Open Problems in Discrete Geometry
Collected by Günter Rote

PROBLEM 1 (Karim Adiprasito). Repeated halving of simplices
Start with a d-dimensional simplex. Subdivide the longest edge and cut the simplex into two simplices of half the area. Repeat the process recursively with both halves ad infinitum. (The result is in general not a face-to-face decomposition.)

We assume that the starting simplex is sufficiently generic, so that there is never a tie in choosing the longest edge.

In dimensions 2 and 3, this process stabilizes after finitely many iterations in the sense that every new simplex is homothetic to one of the simplices that have already been seen. In dimension 4, this is true provided that one starts with a suitable starting simplex: a generically perturbed orthoscheme.

Question. *Does this stabilizing behavior occur for any starting simplex, in any dimension?*

Or, on the contrary, can this process lead to arbitrarily badly shaped simplices, where the ratio between the inradius and the circumradius approaches 0*?*

The question has applications in scientific computing.

PROBLEM 2 (Gil Kalai). Complicated intersections
Consider a fixed real algebraic variety N in $\mathbb{R}^d$, and real algebraic varieties M of a certain type.

We are looking for general statement of the form: If all intersections between M and affine transformations of N are "complicated" then the dimension of the affine hull of M is "small".

The model statement for this is the Sylvester–Gallai Theorem: Here, M is a set of points and N is a line. If all nontrivial intersections have more then 2 points, then the affine hull of M has dimension 1.

PROBLEM 3 (Tomasz Szemberg). Absolute linear Harbourne constants
Harbourne constants have been introduced in connection with the Bounded Negativity Conjecture in algebraic geometry [1]. However, they can be considered purely combinatorially.

Let $\mathbb{K}$ be a field and $\mathcal{L}$ be a set of d lines in the projective plane $\mathbb{P}^2(\mathbb{K})$. Let $\{P_1, \ldots, P_s\}$ be the set of points where at least two lines of $\mathcal{L}$ intersect, and let m_i be the number of lines passing through P_i.

The rational number

$$H(\mathbb{K}, \mathcal{L}) = \frac{d^2 - \sum_{i=1}^{s} m_i^2}{s}$$

is the Herbourne constant of the arrangement $\mathcal{L}$.

Taking the minimum over all arrangements of d lines in $\mathbb{P}^2(\mathbb{K})$, we obtain the linear Harbourne constant of d lines over $\mathbb{K}$

$$H(\mathbb{K}, d) = \min_{|\mathcal{L}|=d} H(\mathbb{K}, \mathcal{L}).$$

Finally, taking the minimum over all fields $\mathbb{K}$ we arrive at the absolute linear Harbourne constant of d lines

$$H(d) = \min_{\mathbb{K}} H(\mathbb{K}, d).$$

Question. *Compute the numbers $H(d)$.*

This has been done for $1 \leq d \leq 31$ and for d of the form $d = q^2 + q + 1$ [2]. The article contains also a conjectural formula for these numbers.

[1] T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora, and T. Szemberg, *Bounded negativity and arrangements of lines*, Int. Math. Res. Not. IMRN **19** (2015), 9456–9471.

[2] M. Dumnicki, D. Harrer, and J. Szpond, *On absolute linear Harbourne constants*, Finite Fields Appl. **51** (2018), 371–387.

PROBLEM 4 (Tomasz Szemberg). Projective plane of order 10

A computer proof of the non-existence of a projective plane of order 10 was announced in 1989 [1]. Has this proof ever been independently verified?

Note. (Konrad Swanepoel) It seems that the non-existence of a projective plane of order 10 was independently checked as described in a M.Sc. thesis from 2010 [3]. There are other verifications of parts of the search [2].

Curtis Bright and coworkers are busy using SAT solvers to produce more rigorous computer-based proofs, see `https://cs.uwaterloo.ca/~cbright/#writings`. There is no complete formal proof yet, but it looks as if this is not very far away. The most enlightening of his papers is [4].

[1] C. W. H. Lam, L. Thiel and S. Swiercz, *The nonexistence of finite projective planes of order 10*. Canad. J. Math. **41** (1989), 1117–1123.

[2] X. Perrott, *Existence of projective planes*, arXiv:1603.05333, (2016).

[3] D. J. Roy, *Confirmation of the non-existence of a projective plane of order 10*, Masters dissertation, Carleton University, 2010.

[4] C. Bright, *A SAT-based Resolution of Lam's Problem*, `https://cs.uwaterloo.ca/~cbright/reports/lams-preprint.pdf`, September 2020.

PROBLEM 5 (Luis Montejano). Sections that are bodies of revolution

If all hyperplane sections of a convex body of dimension at least 4 are either single points or bodies of revolution, prove that the body is itself a body of revolution.

PROBLEM 6 (Gil Kalai). 4-polytopes with dense graphs

Suppose a simplicial 4-polytope P with n vertices has the property that, among every three vertices, at least two of them are joined by an edge. Does it follow that the graph of P contains a "large" complete subgraph, say, of size $n/10$?

PROBLEM 7 (Arseniy Akopyan).
Unbalanced ham-sandwich cuts for spherical caps

We are given two continuous probability measures on the 2-dimensional sphere and a parameter $0 < \alpha < 1/2$. Is there always a spherical cap (intersection with a half-space) that has measure α for both measures?

PROBLEM 8 (Raphael Steiner).
Bichromatic triangles in arrangements of pseudolines

A pseudoline is a non-self-intersecting infinite curve in $\mathbb{R}^2$ dividing the plane into two connected components. A *simple* arrangement of pseudolines is a set of pseudolines such that any two distinct pseudolines intersect in a point, and no point is contained in three or more pseudolines.

Question. *In a simple arrangement of red and blue pseudolines, is there always a bounded triangular face that is incident to a red and a blue pseudoline?*

It is easy to see that this is true for planar *line* arrangements. It holds also for the more general class of *approaching* pseudoline arrangements [1].

[1] Stefan Felsner, Alexander Pilz, and Patrick Schnider, *Arrangements of approaching pseudolines.* arXiv:2001.08419, (2020).

PROBLEM 9 (Balázs Keszegh). Hereditary polychromatic k-colorings

For a hypergraph $\mathcal{H}$ denote by m_k the smallest number for which we can k-color the vertices such that on every hyperedge of size at least m_k, all k colors appear. Denote by m_k^* the maximum of m_k over every induced subhypergraph of $\mathcal{H}$.

Berge showed that if for a hypergraph $m_2^* = 2$ then $m_k^* = k$ for all k. What about larger m_2^*? Does $m_2^* = 3$ imply that m_3^* is finite?

PROBLEM 10 (Emo Welzl). Minimum number of partial triangulations

A partial triangulation of a set of n points in the plane is a triangulation of the convex hull that may use the interior points as vertices, but does not have to use all of them.

Question. *What is the smallest number of partial triangulations that a set of n points in general position can have? Is it the Catalan number $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$?*

For full triangulations, where all interior points have to be used, smallest known number of full triangulations, roughly $\sqrt{12}^n$, is obtained by the so-called *double circle*, which is constructed by putting an interior point near the midpoint of every edge of a regular $\frac{n}{2}$-gon. By contrast, n points in convex position have $C_{n-2} \sim 4^n$ full (or equivalently, partial) triangulations. Interestingly, the double circle has exactly the same number C_{n-2} of partial triangulations.

PROBLEM 11 (Karim Adiprasito).
The compact part of a polyhedral subdivision

Take a polyhedral subdivision of $\mathbb{R}^3$ into finitely many parts, none of which contains a line, and look at the union of all bounded faces. This set is contractible. Is it collapsible?

PROBLEM 12 (Stefan Langerman).
The centerpoint constant for complete intersections

For every set of n lines in the plane, there is a point p such that for every halfspace H containing p there is a subset of at least $\sqrt{n/3}$ of lines all of whose intersections lie in H. There are examples that show that the bound cannot be improved to more than $\sqrt{n}$. What is the right constant?

PROBLEM 13 (Gil Kalai). Sets consisting of two convex pieces

Suppose there is a family of sets in d dimensions, each of which is the disjoint union of two nonempty closed convex sets. Moreover, the intersection of any $2, 3, \dots$ or $d+1$ sets from the family has the same property of consisting of exactly two convex pieces. Does it follow that the whole family has a nonempty intersection?

Micha Perles constructed an example that shows that the statement is not true in the plane if the number "two" of convex pieces is replaced by 48.

PROBLEM 14 (Michael Dobbins). Extending piecewise-linear maps from the boundary to the interior in a continuous manner

Take a fixed reference triangle ABC, and consider a piecewise-linear (PL) one-to-one map from the boundary of the triangle ABC into the plane (in other words, a PL parameterization of a simple polygon).

Such a map can be easily extended to a PL one-to-one map from whole triangular area ABC into the plane.

Can this be done in a way that depends continuously on the boundary map? In other words, is there a continuous function that assigns to every PL one-to-one map $\partial ABC \to \mathbb{R}^2$ a PL one-to-one map $ABC \to \mathbb{R}^2$ extending it?