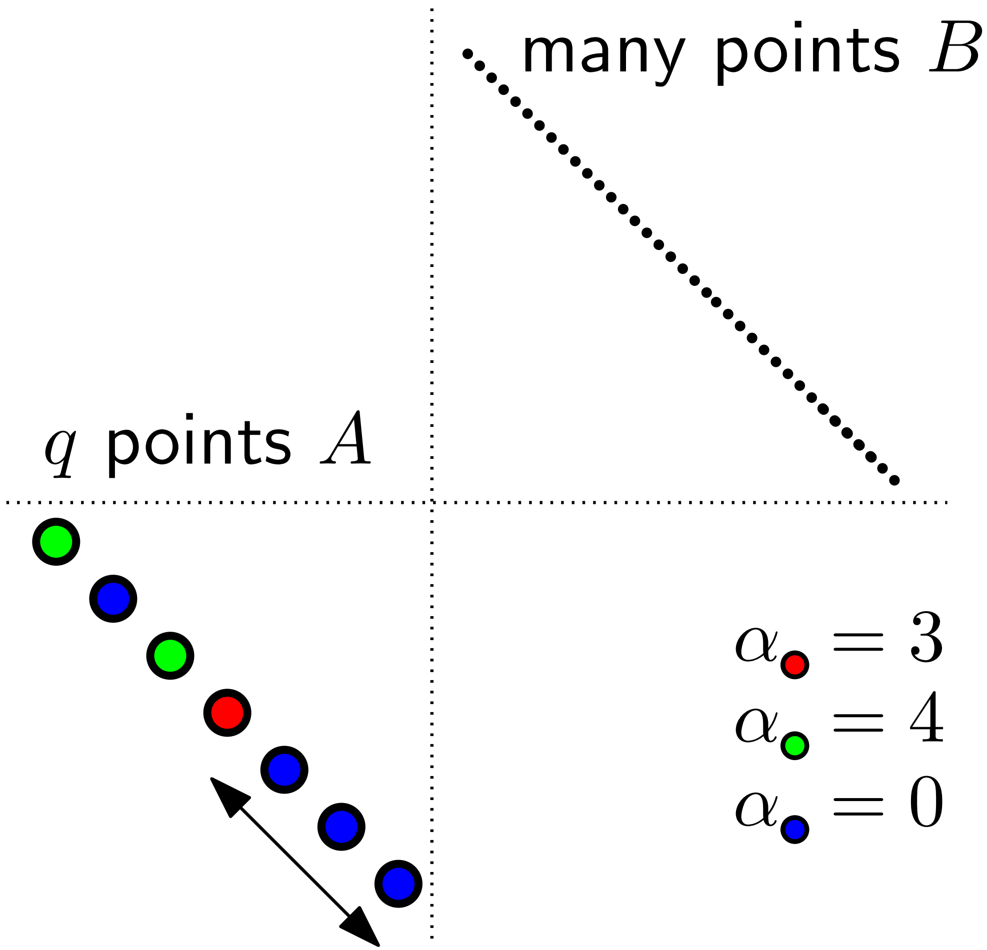
INVARIANT: $\boxed{k - 1 \leq \text{gap} \leq 3k - 3}$ for every color



Each of the remaining $k - 1$ colors can occur at most once in the middle part. □



(Weaker) Lower Bound: $f(k) \geq 1.58k$



α_j = gap before the first occurrence of color j

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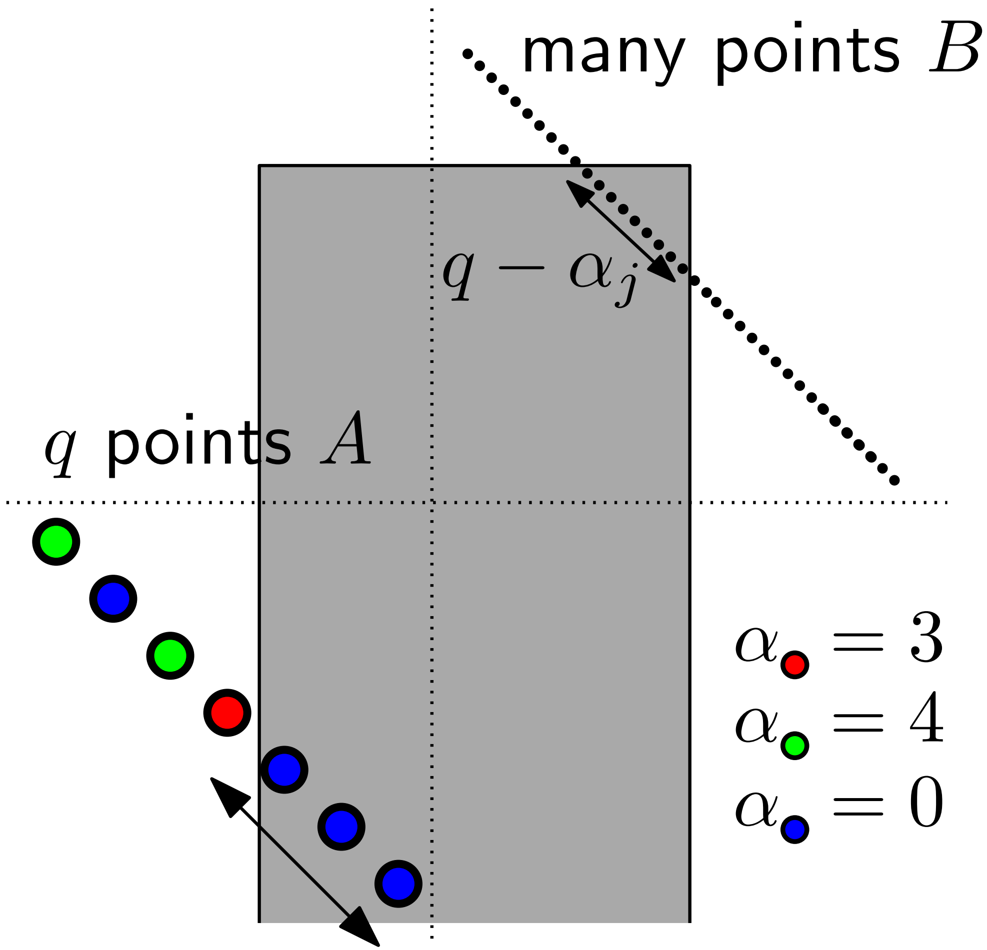

α_j = gap before the first occurrence of color j

Any $q - \alpha_j$ consecutive points of B must contain color j:
frequency of j is $\geq \frac{1}{q-\alpha_j}$

FREQUENCY condition

$$\sum_{j=1}^{k} \frac{1}{q - \alpha_j} \leq 1$$

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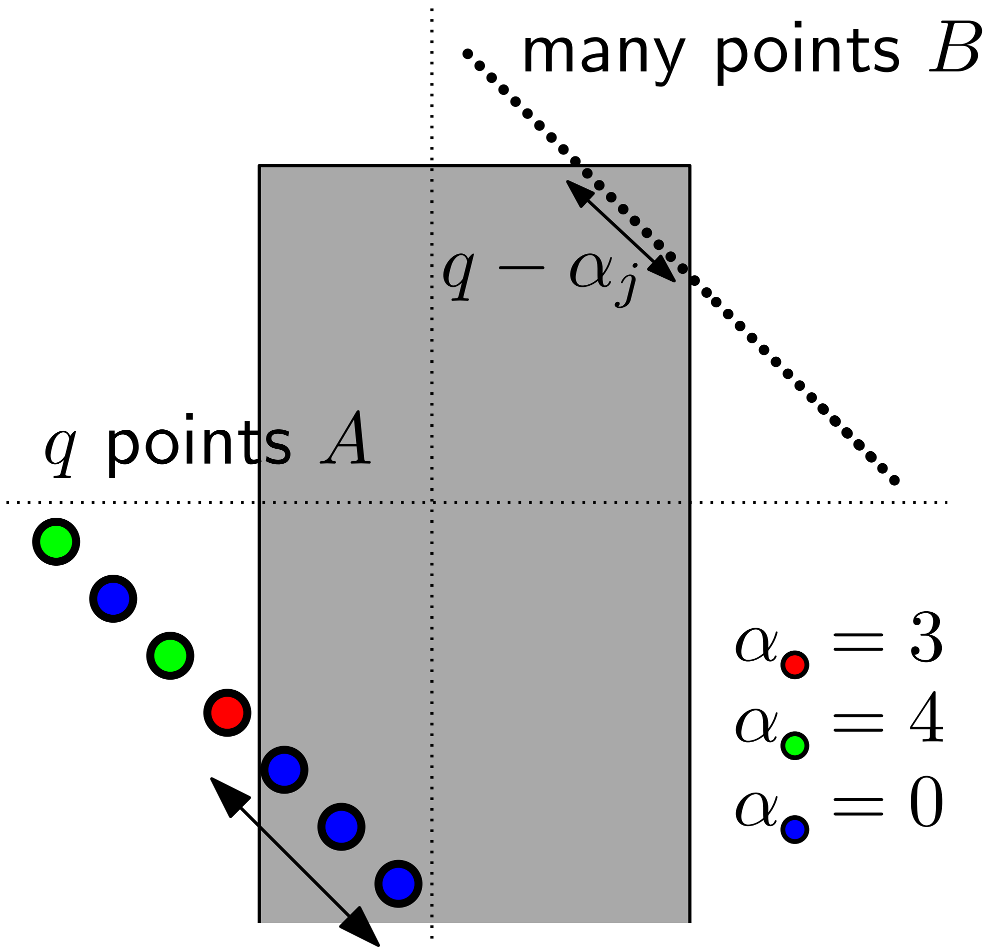

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$\alpha_1 \geq 0$, w.l.o.g.
$\alpha_2 \geq 1$,
$\alpha_3 \geq 2, \ldots$

$$\sum_{j=1}^{k} \frac{1}{q - j + 1} \leq 1$$

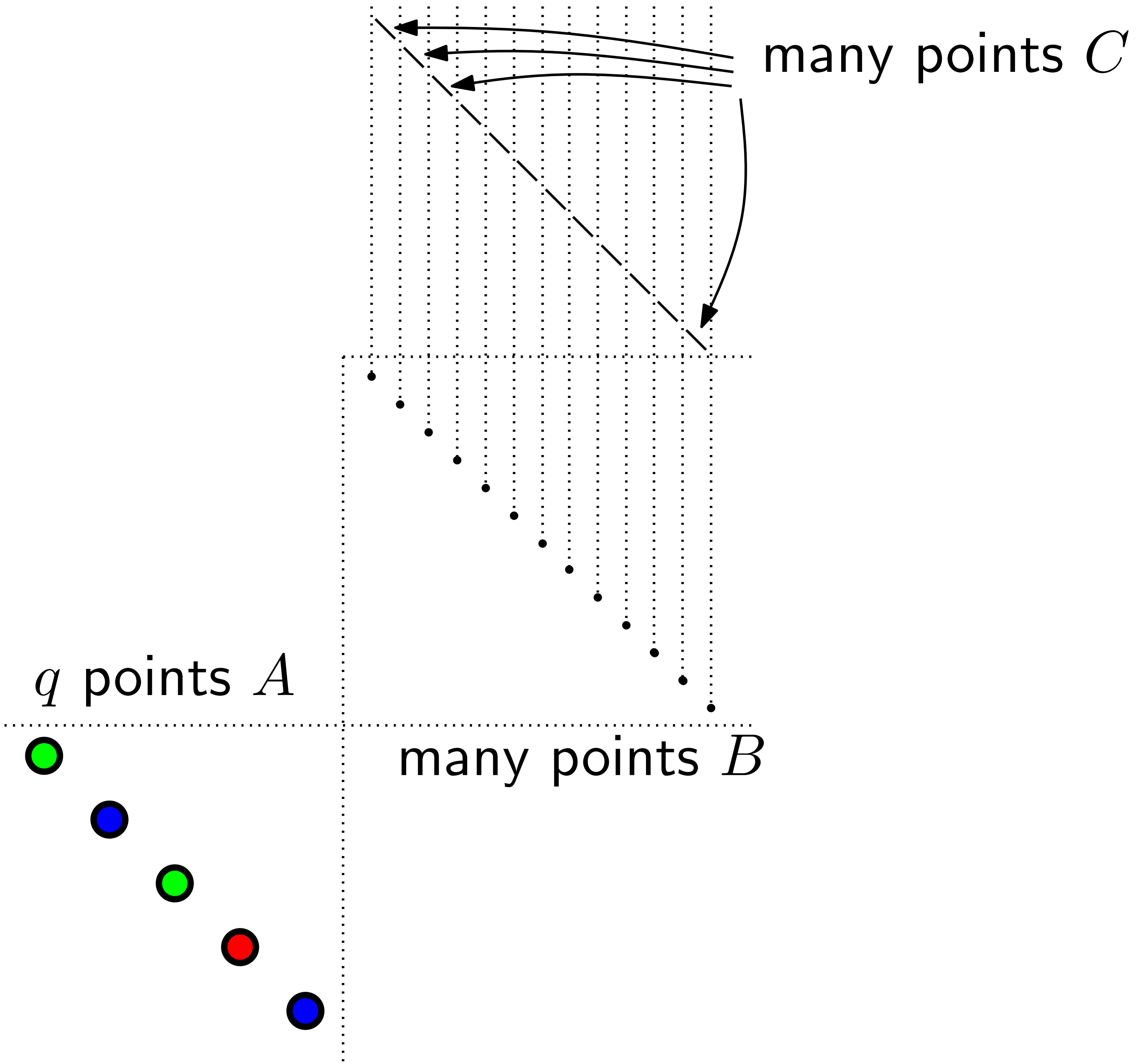
Lower Bound: $f(k) \geq 1.58k$

$$\sum_{j=1}^{k} \frac{1}{q-j+1} \leq 1 \text{ !}$$

$$\frac{1}{q} + \frac{1}{q-1} + \cdots + \frac{1}{q-k+1} \approx \ln q - \ln(q-k) = \ln \frac{q}{q-k} = 1$$

$$\Longrightarrow \; q = \frac{e}{e-1} k \approx 1.58k \qquad \square$$

Three Lines: $f(k) \geq 1.63k$



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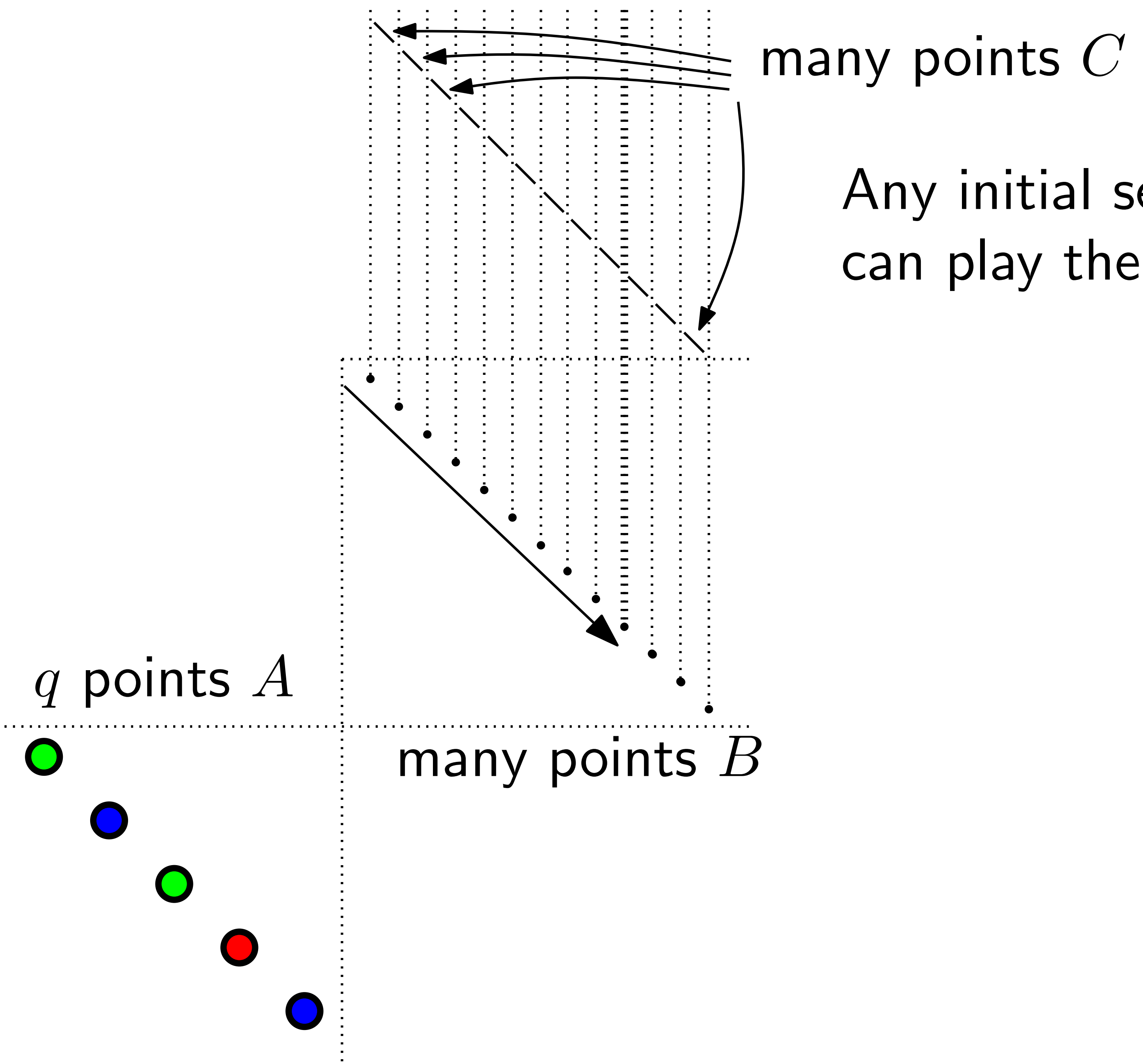
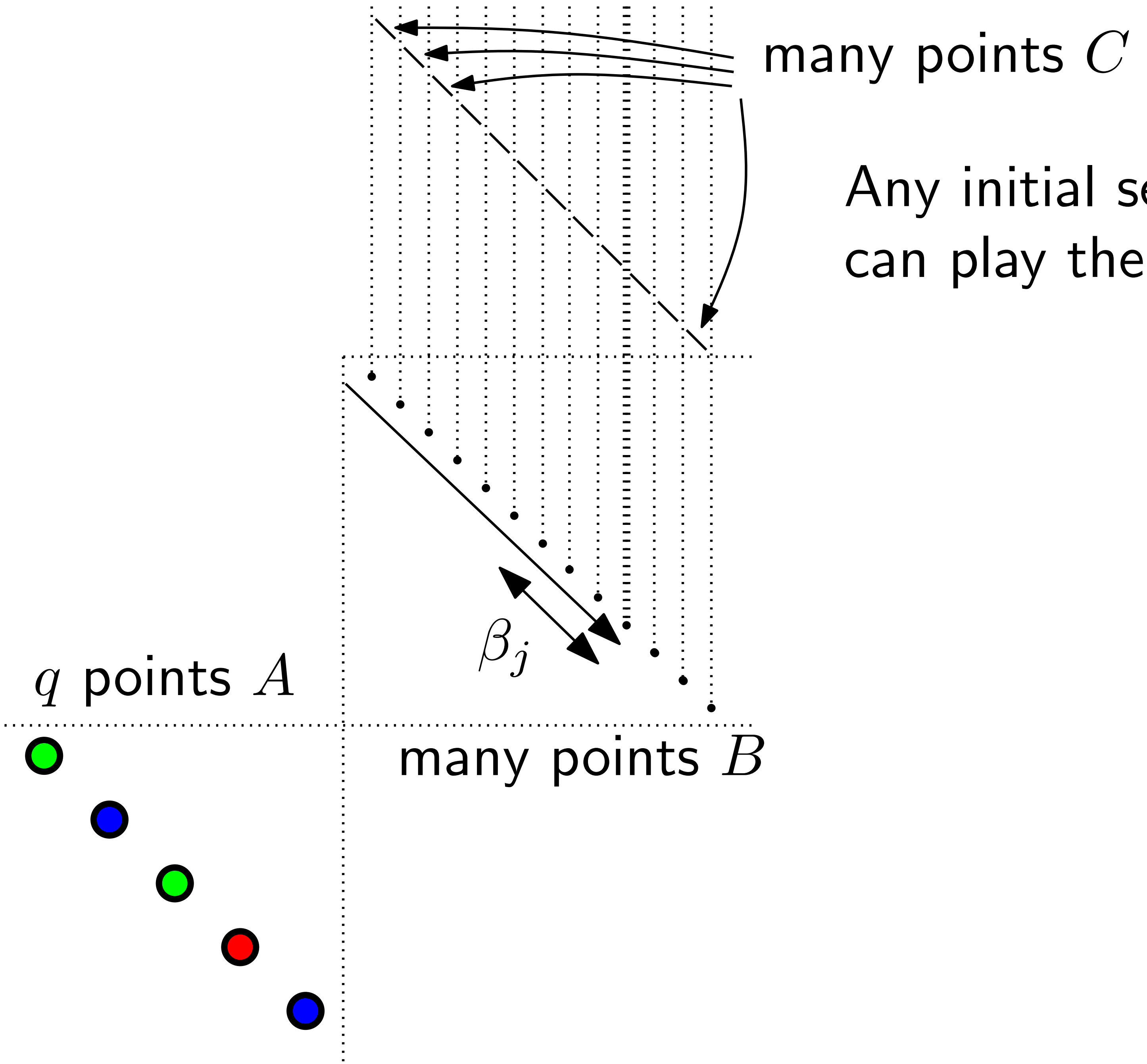


Any initial seqment of B can play the role of A:

Three Lines: $f(k) \geq 1.63k$



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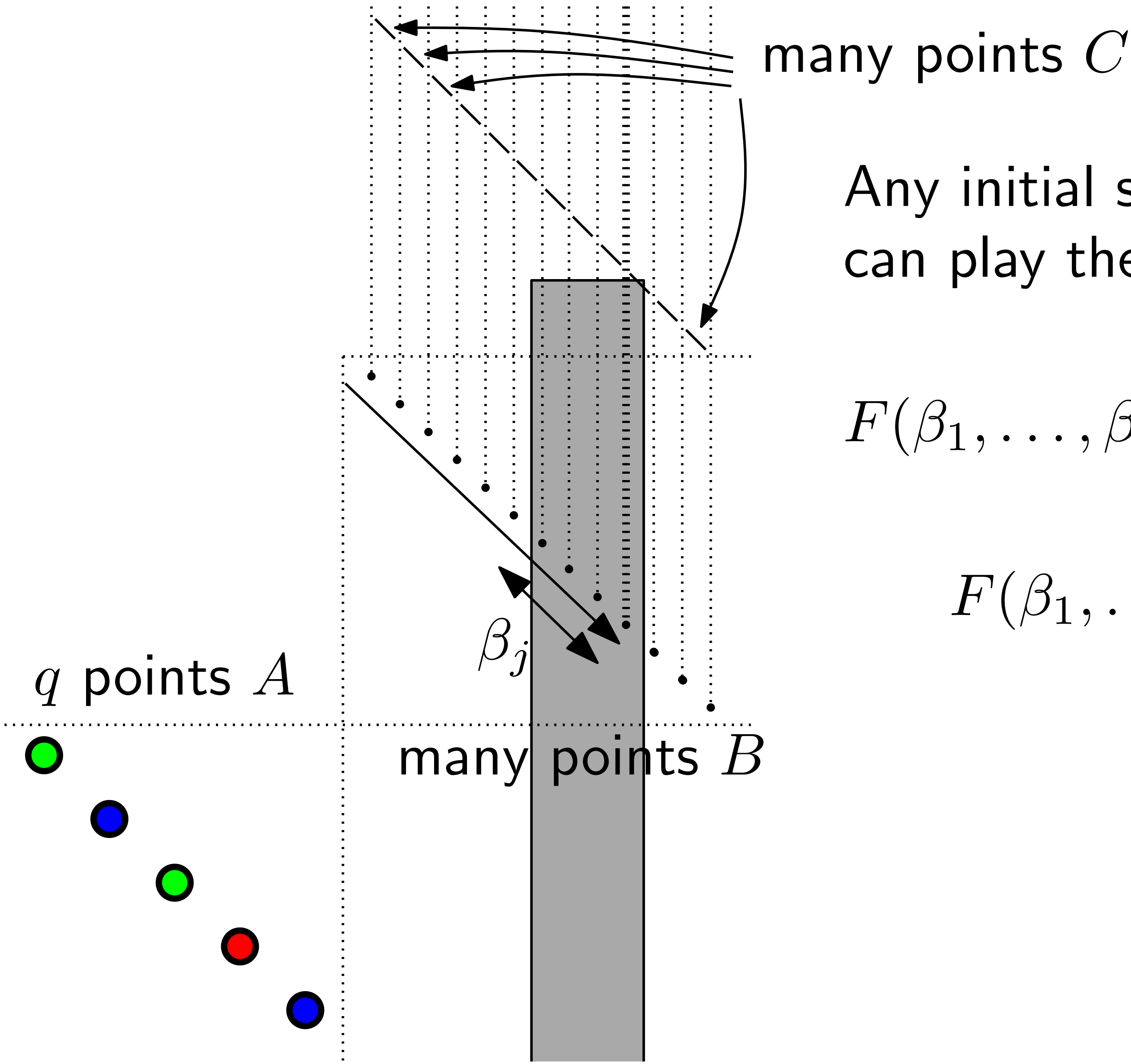

Any initial seqment of B can play the role of A:

$$F(\beta_1, \ldots, \beta_k) := \sum_{j=1}^{k} \frac{1}{q - \beta_j}$$

$$F(\beta_1, \ldots, \beta_k) \leq 1 \; !$$

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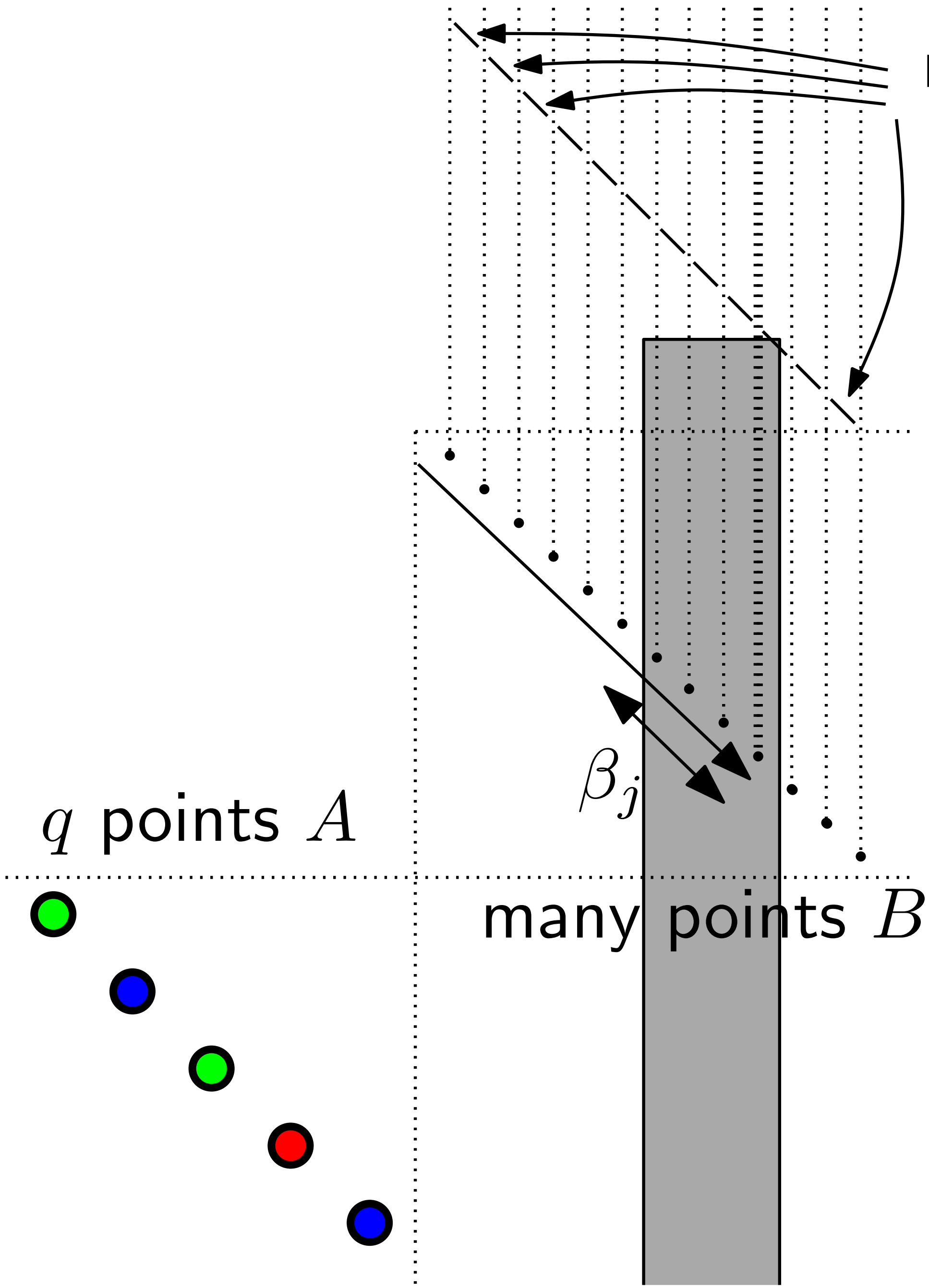

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$$F(\beta_1, \ldots, \beta_k) \leq 1 \ !$$

We know more about β_j than about α_j:

$$\beta_j \leq q - (j-1)$$

We can *pick* an initial segment of B.

Three Lines: $f(k) \geq 1.63k$

Any initial seqment of B can play the role of A:

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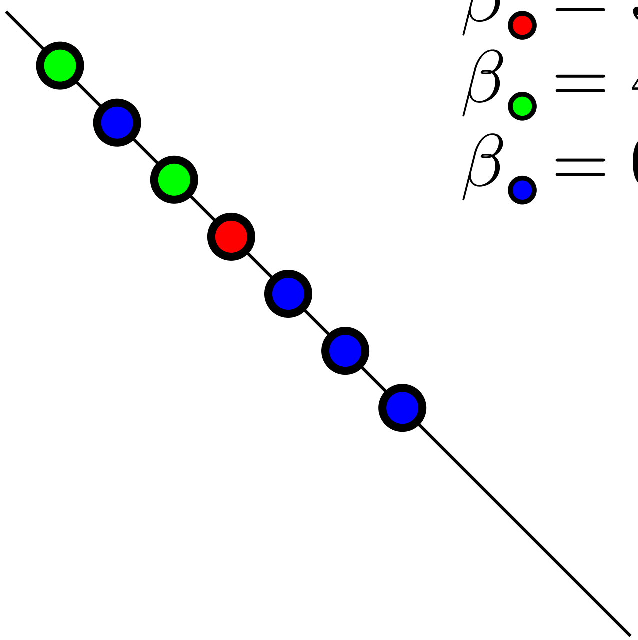
We know more about β_j than about α_j:

$$\beta_j \leq q - (j - 1)$$

We can *pick* an initial segment of B.

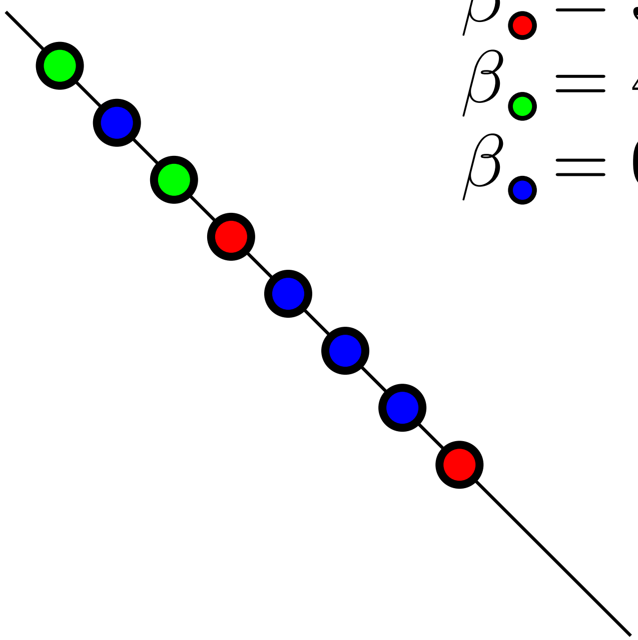
IDEA:
If $q < 1.63k$, then the *average* value of $F(\beta_1, \ldots, \beta_k)$ over all initial segments is > 1.

Evolution of β_j



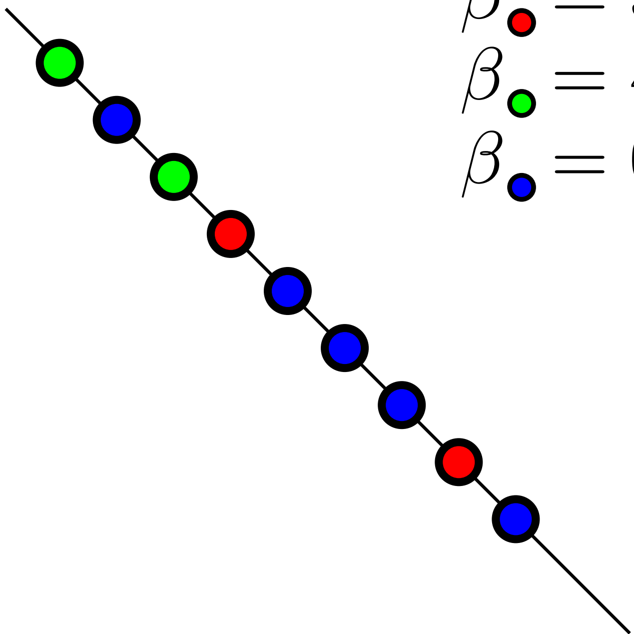
$$\beta_{\color{red}\bullet} = 3$$
$$\beta_{\color{green}\bullet} = 4$$
$$\beta_{\color{blue}\bullet} = 0$$

Evolution of β_j



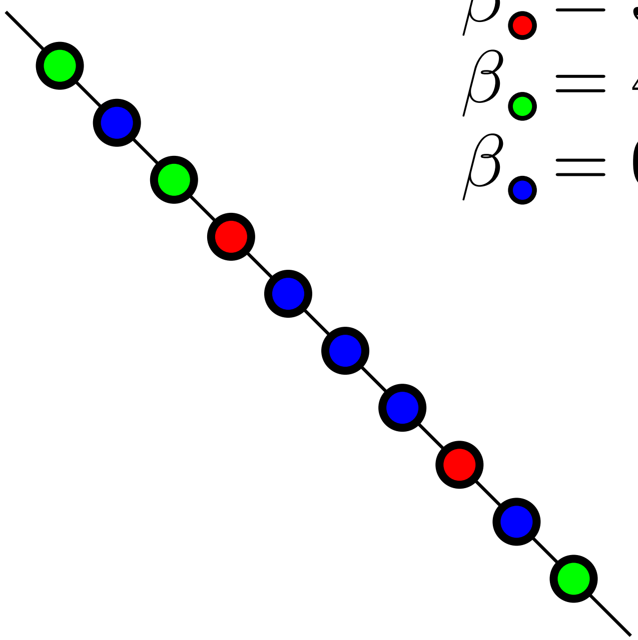
$$\beta_{\color{red}\bullet} = 3 \quad 0$$
$$\beta_{\color{green}\bullet} = 4 \quad 5$$
$$\beta_{\color{blue}\bullet} = 0 \quad 1$$

Evolution of β_j



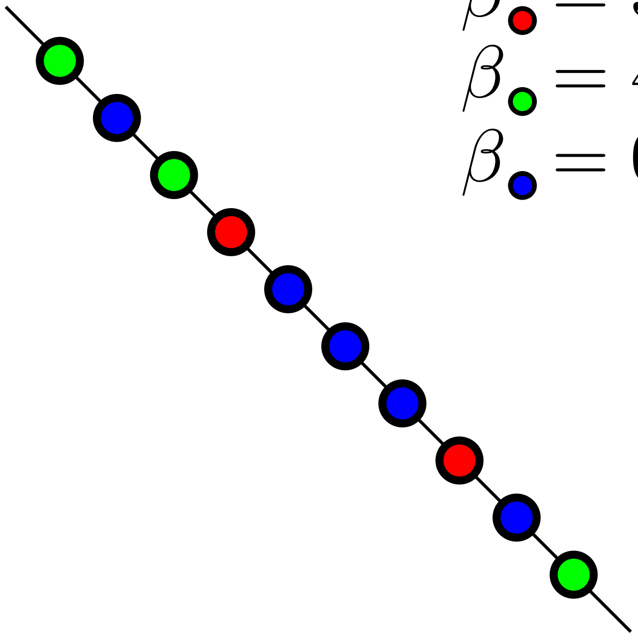
$$\beta_{\color{red}\bullet} = 3 \quad 0 \quad 1$$
$$\beta_{\color{green}\bullet} = 4 \quad 5 \quad 6$$
$$\beta_{\color{blue}\bullet} = 0 \quad 1 \quad 0$$

Evolution of β_j



$$\beta_{\color{red}\bullet} = 3 \quad 0 \quad 1 \quad 2$$
$$\beta_{\color{green}\bullet} = 4 \quad 5 \quad 6 \quad 0$$
$$\beta_{\color{blue}\bullet} = 0 \quad 1 \quad 0 \quad 1$$

Evolution of β_j



$$\beta_{\bullet} = 3\ 0\ 1\ 2\ 0\ 1\ 0\ 1\ 2\ 0\ 1\ 2\ 3\ 4\ 5\ 0\ 1 \ldots$$
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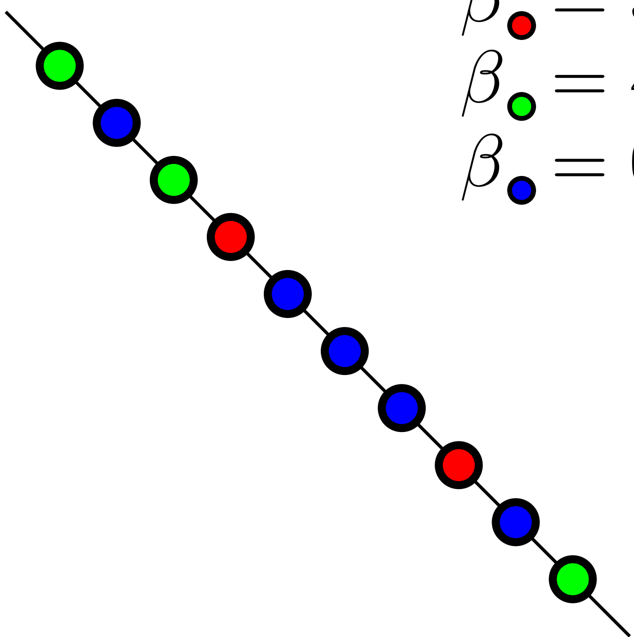

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$$\beta_{\color{blue}\bullet} = 0\ \ 1\ \ 0\ \ 1\ \ 2\ 3\ 4\ 0\ 1\ 2\ 0\ 1\ 0\ 0\ 1\ 2\ 3 \ldots$$

x_{jr} := the relative frequency of r as a value of β_j

$$x_{j0} + x_{j1} + x_{j2} + \cdots + x_{j,q-j} = 1$$
$$(\beta_j \le q - j)$$

$$x_{jr} \ge x_{j,r+1}$$

$$x_{1r} + x_{2r} + \cdots + x_{jr} \le 1$$ (all β_j values are distinct.)

The average value of $F(\beta_1, \ldots, \beta_k) = \sum_{j=1}^{k} \frac{1}{q-\beta_j}$ is

$$\sum_{j=1}^{k} \sum_{r \ge 0} x_{jr} \frac{1}{q-r} \to \text{MIN!}$$

If min>1, then q is too small.

A Linear Programming Problem

	$r{=}0$	$r{=}1$	$\cdots$	$q-k+1$	$\cdots$	$q-3$	$q-2$	$q-1$	
color 1:	x_{10}	x_{11}	$\cdots$	$x_{1,q-k+1}$	$\cdots$	$x_{1,q-3}$	$x_{1,q-2}$	$x_{1,q-1}$	=
color 2:	x_{20}	x_{21}	$\cdots$	$x_{2,q-k+1}$	$\cdots$	$x_{2,q-3}$	$x_{2,q-2}$		=
color 3:	x_{30}	x_{31}	$\cdots$	$x_{3,q-k+1}$	$\cdots$	$x_{3,q-3}$			=
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$					
color k:	x_{k0}	x_{k1}	$\cdots$	$x_{k,q-k+1}$					=
	≤ 1	≤ 1	$\cdots$	≤ 1	$\cdots$	≤ 1	≤ 1	≤ 1	

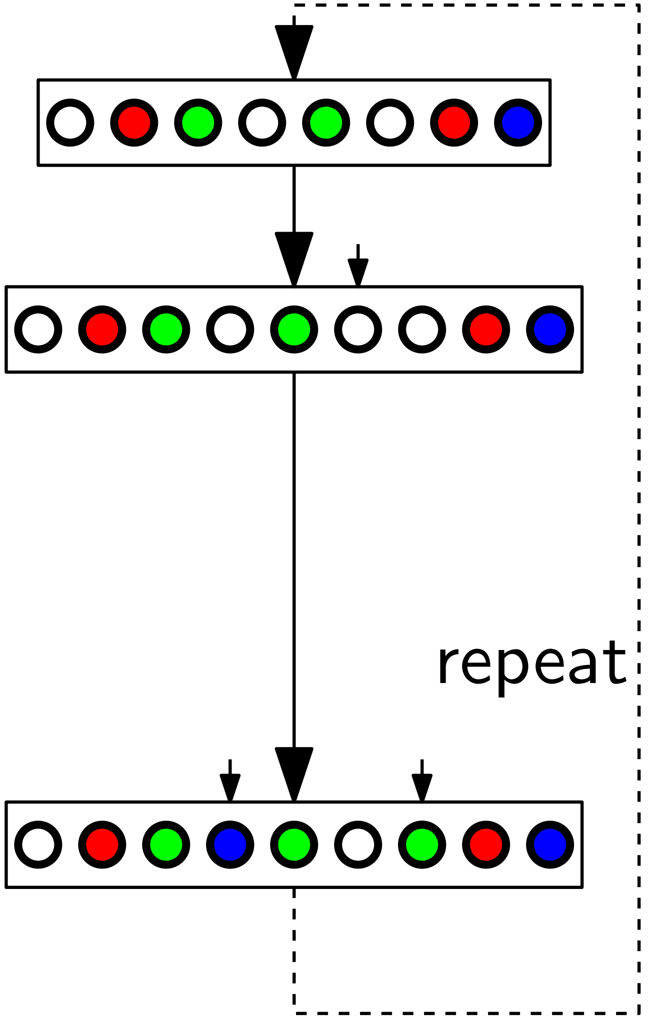
x_{jr} decreasing in rows

$$\sum_{j=1}^{k} \sum_{r=0}^{q-j} x_{jr} \frac{1}{q-r} \to \text{MIN!}$$

The solution can be worked out explicitly. □

Semi-Online Coloring as a Game

ADVERSARY inserts an uncolored point.



COLORER colors uncolored points, *must* make the coloring legal.

Semi-Online Coloring as a Game

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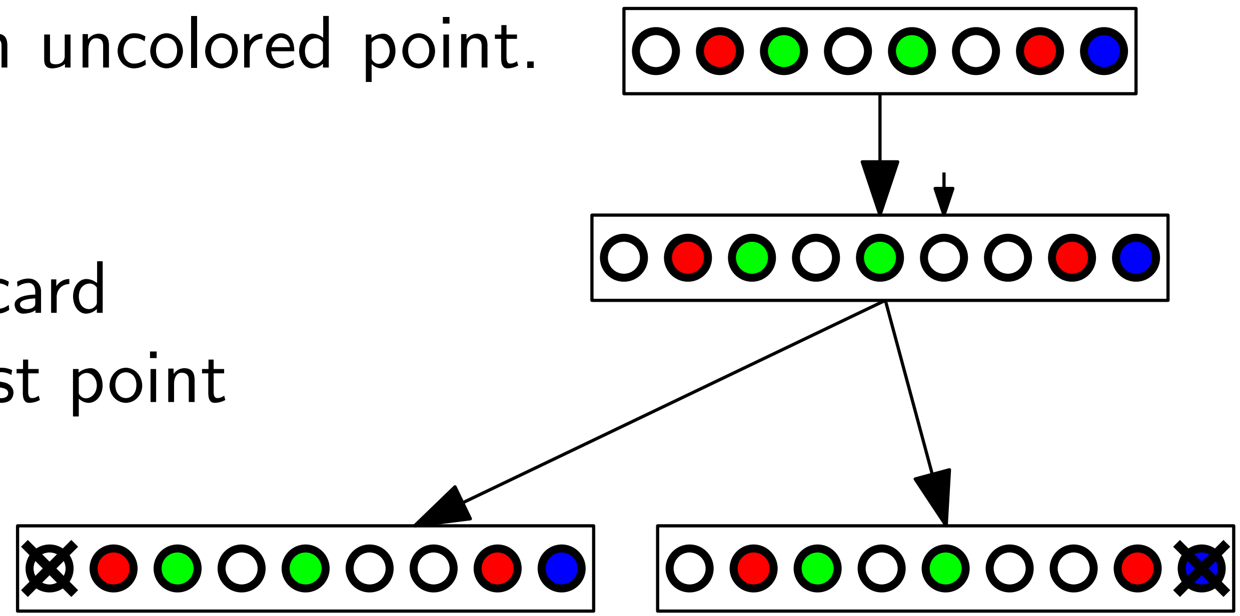
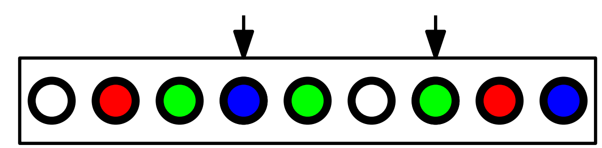


If more than s points,
ADVERSARY must discard
the leftmost or rightmost point

COLORER colors uncolored points,
must make the coloring legal.



This becomes a game on a *finite* bipartite graph.

ADVERSARY wins for $k = 2,\ q = 3,\ s = 5 \implies f'(2) \geq 4$

$k = 3,\ q = 6,\ s = 10 \implies f'(3) \geq 7$

$k = 4,\ q = 8,\ s = 11 \implies f'(4) \geq 9$

ADVERSARY loses for $k = 4,\ q = 9,\ s = 13.$ ($> 10^8$ edges)