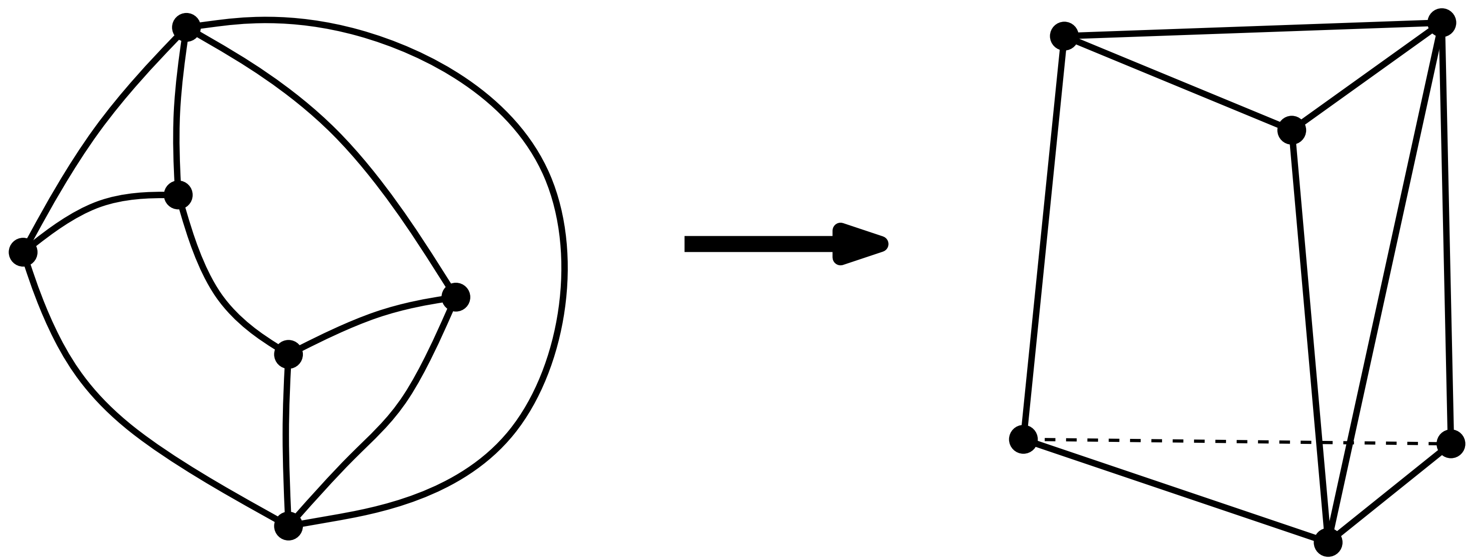
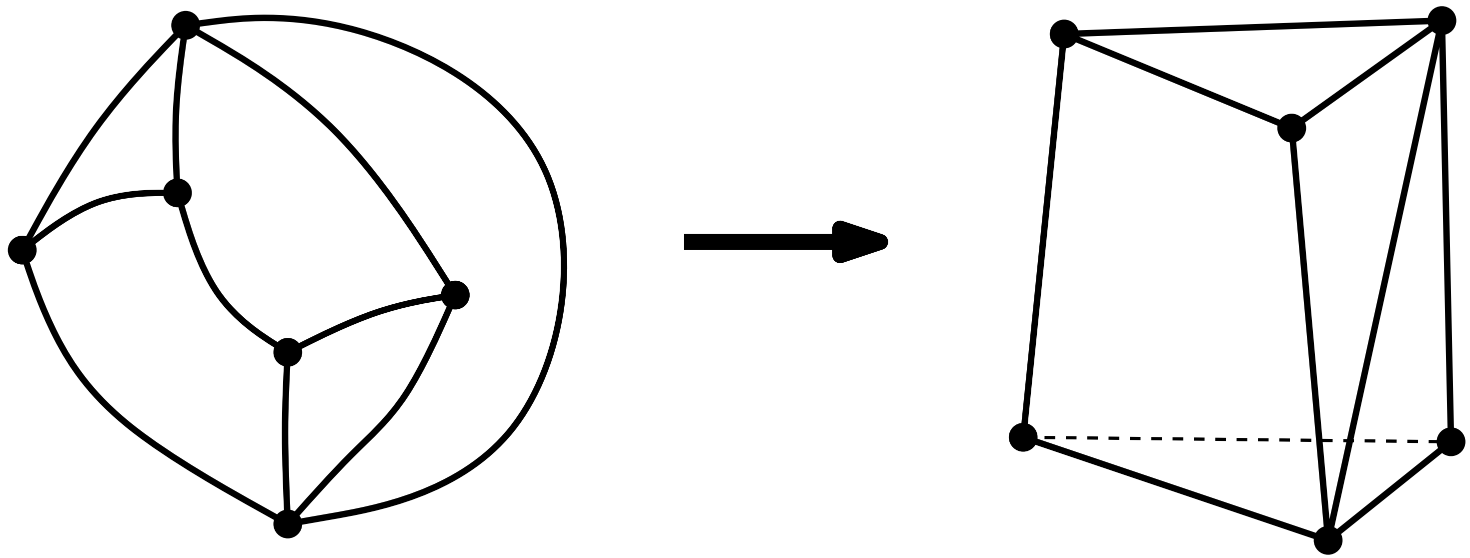
Realizing Planar Graphs as Convex Polytopes

Günter Rote
Freie Universität Berlin



General Problem Statement



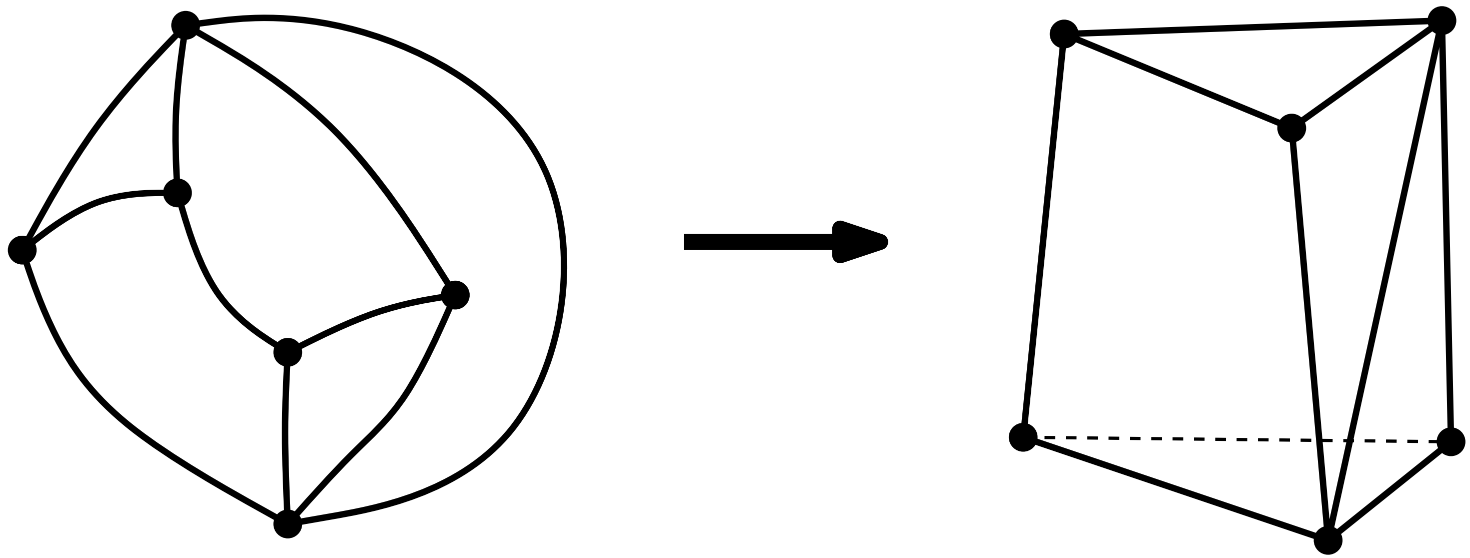
GIVEN:
a combinatorial type of 3-dimensional polytope (a 3-connected planar graph)

[+ additional data]

CONSTRUCT:
a geometric realization of the polytope

[with additional properties]

General Problem Statement



GIVEN:
a combinatorial type of
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[+ additional data]

CONSTRUCT:
a geometric realization of
the polytope

[with additional properties]

e.g.: *small integer vertex coordinates*

Polytopes with Small Vertex Coordinates

Every polytope with n vertices can be realized with integer coordinates less than 148^n.

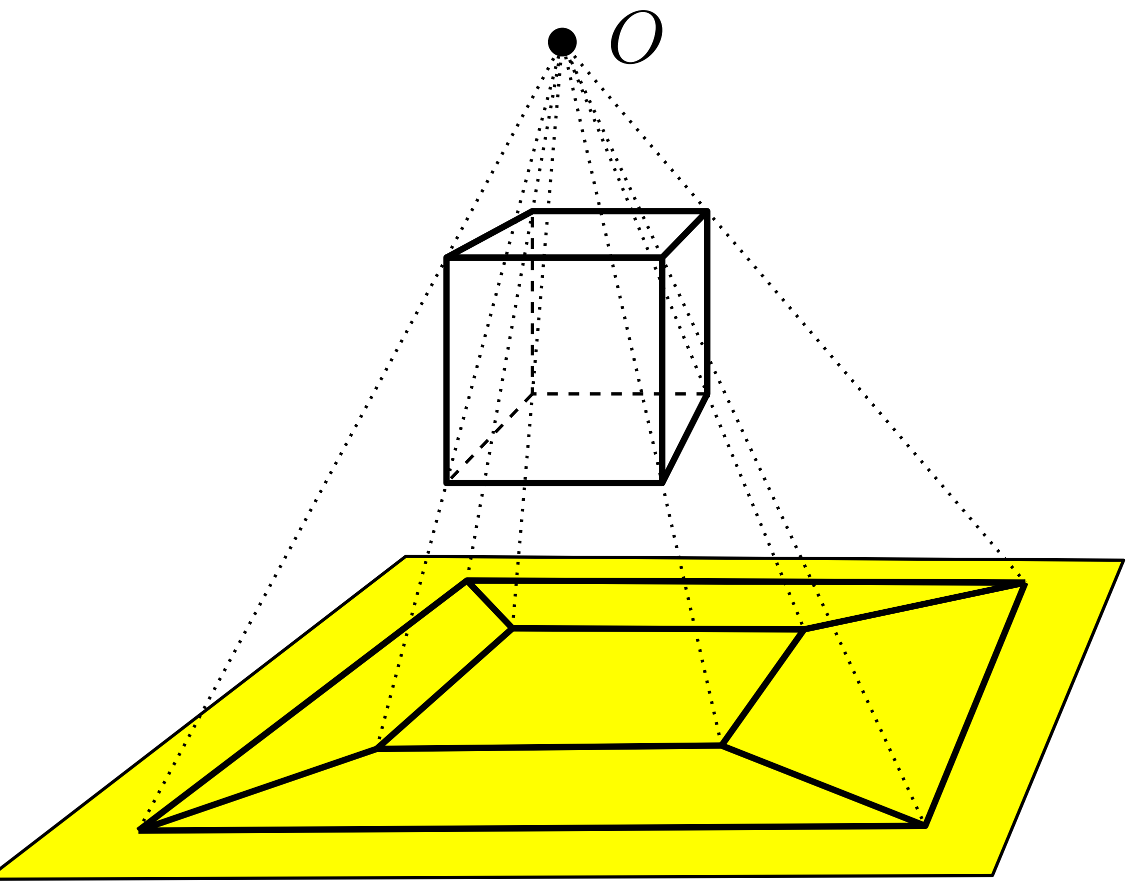
[Ribó, Rote, Schulz 2011, Buchin & Schulz 2010]

Lower bounds: $\Omega(n^{1.5})$

Better bounds for special cases:

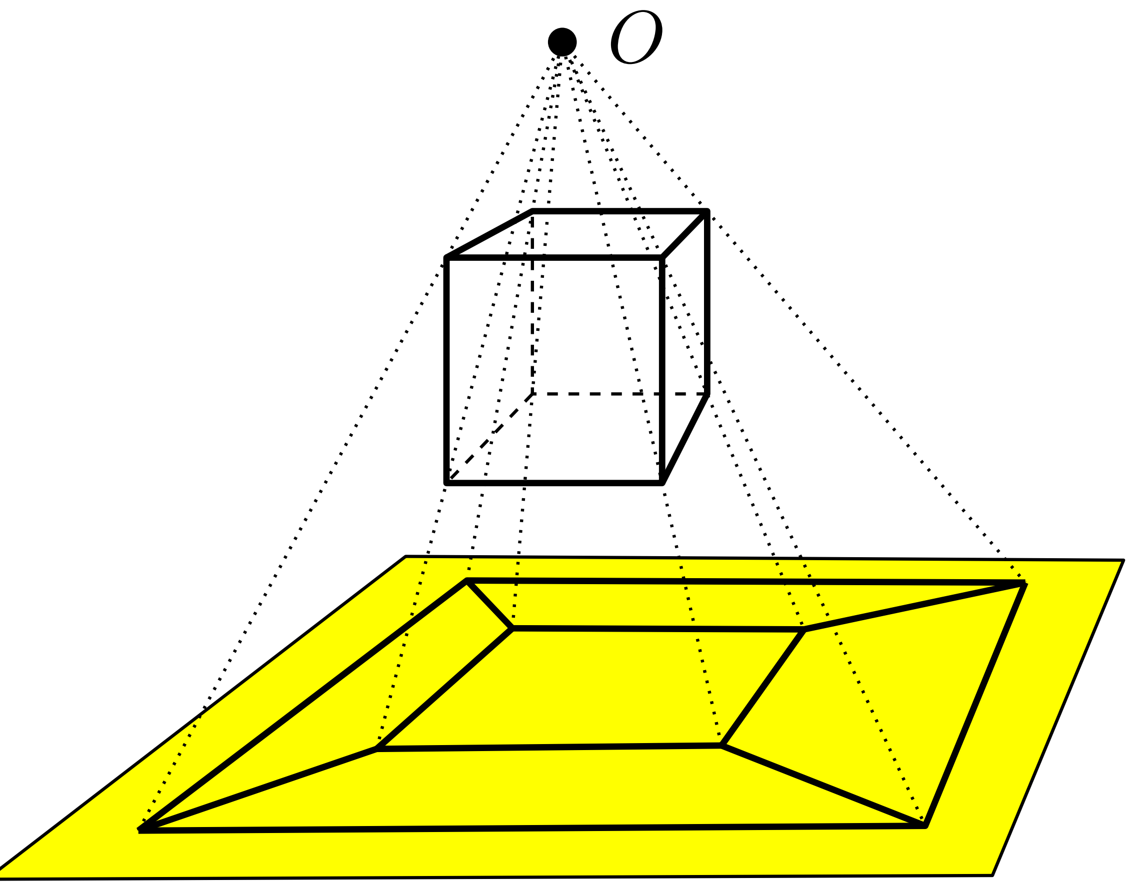
$O(n^{18})$ for *stacked polytopes* [Demaine & Schulz 2011]

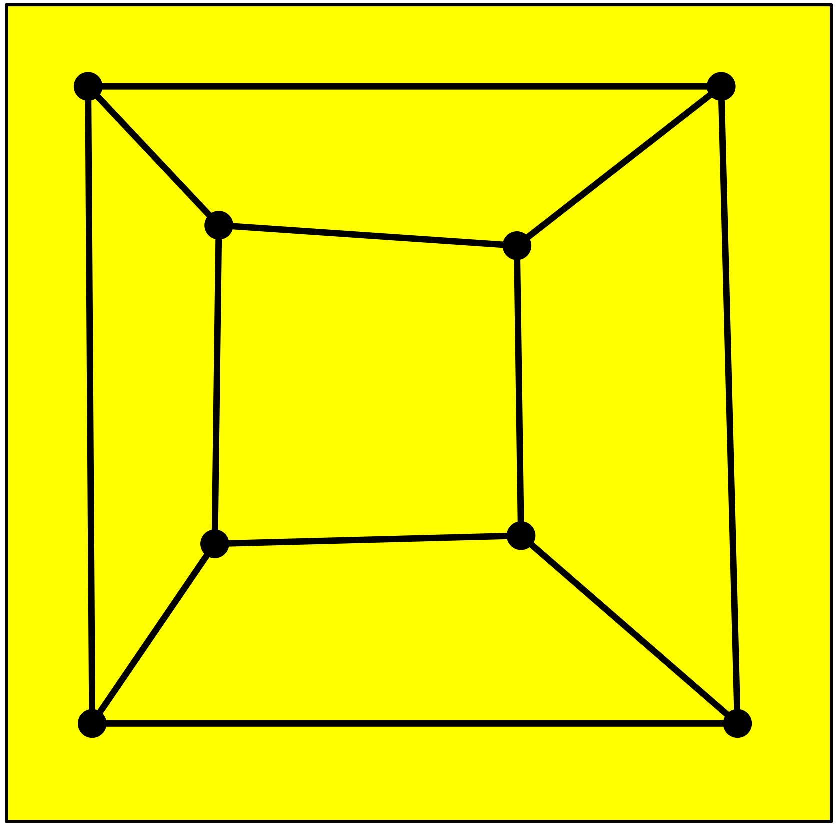
Schlegel Diagrams



project from a center O
close enough to a face

Schlegel Diagrams





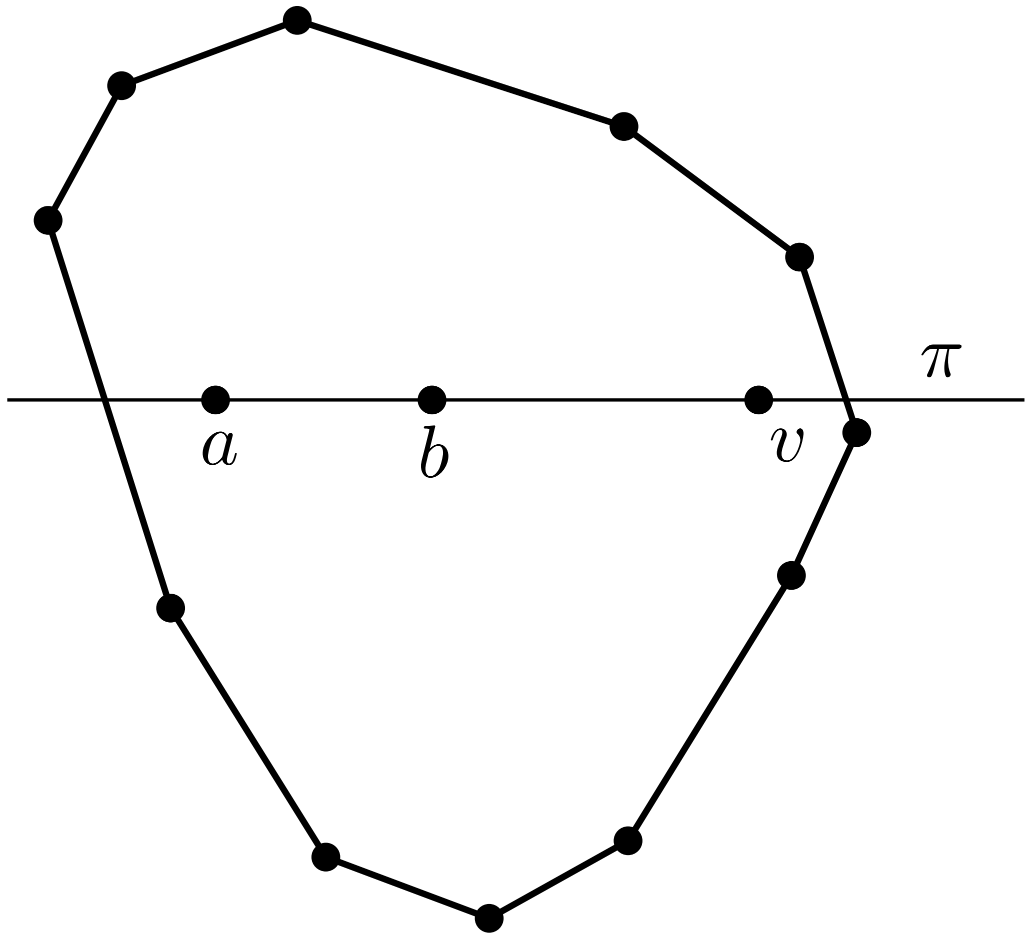
project from a center O
close enough to a face

a Schlegel diagram:
a planar graph with
convex faces

3-Connectivity

Assume a, b separate the graph G.

Choose a third vertex v. Take a plane π through a, b, v.



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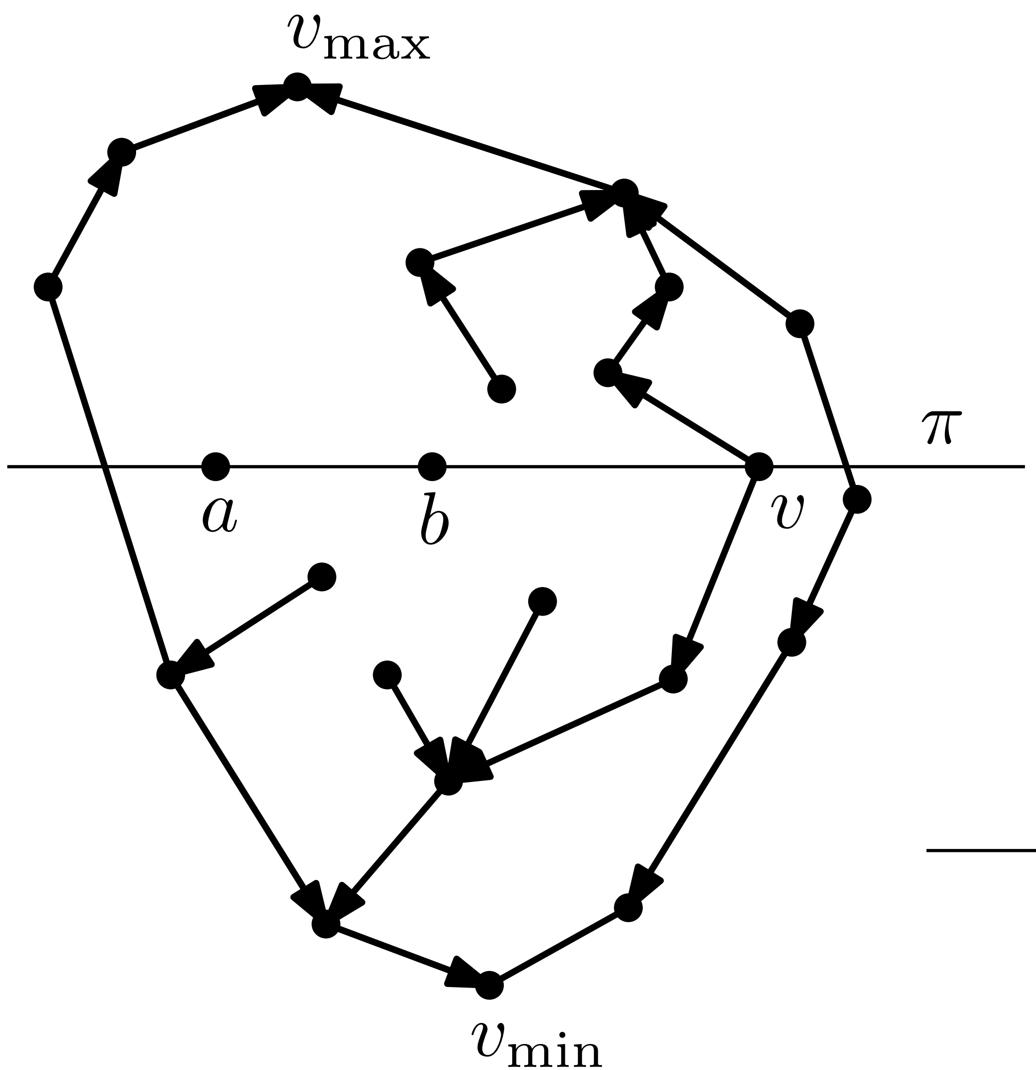

Every vertex has a monotone path to $v_{\max}$ or $v_{\min}$.

v has both paths.

$\Longrightarrow$

$G - \{a, b\}$ is connec

---

d-connected in d dimensions
[Balinski 1961]

[this proof: Grünbaum]

The Theorem of Steinitz (1916)

The graphs of convex three-dimensional polytopes are exactly the *planar*, *3-connected* graphs.

We have seen "$\Longrightarrow$".

Whitney's Theorem:

3-connected planar graphs have a unique face structure.

($\Longrightarrow$ they have a combinatorially unique plane drawing up to *reflection* and the choice of the *outer* face.)

$\Longrightarrow$ The combinatorial structure of a 3-polytope is given by its graph.

Constructive Approaches

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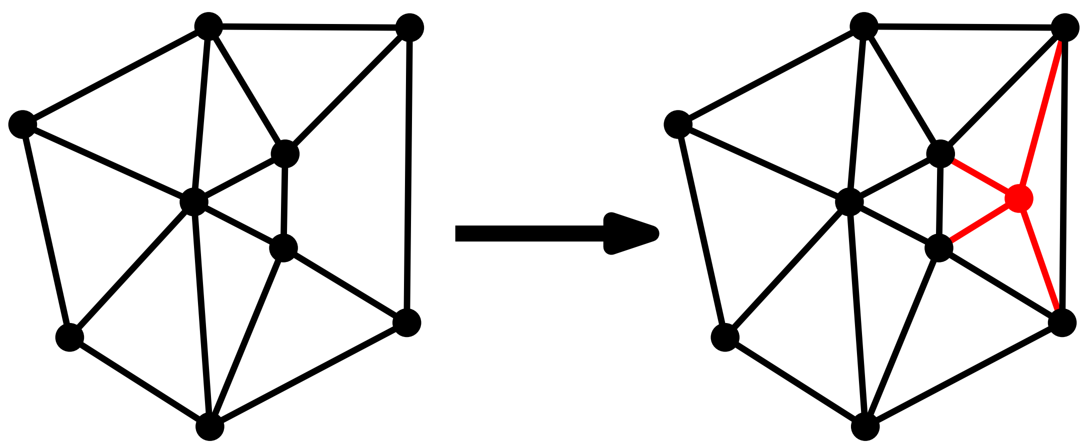
1. INDUCTIVE

Start with the simplest polytope and make local modifications.



[Steinitz]

[Das & Goodrich 1995]

2. DIRECT

Obtain the polytope as the result of

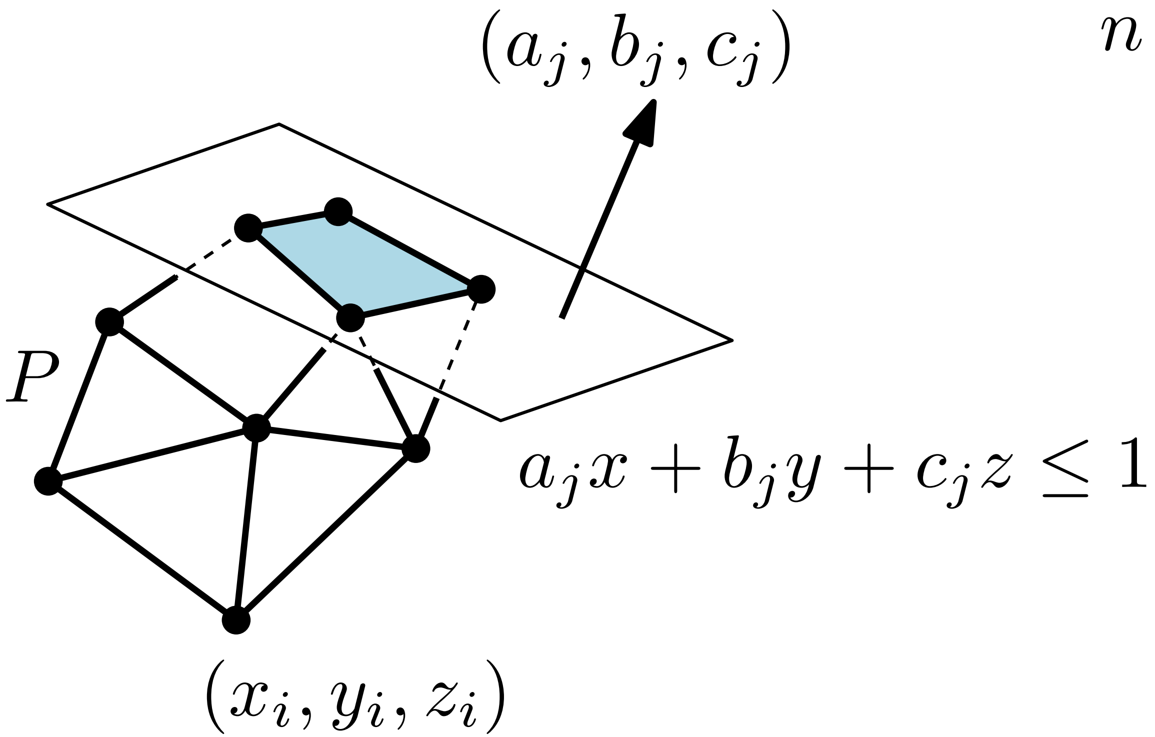
- a system of equations [Tutte]
- an optimization problem
- an iterative procedure
- (and existential argument)

} [Koebe–Andreyev–Thurston]

The Realization Space

assume: origin in the interior of P.

n vertices, m edges, f faces



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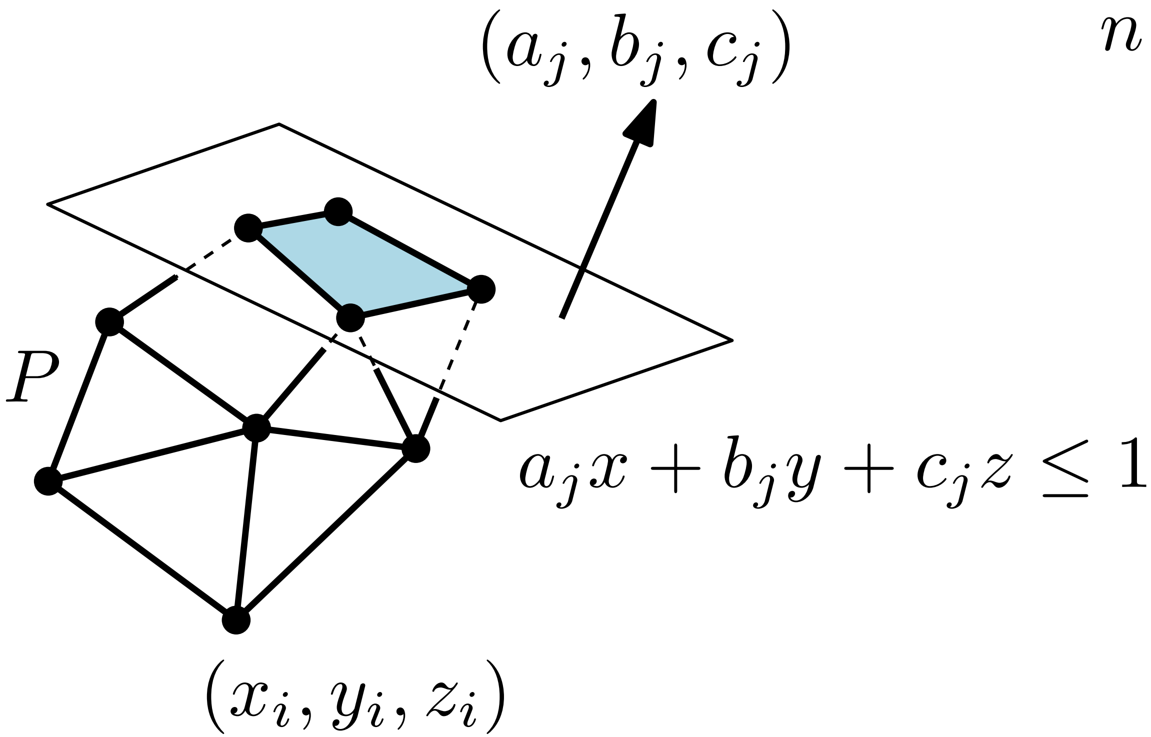

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \dots & & \\ x_n & y_n & z_n \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \dots & & \\ a_f & b_f & c_f \end{pmatrix}$$

$$(a_j, b_j, c_j) \cdot (x_i, y_i, z_i) \begin{cases} = 1, & \text{if face } j \text{ contains vertex } i \\ < 1, & \text{otherwise} \end{cases}$$

The Realization Space

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n vertices, m edges, f faces

$$\mathcal{R}^0 = \{ \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \dots & & \\ x_n & y_n & z_n \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \dots & & \\ a_f & b_f & c_f \end{pmatrix} \in \mathbb{R}^{(n+f)\times 3} :$$

$$(a_j, b_j, c_j) \cdot (x_i, y_i, z_i) \begin{cases} = 1, & \text{if face } j \text{ contains vertex } i \\ < 1, & \text{otherwise} \end{cases}$$

$3n + 3f$ variables, $2m$ equations

THEOREM: $\dim \mathcal{R}^0 = 3n + 3f - 2m = m + 6.$

$\mathcal{R}^0$ is contractible.

In 4 and higher dimensions, realization spaces can be arbitrarily complicated. [Mnëv 1988, Richter-Gebert 1996]

The Realization Space

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n vertices, m edges, f faces

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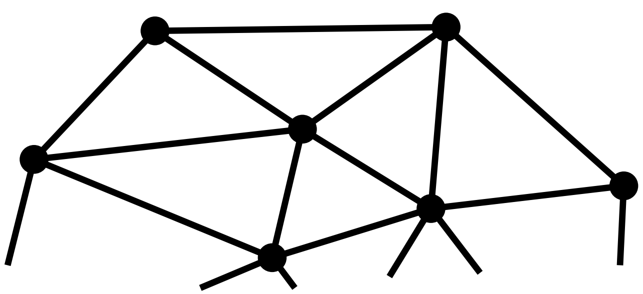
- triangulated (simplicial) polytopes



vertices can be perturbed.
(a_j, b_j, c_j) variables are redundant.

The Realization Space

n vertices, m edges, f faces

$$\mathcal{R}^0 = \left\{ \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \ldots & & \\ x_n & y_n & z_n \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \ldots & & \\ a_f & b_f & c_f \end{pmatrix} \in \mathbb{R}^{(n+f)\times 3} : \right.$$

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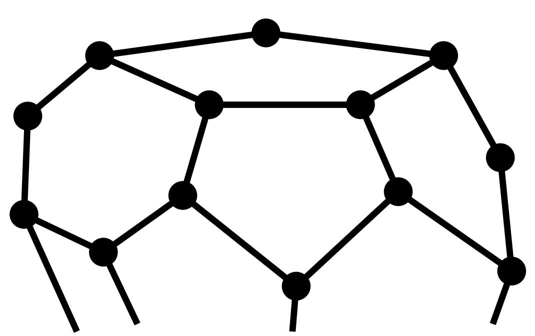
- simple polytopes (with 3-regular graphs)



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The Realization Space

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$$(a_j, b_j, c_j) \cdot (x_i, y_i, z_i) \begin{cases} = 1, & \text{if face } j \text{ contains vertex } i \\ < 1, & \text{otherwise} \end{cases}$$

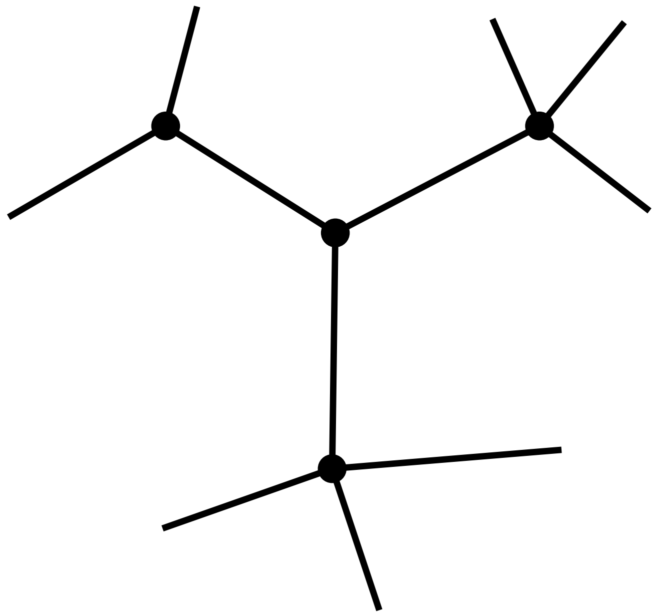
Polarity:

interpret (a_j, b_j, c_j) as vertices and (x_i, y_i, z_i) as half-spaces.

$\rightarrow$ the polar polytope: VERTICES $\leftrightarrow$ FACES exchange roles.

$\rightarrow$ the (planar) dual graph

Inductive Constructions of Polytopes



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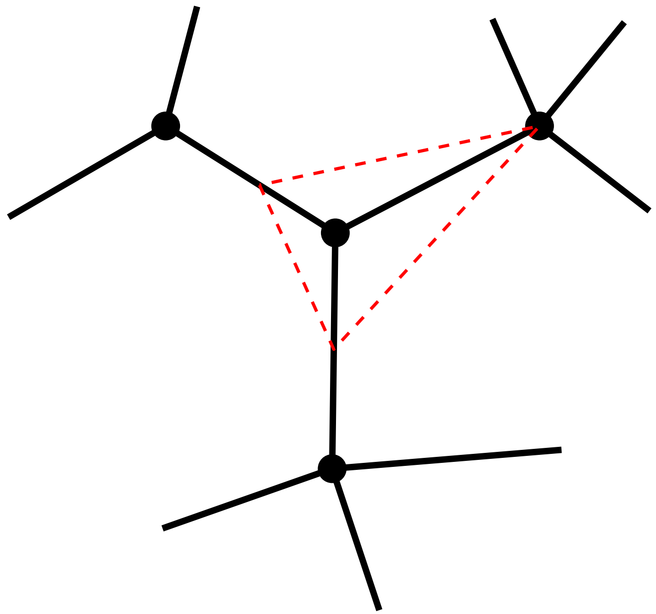

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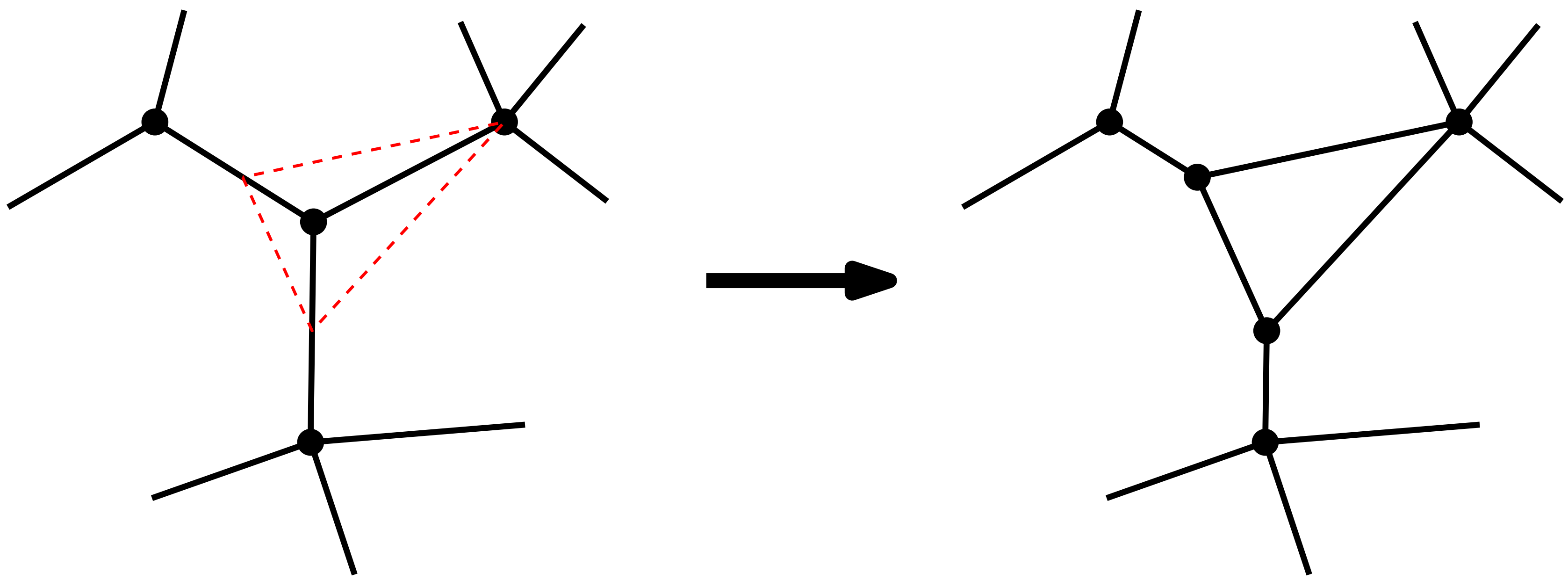


an additional (triangular) face

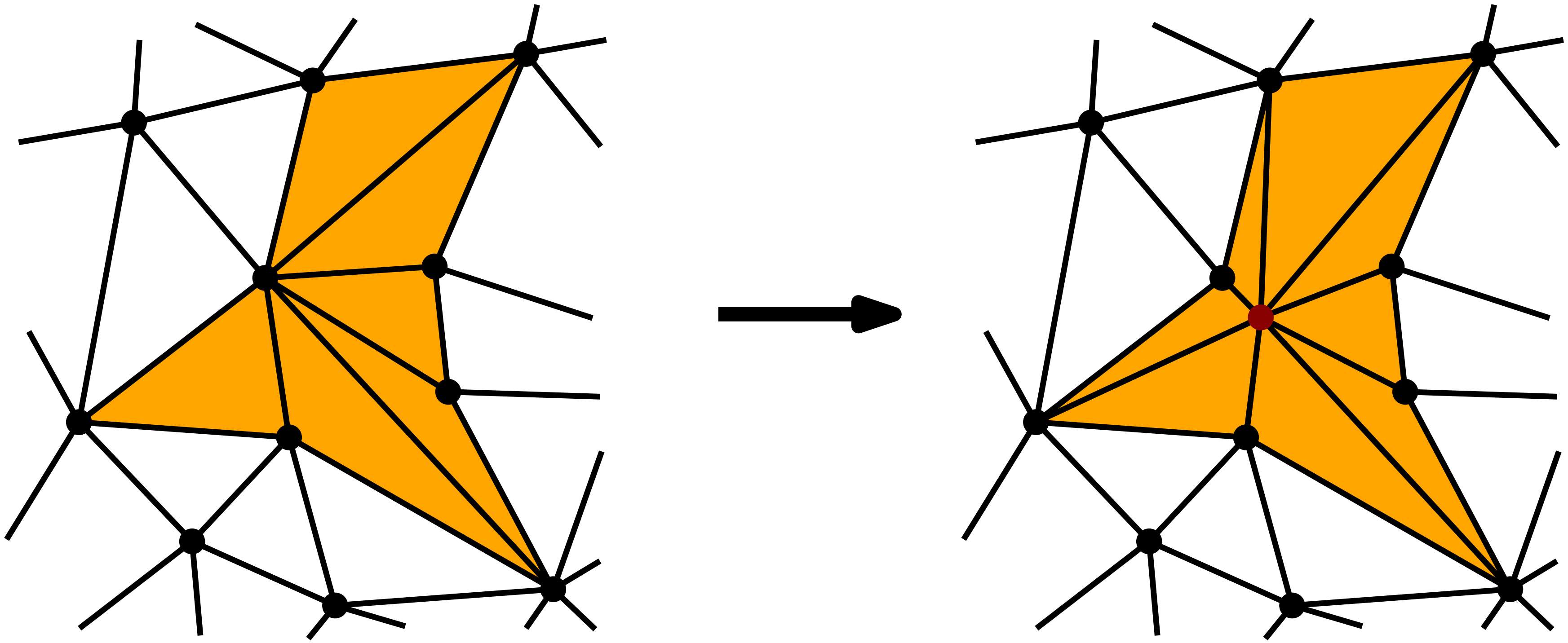
$+$ apply polarity when necessary [Steinitz 1916]

Everything can be done with rational coordinates.
$\rightarrow$ integer coordinates of size $2^{\exp(n)}$

COMBINATORIAL $+$ GEOMETRIC arguments

Inductive Constructions of Polytopes

Das & Goodrich [1997]: $2^{\mathrm{poly}(n)}$ for *triangulated* polytopes



perform this operation on $n/24$ independent vertices in parallel

$\rightarrow O(\log n)$ rounds
Each round multiplies the number of bits by a constant factor.

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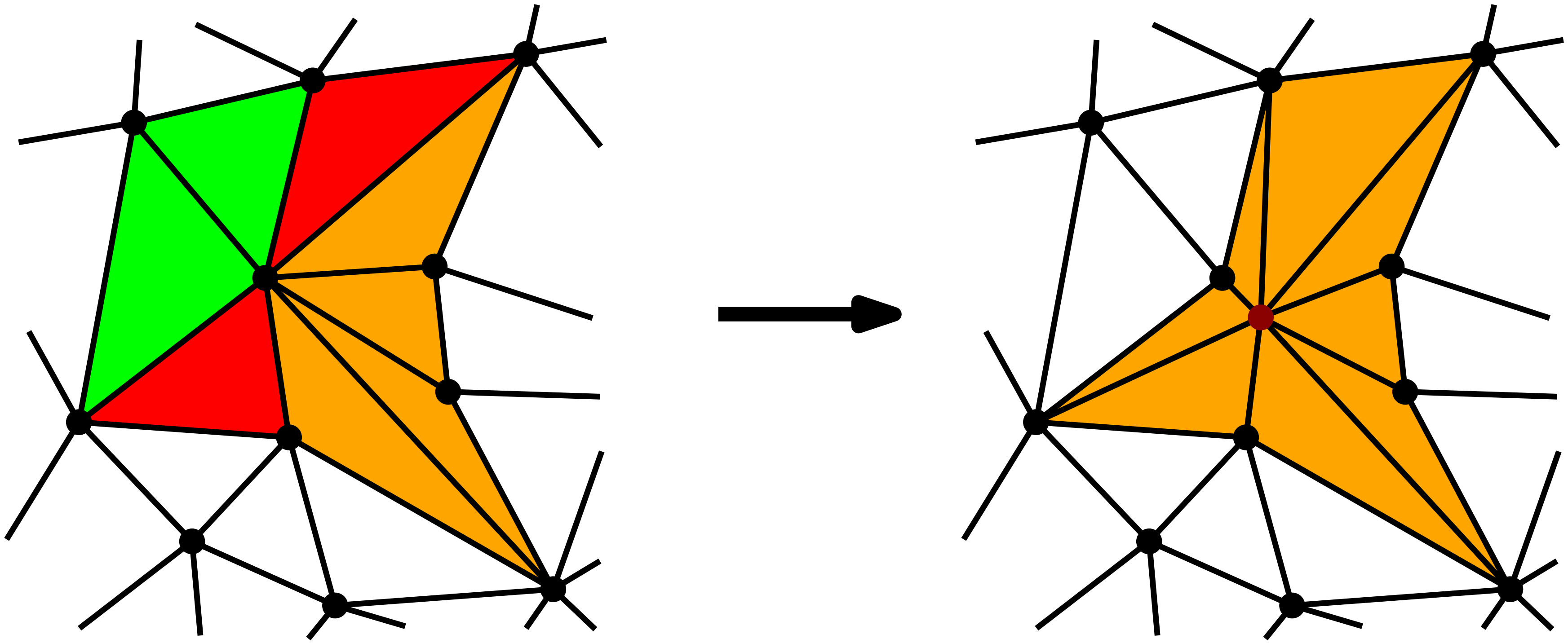

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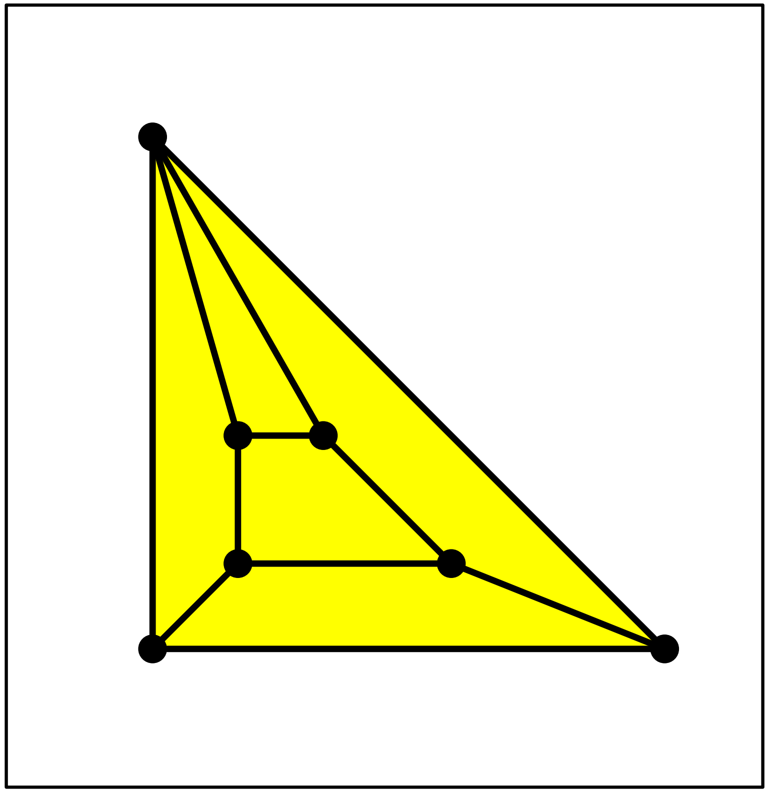
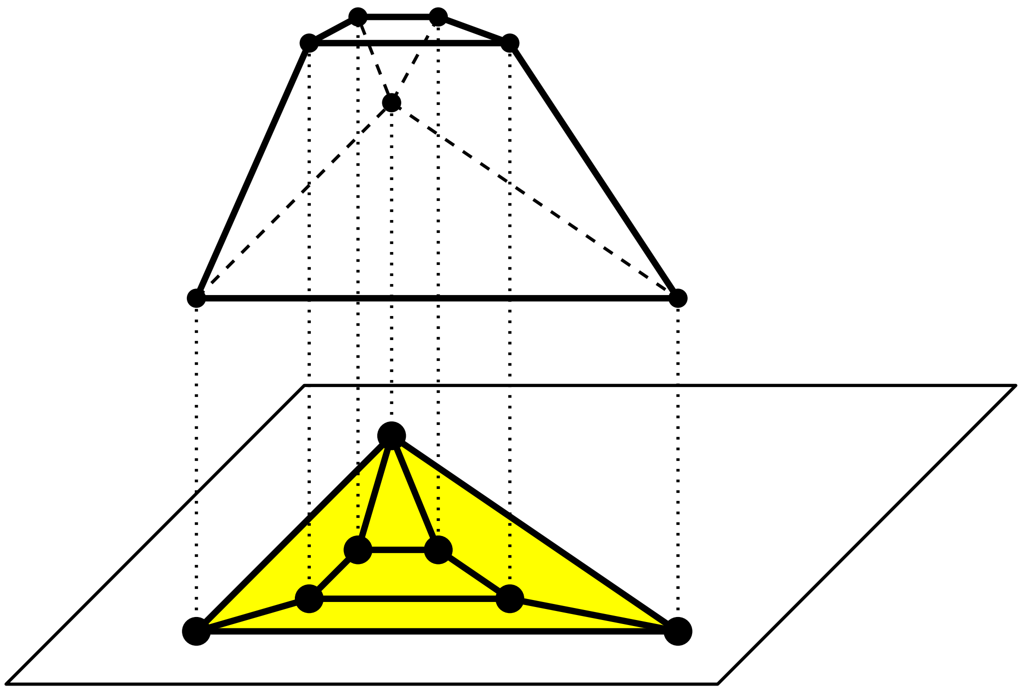
Direct Constructions of Polytopes





A) construct the Schlegel diagram in the plane.

B) *Lift* to three dimensions.

When is a Drawing a Schlegel Diagram?

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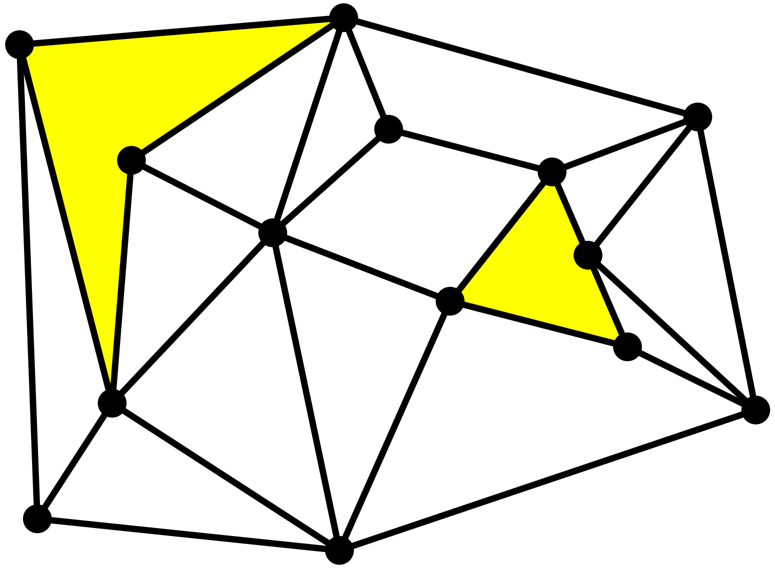
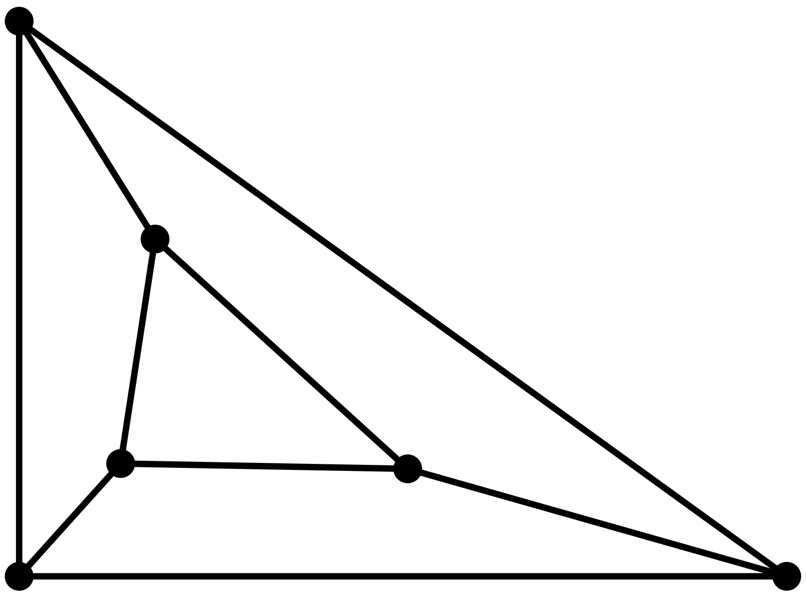


strictly convex faces!



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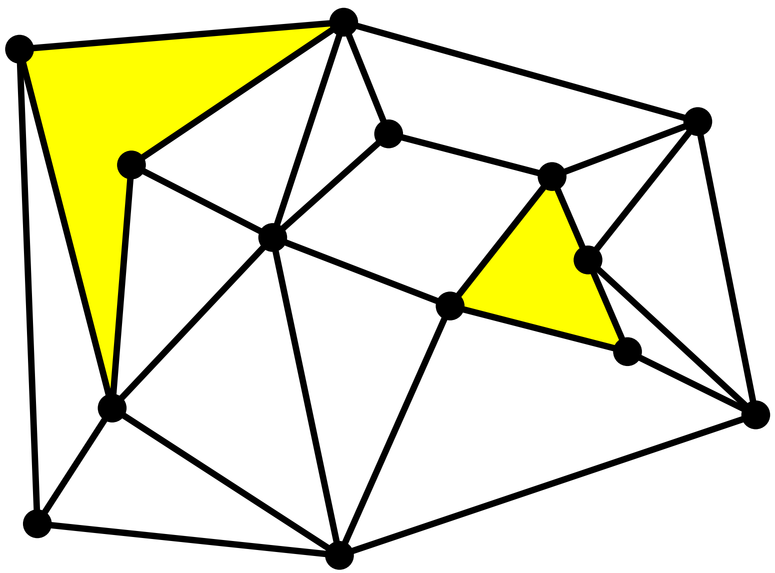

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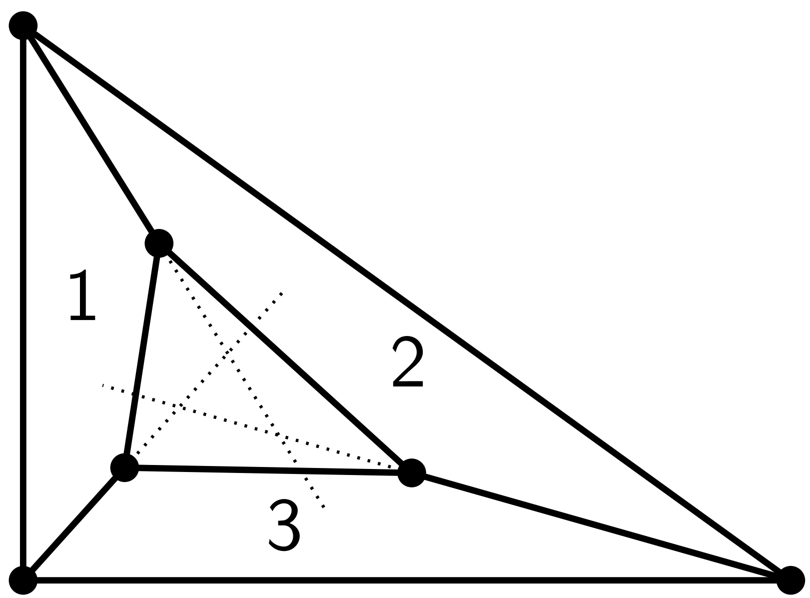

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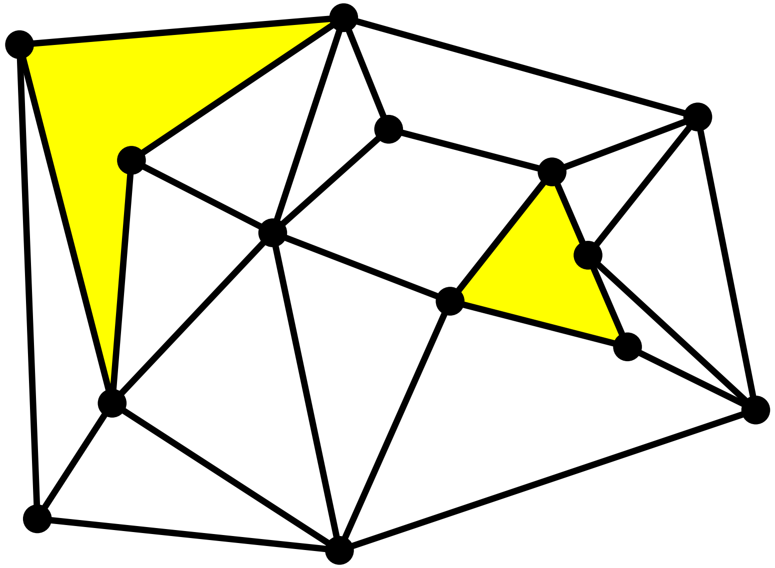

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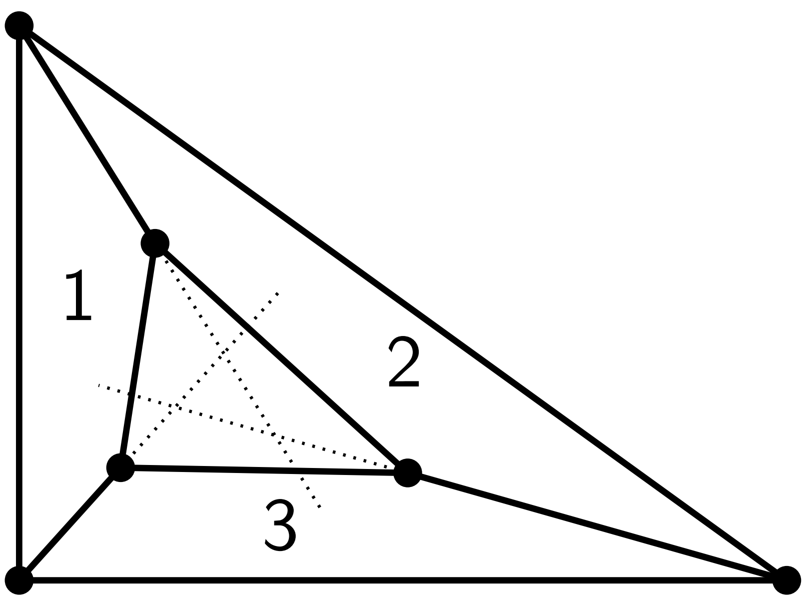

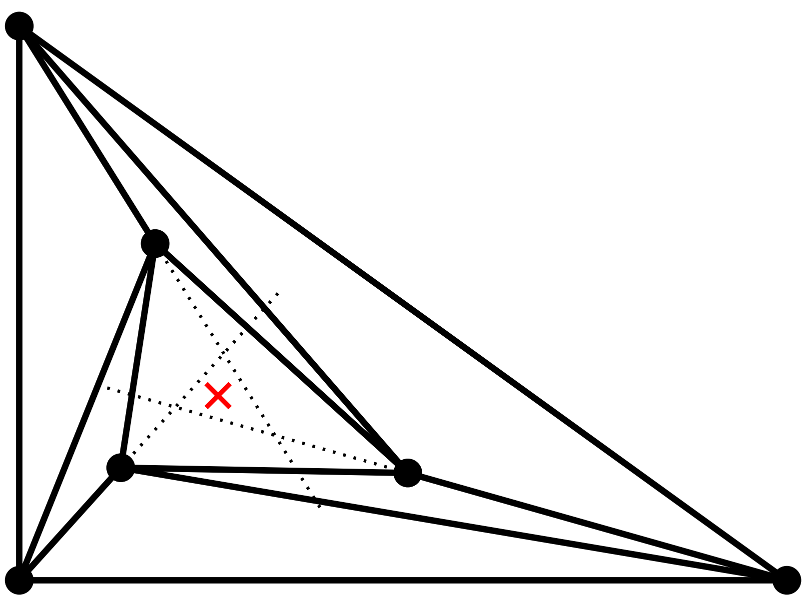

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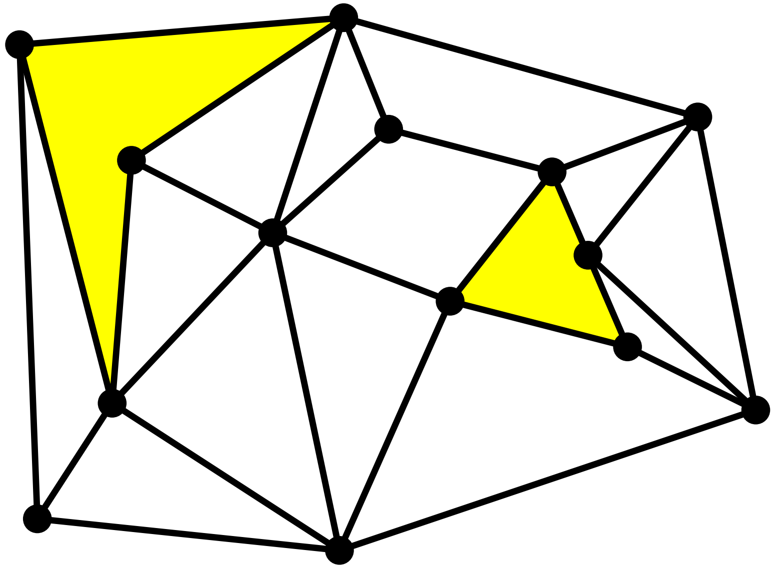

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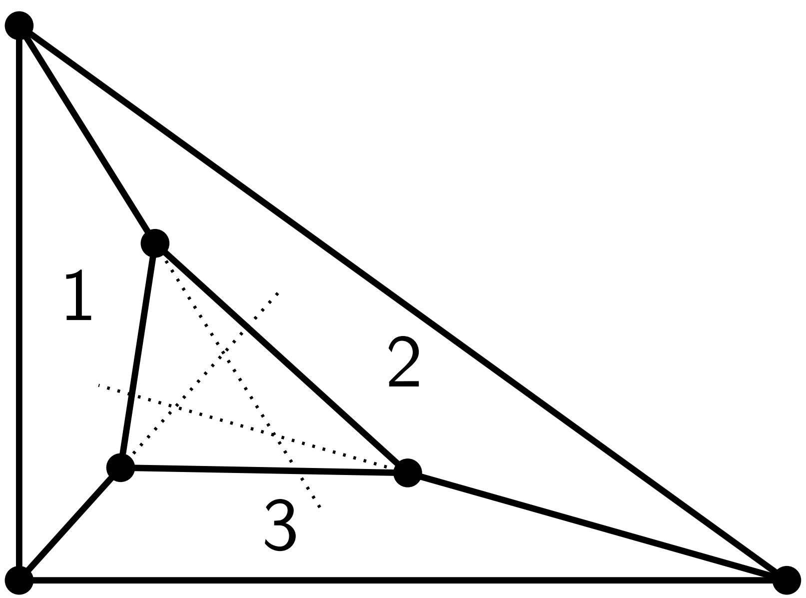
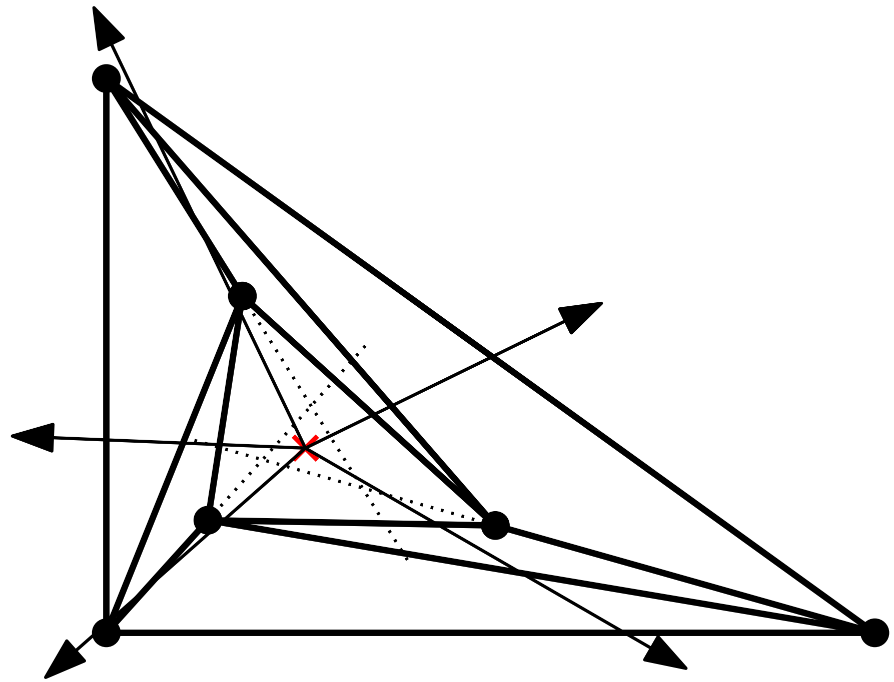
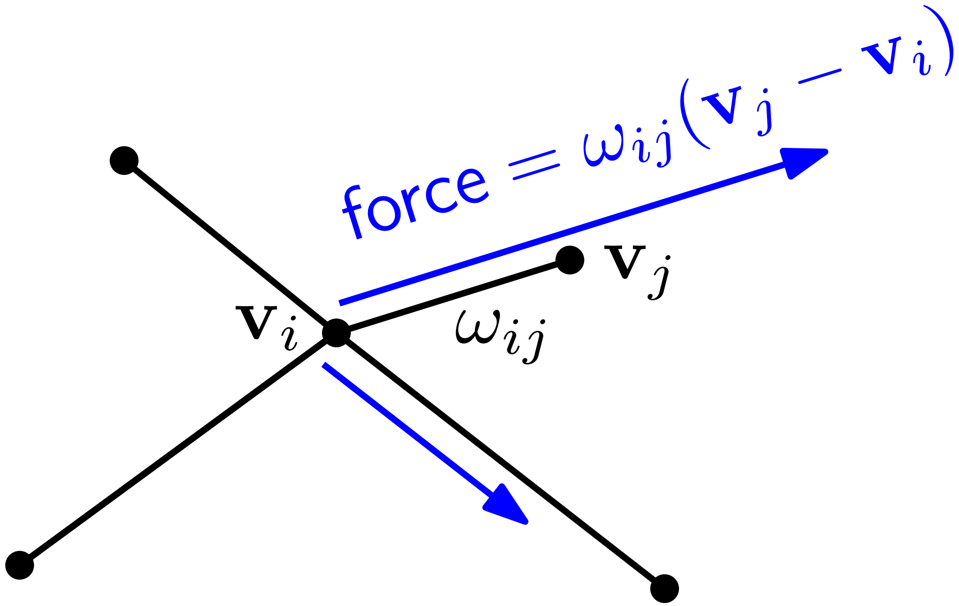


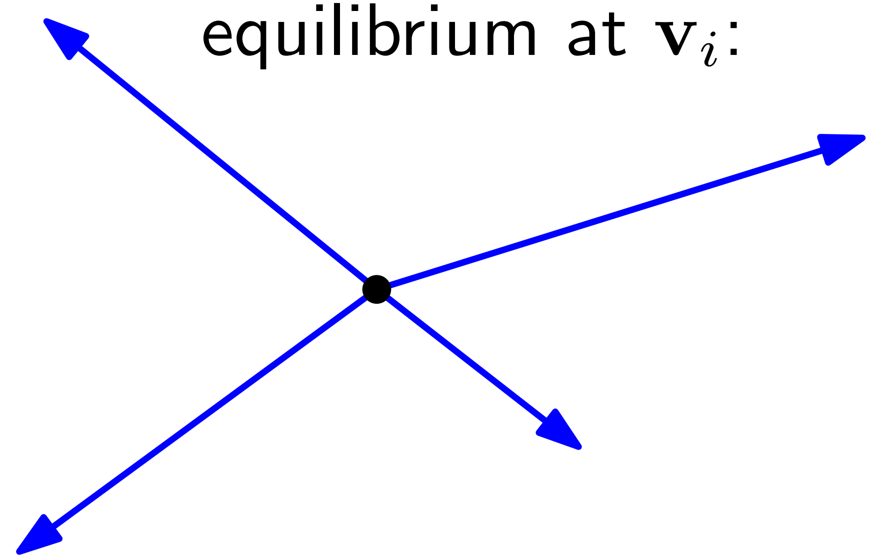


The Maxwell-Cremona Correspondence

Equilibrium stress: Assign a scalar $\omega_{ij} = \omega_{ji}$ to every edge ij.



equilibrium at $\mathbf{v}_i$:



$$(*) \quad \sum_{j:ij\in E} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0$$

Equilibrium stress: equilibrium at every vertex.

THEOREM: [Maxwell 1864, Whiteley 1982]
A drawing is a Schlegel diagram $\iff$ it has an equilibrium stress that is positive on each interior edge.

Tutte Embedding [1960, 1963]

1) Fix the vertices of the outer face

2) Set $\omega_{ij} \equiv 1.$ Compute positions of interior vertices by $(*)$

3) Lift to three dimensions.

$$(*)\ \sum_{j\sim i} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0 \quad \Longrightarrow \quad \mathbf{v}_i = \frac{\sum_{j\sim i} \omega_{ij}\mathbf{v}_j}{\sum_{j\sim i} \omega_{ij}}$$

Every vertex $\mathbf{v}_i$ is the (weighted) barycenter of its neighbors.
SPIDERWEB EMBEDDING

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SPIDERWEB EMBEDDING

If the outer face is a triangle, equilibrium at *interior* vertices is enough.

Tutte Embedding [1960, 1963]

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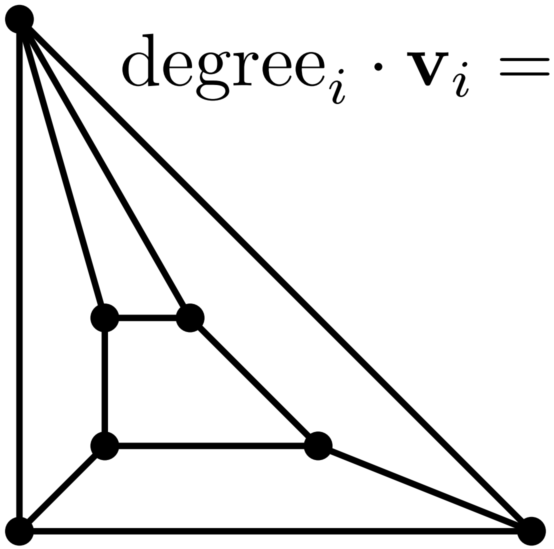
Coefficient matrix (for $\omega \equiv 1$) = the Laplacian Λ



$$\text{degree}_i \cdot \mathbf{v}_i = \sum_{j \sim i} \mathbf{v}_j$$

$$\Lambda = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 4 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{pmatrix}$$

negative adjacency matrix

degrees

$$\mathbf{v}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \qquad x_i, y_i = \frac{\det(\cdot)}{\det \Lambda}$$

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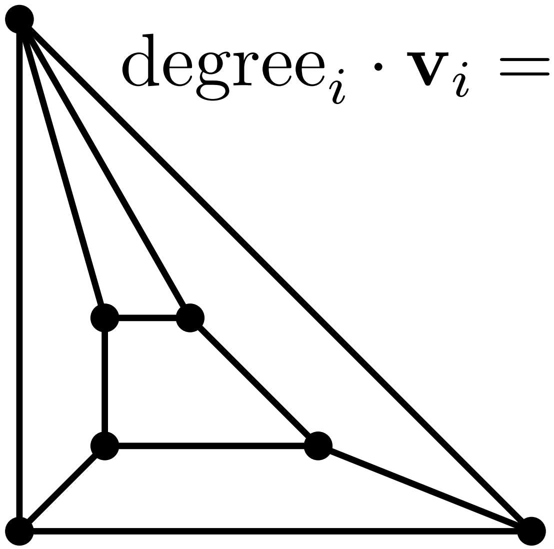

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degrees

negative adjacency matrix

$$\mathbf{v}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \qquad x_i, y_i = \frac{\det(\cdot)}{\det \Lambda'}$$

$\det \Lambda'$ = the number of (certain) spanning forests $< 6^n$

common denominator $< 6^n \implies \ldots$ all coordinates $< \text{const}^n$.

Easy bound on spanning trees

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$$\#T \le \prod_{v=1}^{n} d_v \qquad \text{(product of the degrees)}$$

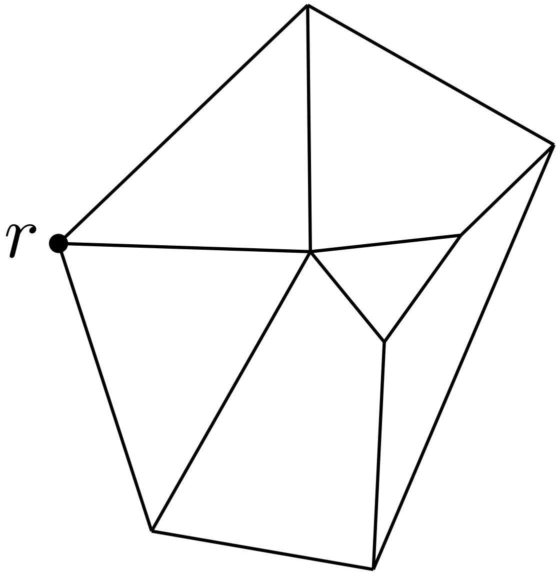
follows from the Hadamard bound for the determinant of positive semidefinite matrices.

For planar graphs: $\#T \le \prod_{v=1}^{n} d_v \le \left(\sum_{v=1}^{n} d_v \; / \; n \right)^{n} < 6^n$

$$\#T \le \prod_{v=1}^{n} d_v \cdot \frac{1}{2m} (1 + \tfrac{1}{n-1})^{n-1} \le \prod_{v=1}^{n} d_v \cdot \frac{e}{2m}$$

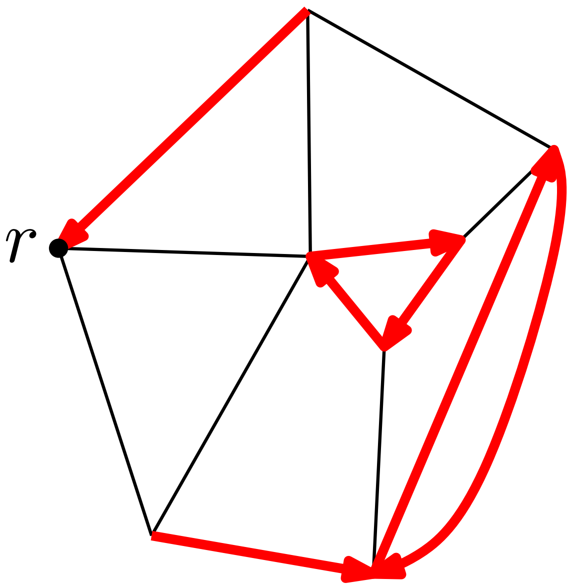
for graphs with m edges [Grone, Merris 1988]

The Outgoing Edge Method



Pick a root r

The Outgoing Edge Method



Pick a root r

Select an arbitrary outgoing edge for each vertex $v \neq r$.

$$\#\text{choices} = \prod_{v \neq r} d_v$$

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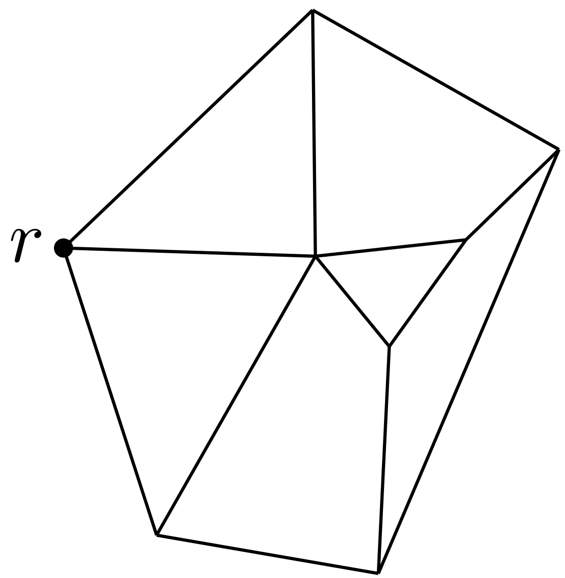

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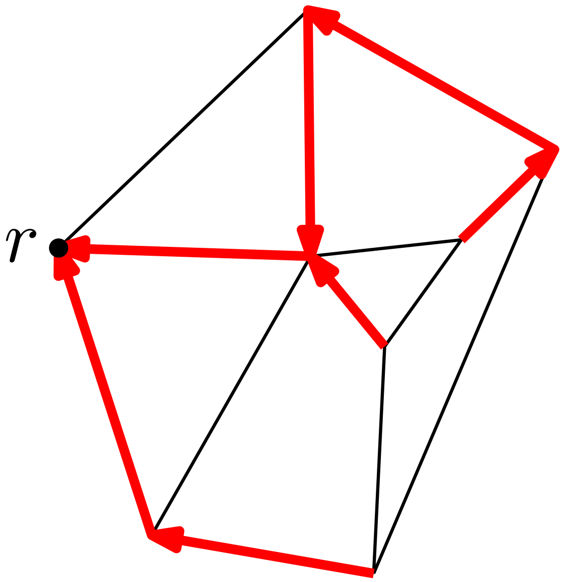

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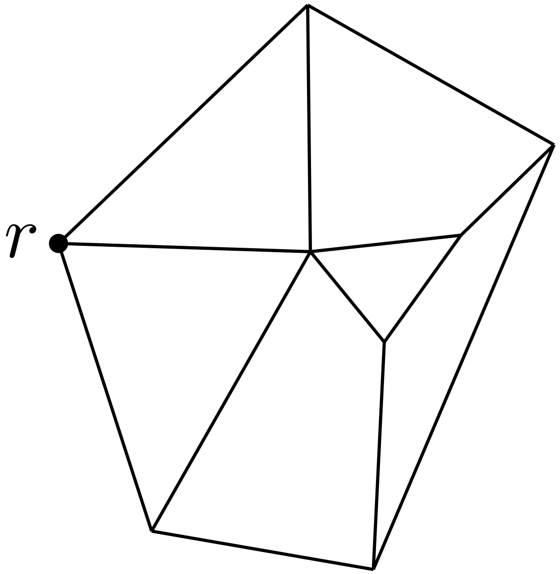

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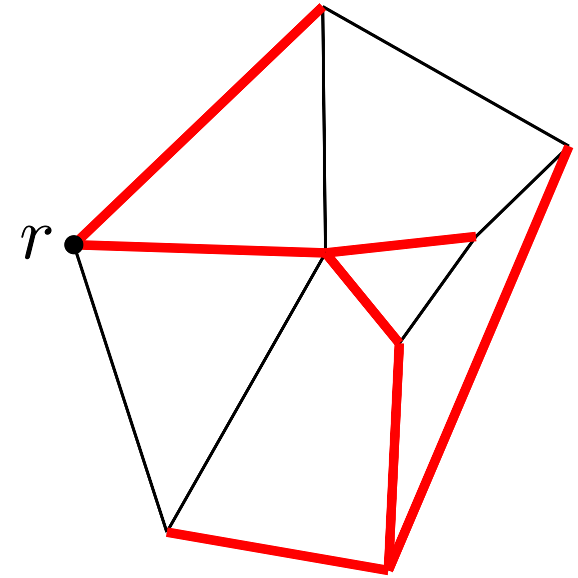
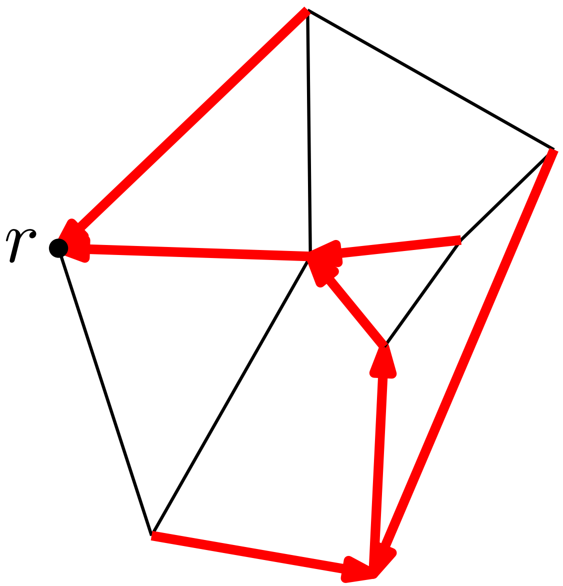


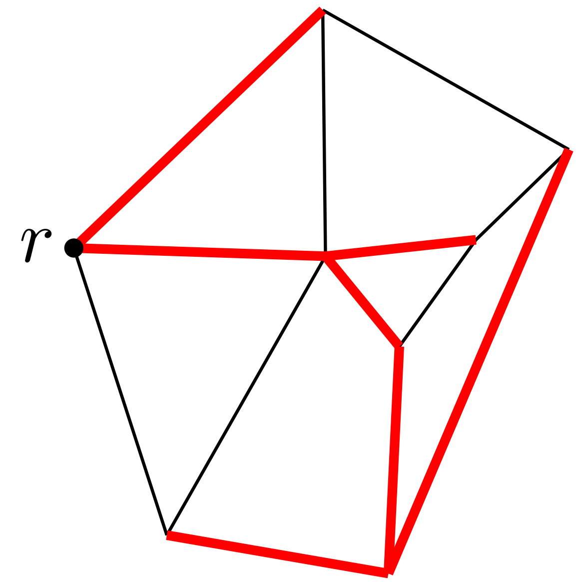
Every spanning tree

arises once as a rooted directed spanning tree

$$\#T \leq \prod_{v \neq r} d_v$$

The Outgoing Edge Method





Pick a root r

Select an arbitrary outgoing edge for each vertex $v \neq r$.

$$\#\text{choices} = \prod_{v \neq r} d_v$$

Every spanning tree arises once as a rooted directed spanning tree

$$\#T \leq \prod_{v \neq r} d_v$$

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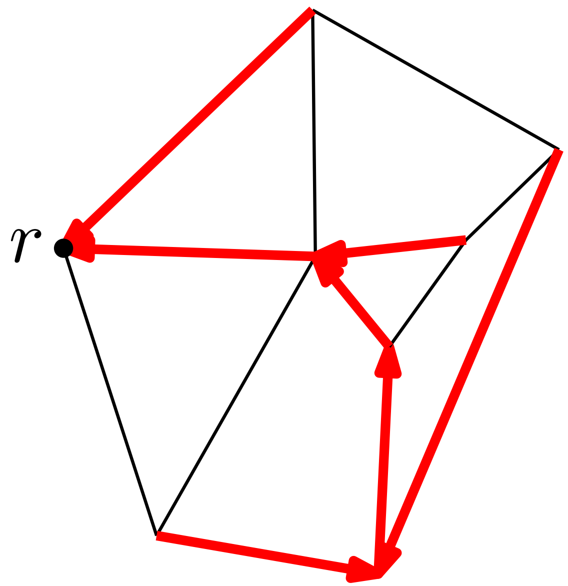

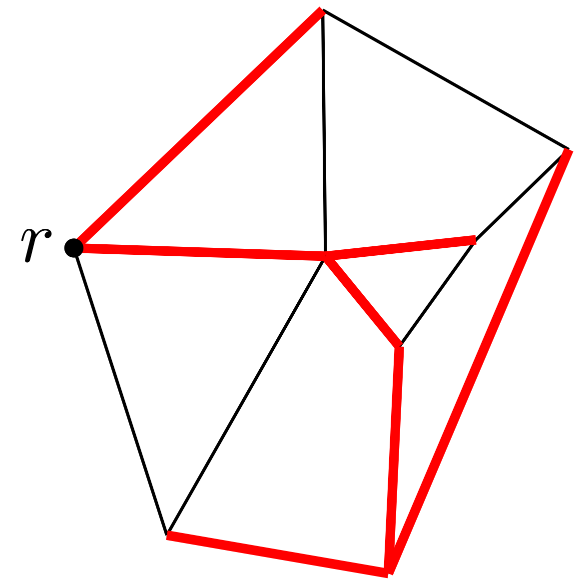

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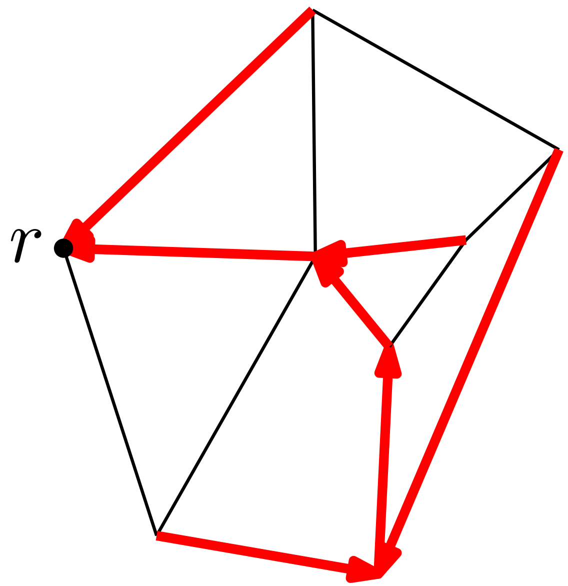

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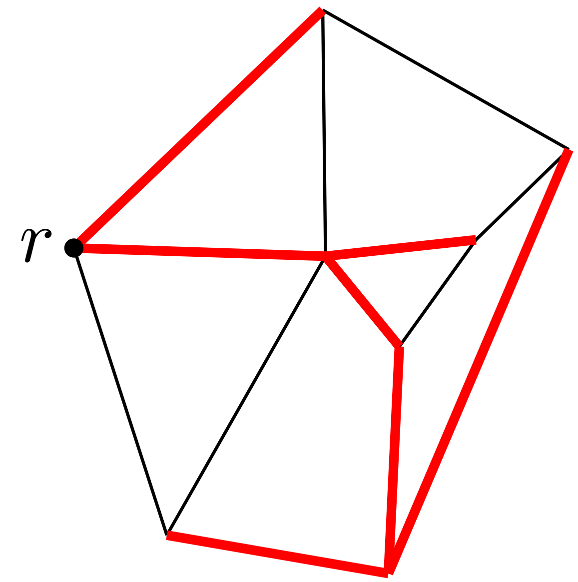

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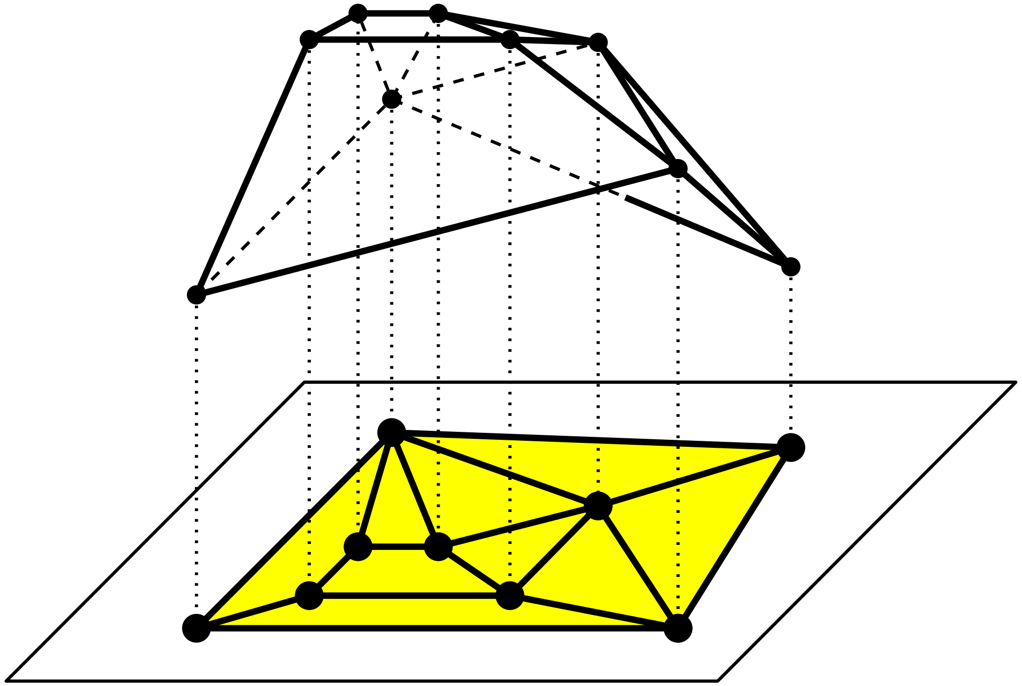
$\#T \leq O(5.29^n)$ [K. Buchin & A. Schulz 2010]

Tutte Embedding [1960, 1963]

If the outer face is NOT a triangle, equilibrium at *interior* vertices is NOT enough.



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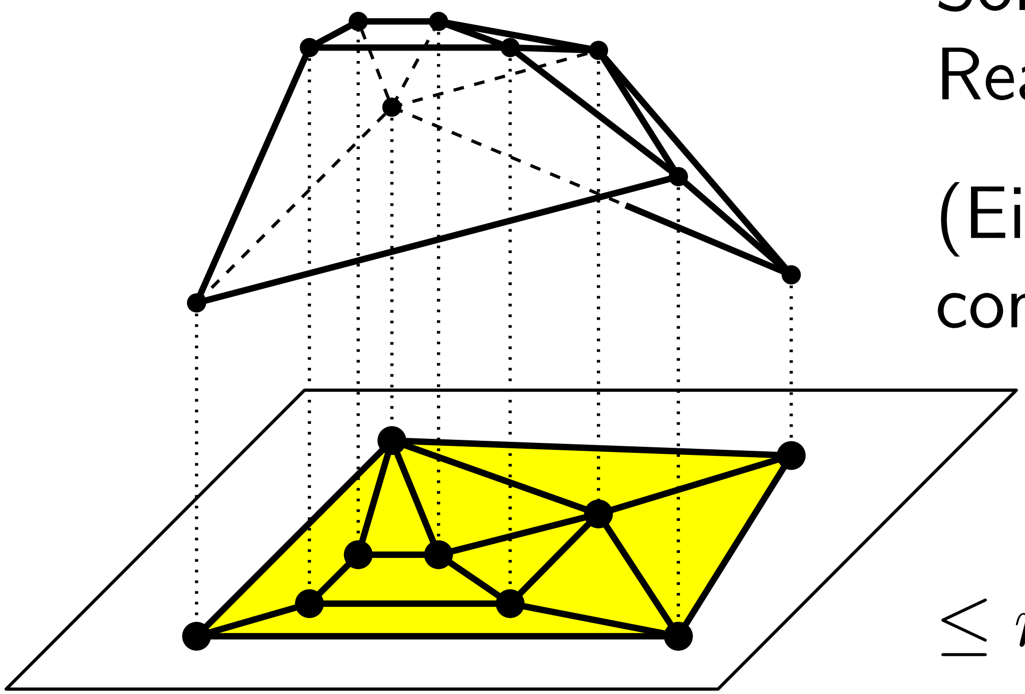

Solution 1)
Realize the polar polytope instead!

(Either the graph or its dual contains a triangle face.)

$\leq n^{169n^3}$ [Onn & Sturmfels 1994]

$\leq 2^{18n^2}$ [Richter-Gebert 1996]

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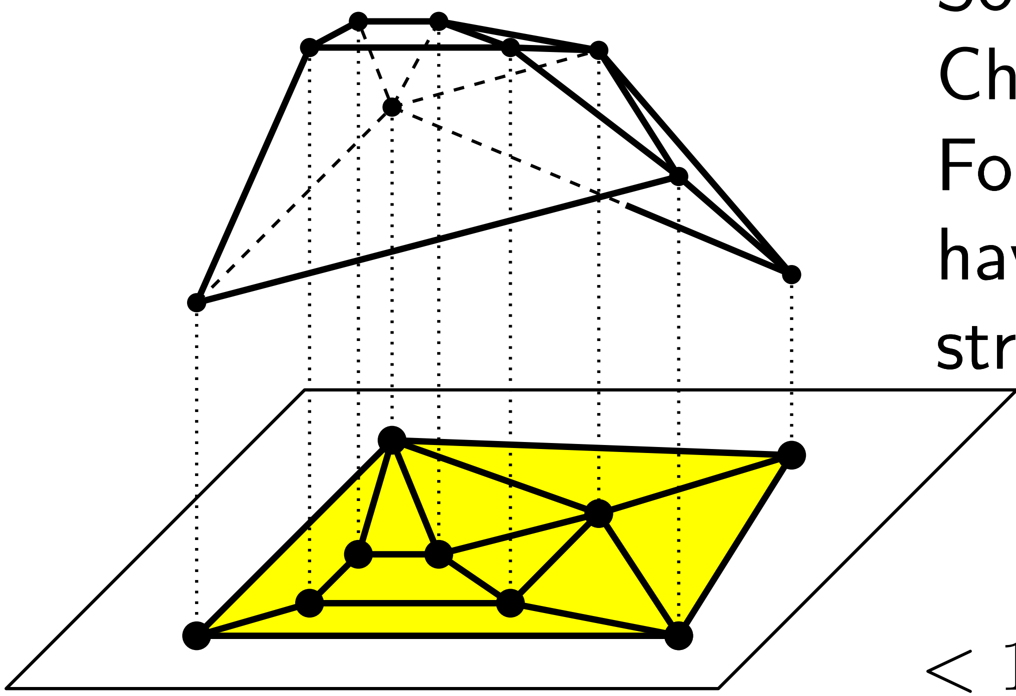


Solution 2)
Choose the outer face carefully.
For the case of 4-gons and 5-gons, have to analyze the resulting stresses on the outer face.

$< 188^n$ [Ribó, Rote, Schulz 2011]

$< 148^n$ [Buchin & Schulz 2010, by better bound on spanning trees]

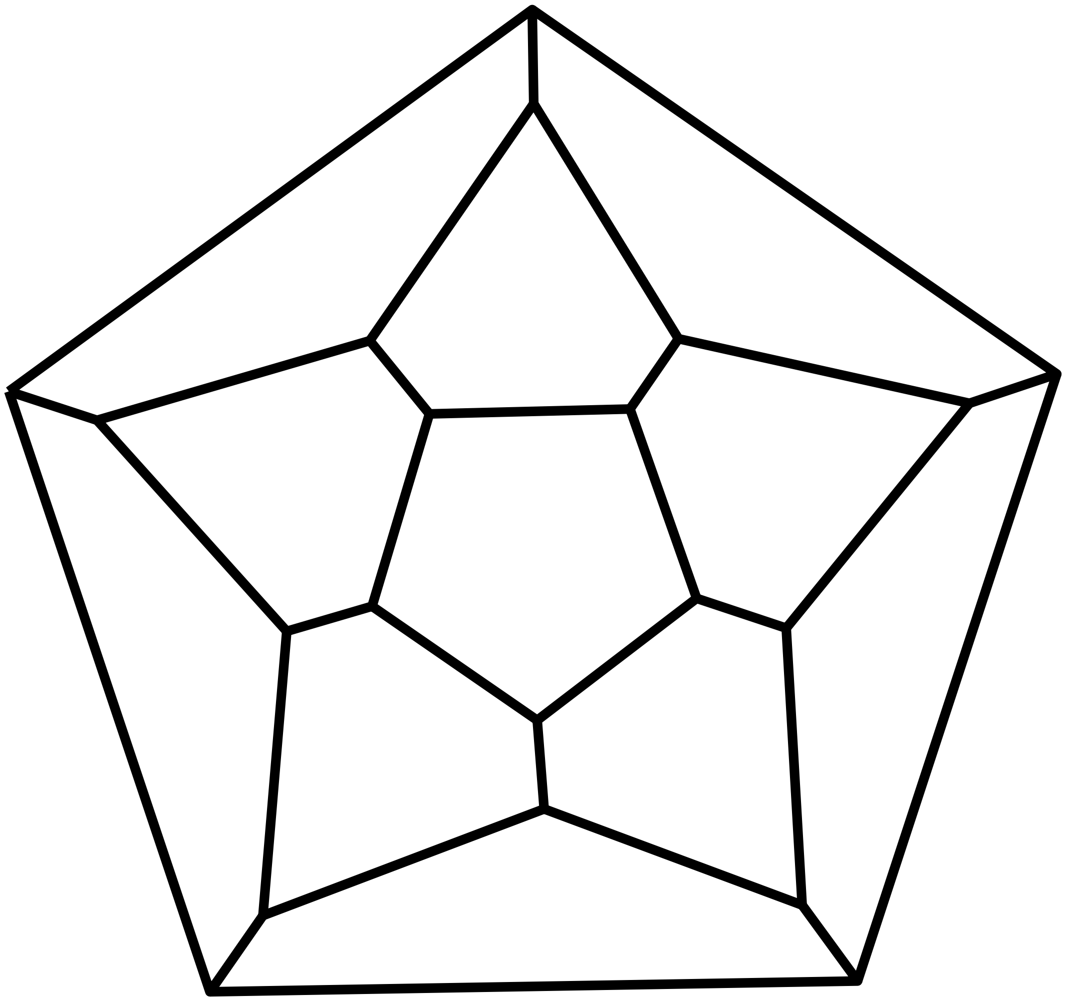
Lower Bounds

Every n-gon with integer vertices needs area $\Omega(n^3)$.
[Andrews 1961, Voss & Klette 1982, Thiele 1991, Acketa & Žunić 1995, Jarník 1929]

$\implies$ side length $\geq \Omega(n^{1.5})$

For comparison:
Strictly convex drawings of 3-connected planar graphs on an $O(n^2) \times O(n^2)$ grid. [Bárány & Rote 2006]

Example: the Dodecahedron



Algorithm gives

$$z \leq 1.11 \times 10^{25}$$

(general bound $\approx 10^{47}$)

remove common factors

$$\Longrightarrow \quad 0 \leq x_i \leq 1374$$
$$0 \leq y_i \leq 898$$
$$0 \leq z_i \leq 406.497$$

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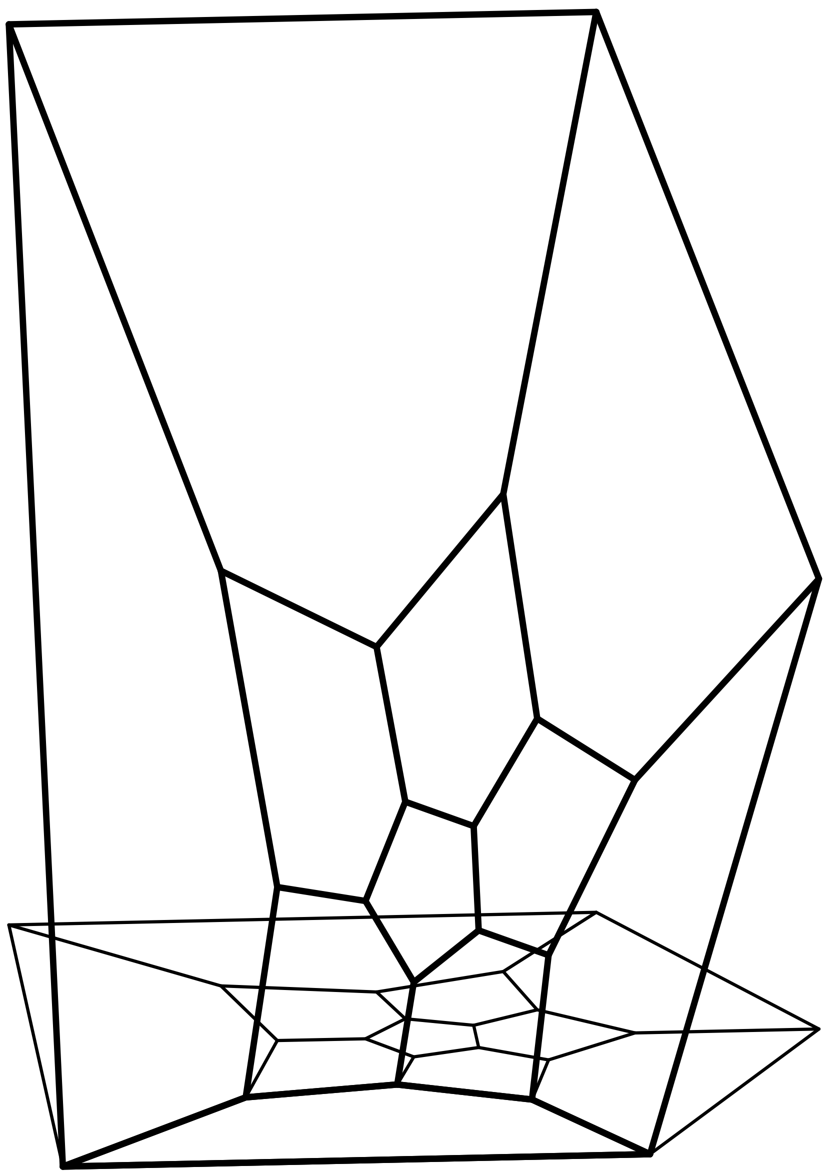

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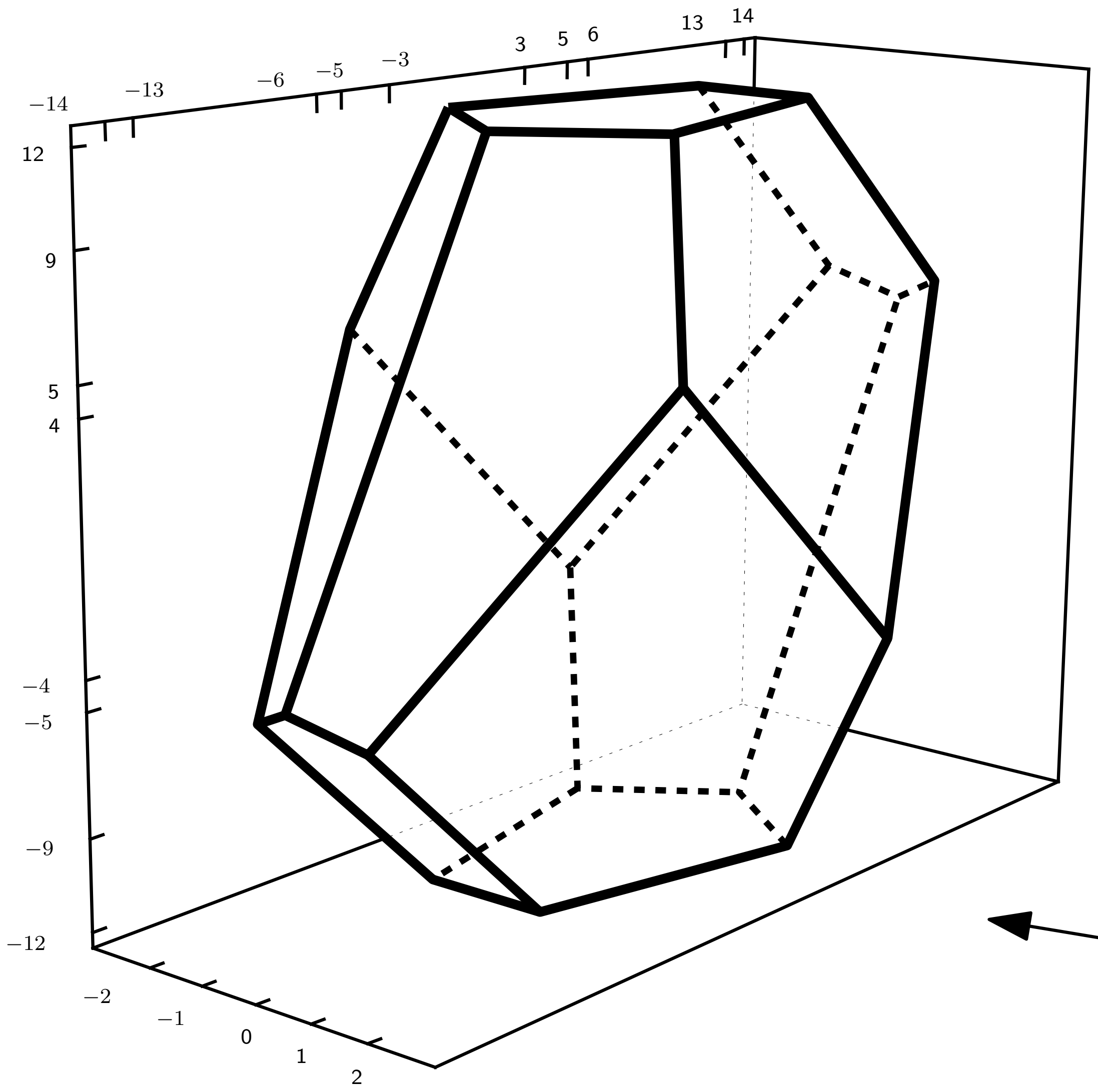

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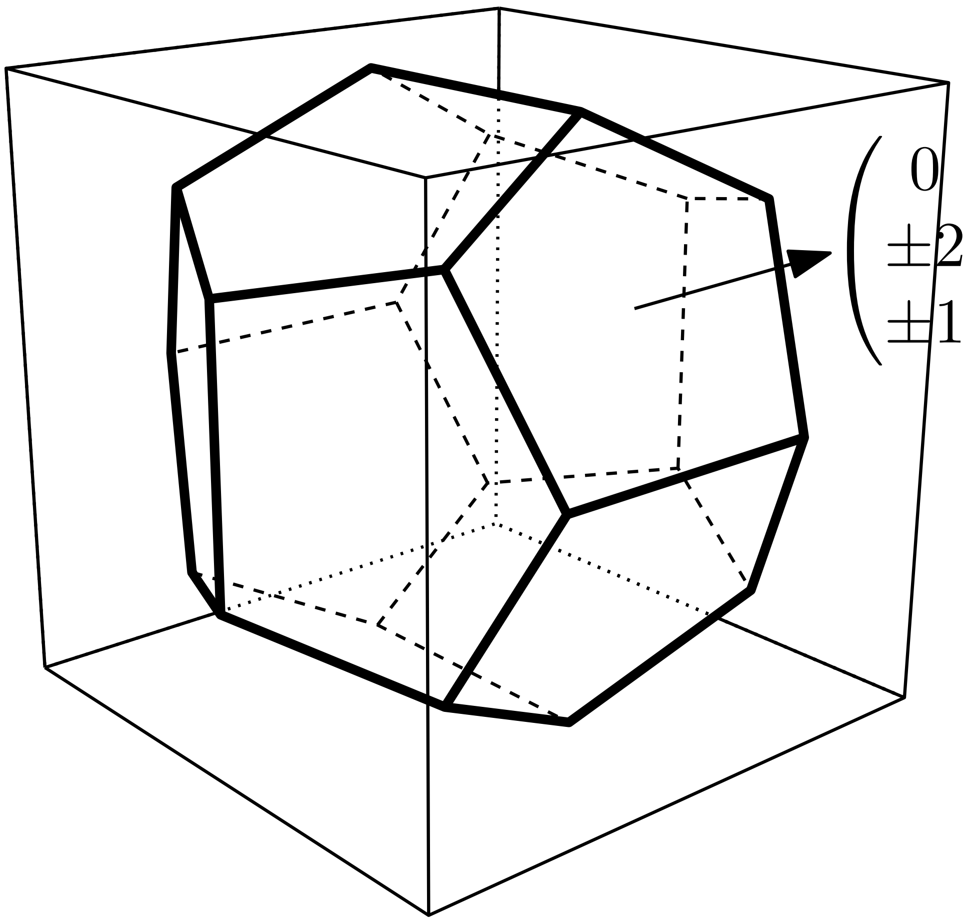
$\leftarrow$ in a $4 \times 24 \times 28$ box (done by hand)

Example: the Dodecahedron



$$\begin{pmatrix} 0 \\ \pm 2 \\ \pm 1 \end{pmatrix}$$

the pyritohedron

$12 \times 12 \times 12$

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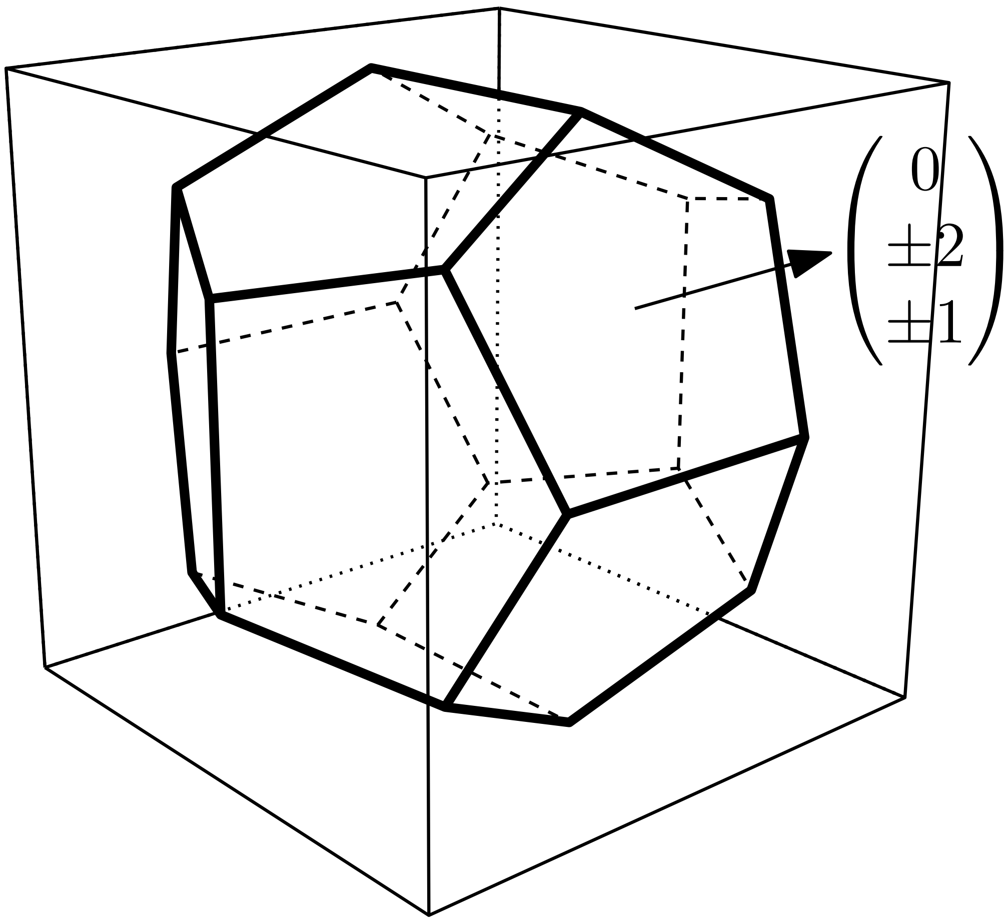
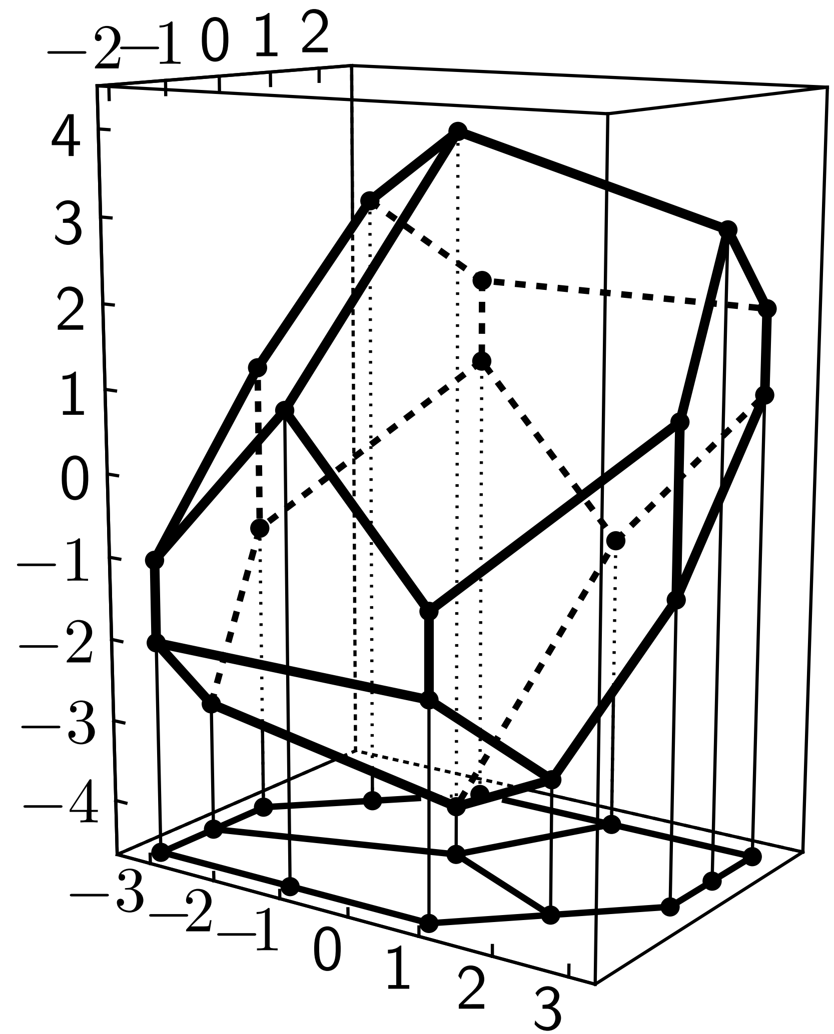




the pyritohedron

$12 \times 12 \times 12$

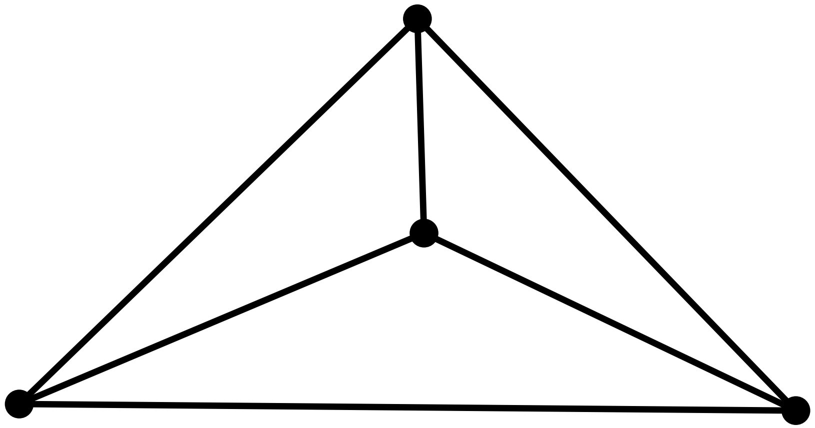
by Francisco Santos

$6 \times 4 \times 8$

Stacked Polytopes (Planar 3-Trees)

Start with K_4

Repeatedly insert a new degree-3 vertex into a face.



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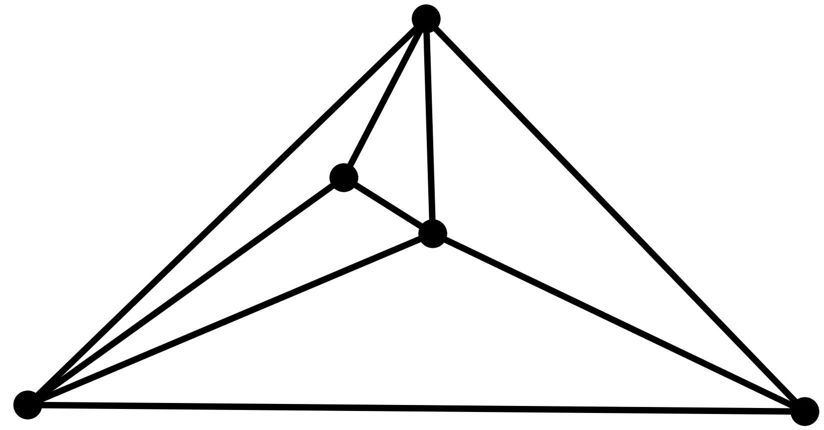

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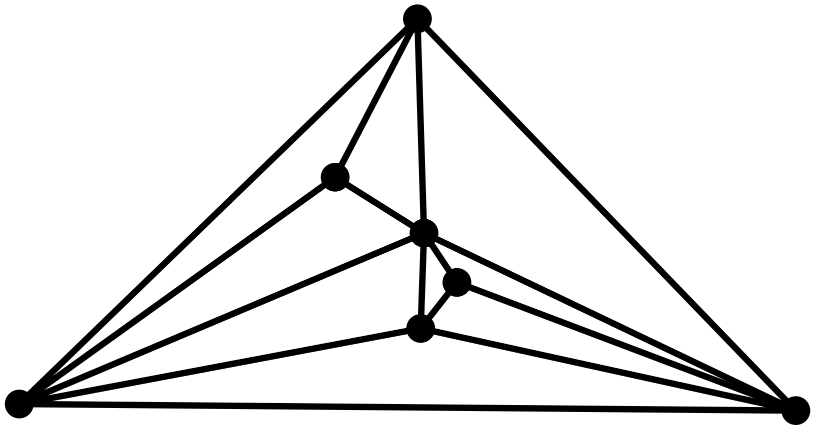

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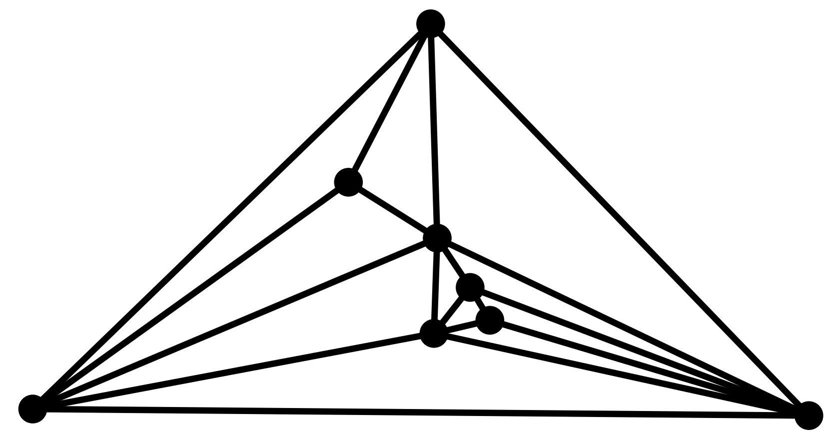

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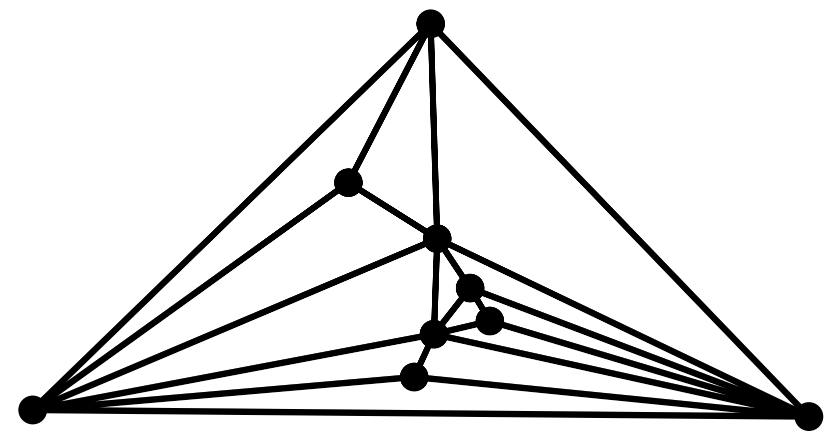

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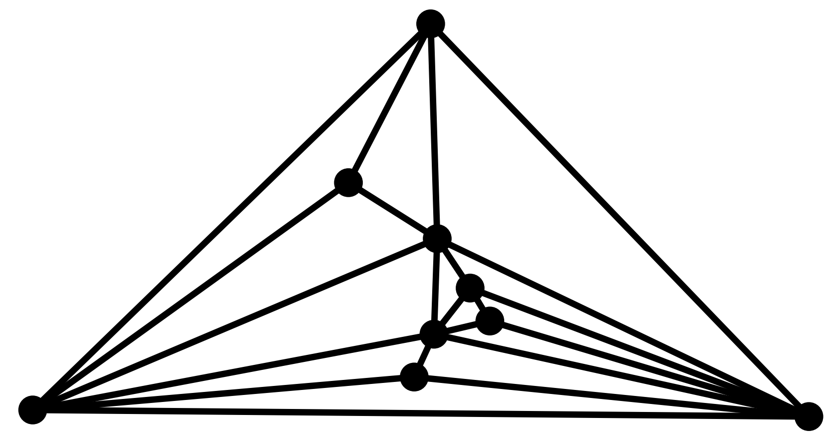

A stacked polytope with n vertices can be realized on an $O(n^4) \times O(n^4) \times O(n^{18})$ grid. [Demaine & Schulz 2011]

Main idea: Recursive bottom-up procedure.
Choose appropriate *areas* for the planar drawing.
Then lift each vertex high enough.

OPEN:
Can every (triangulated) polytope be realized on a polynomial-size grid?

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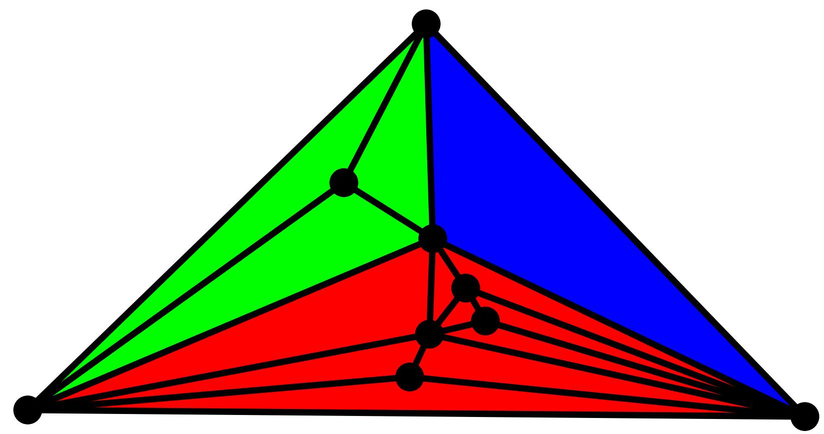

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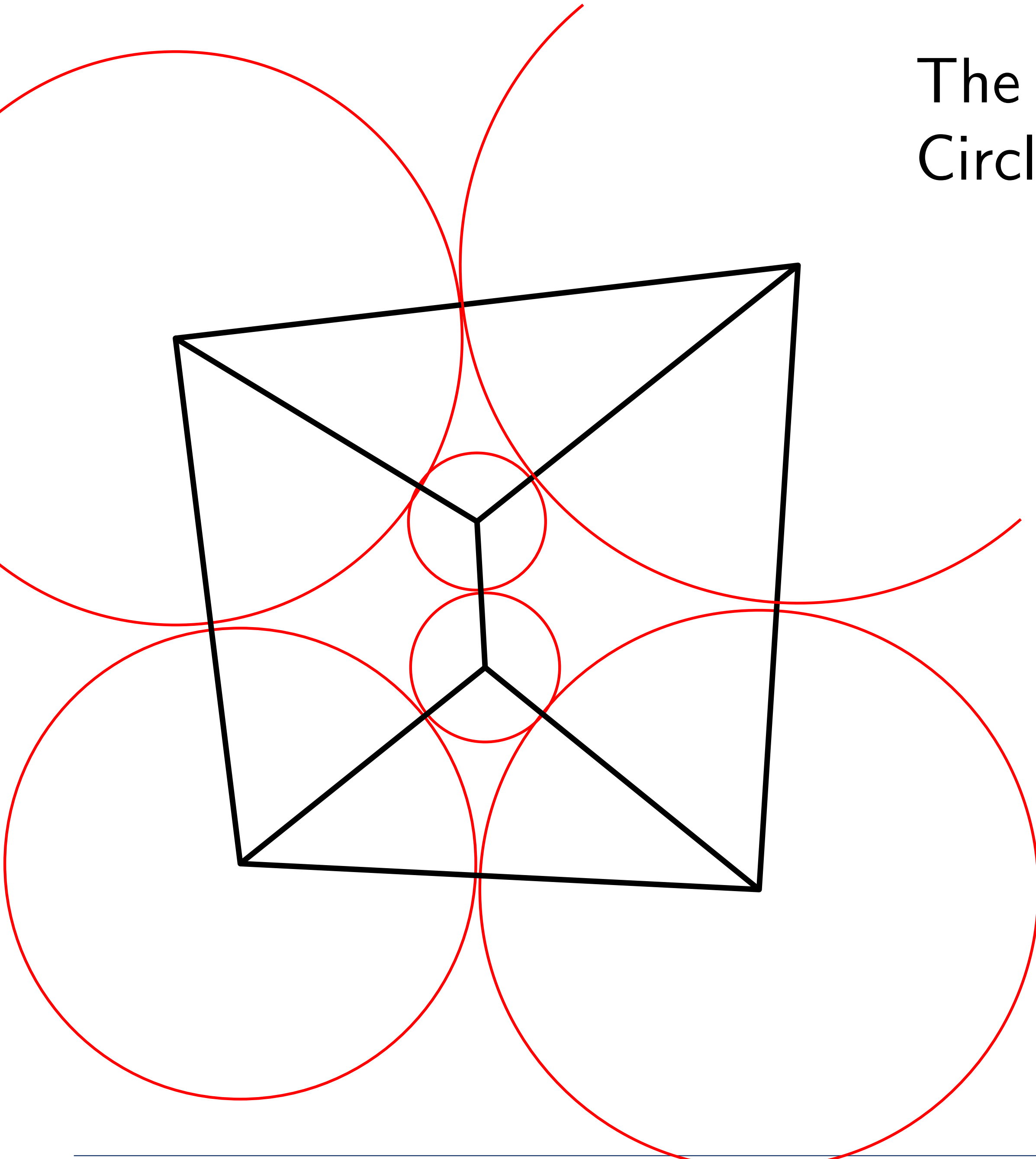
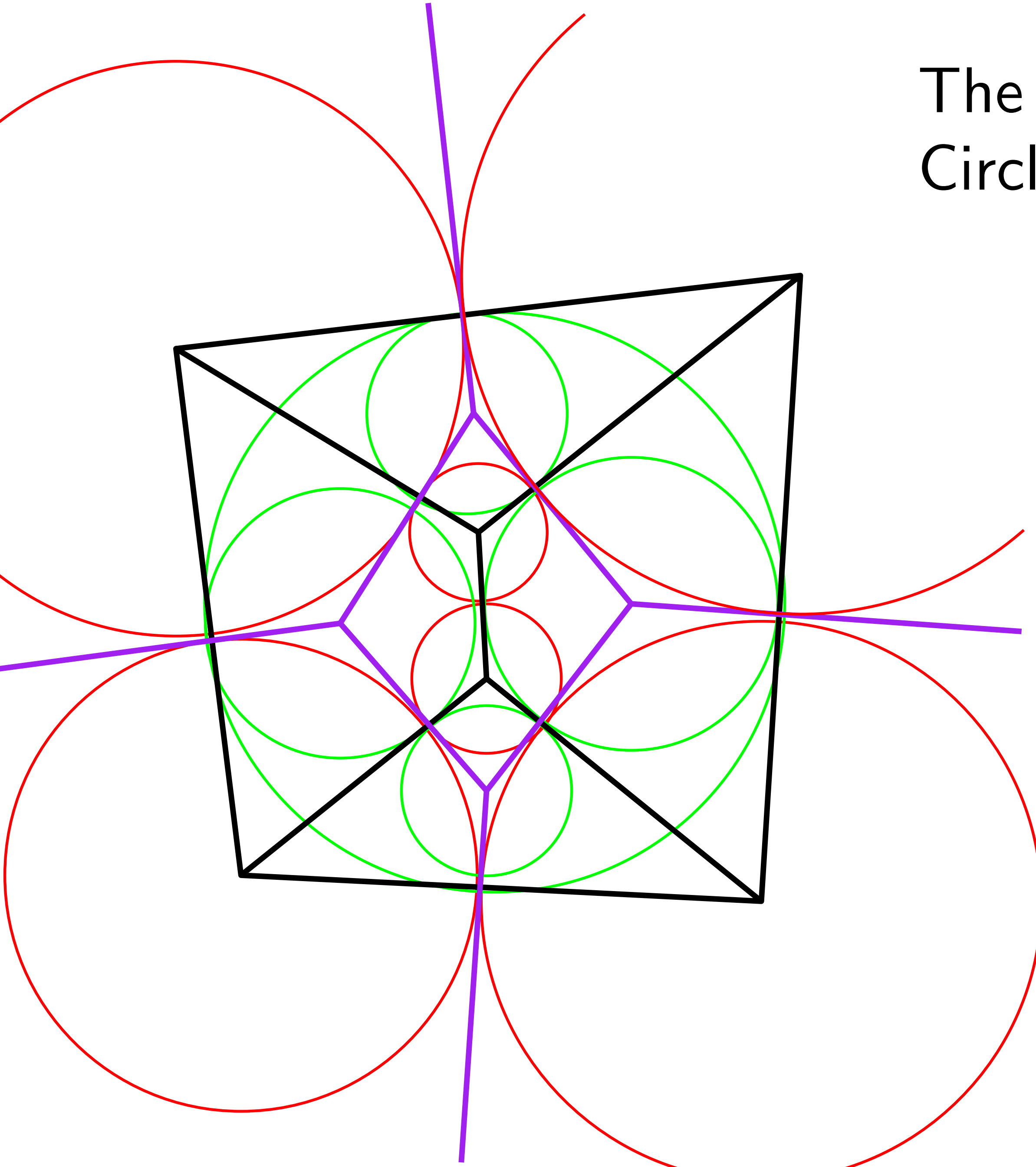
Circle Packings



The Koebe–Andreyev–Thurston Circle Packing Theorem (1936):

Circle Packings

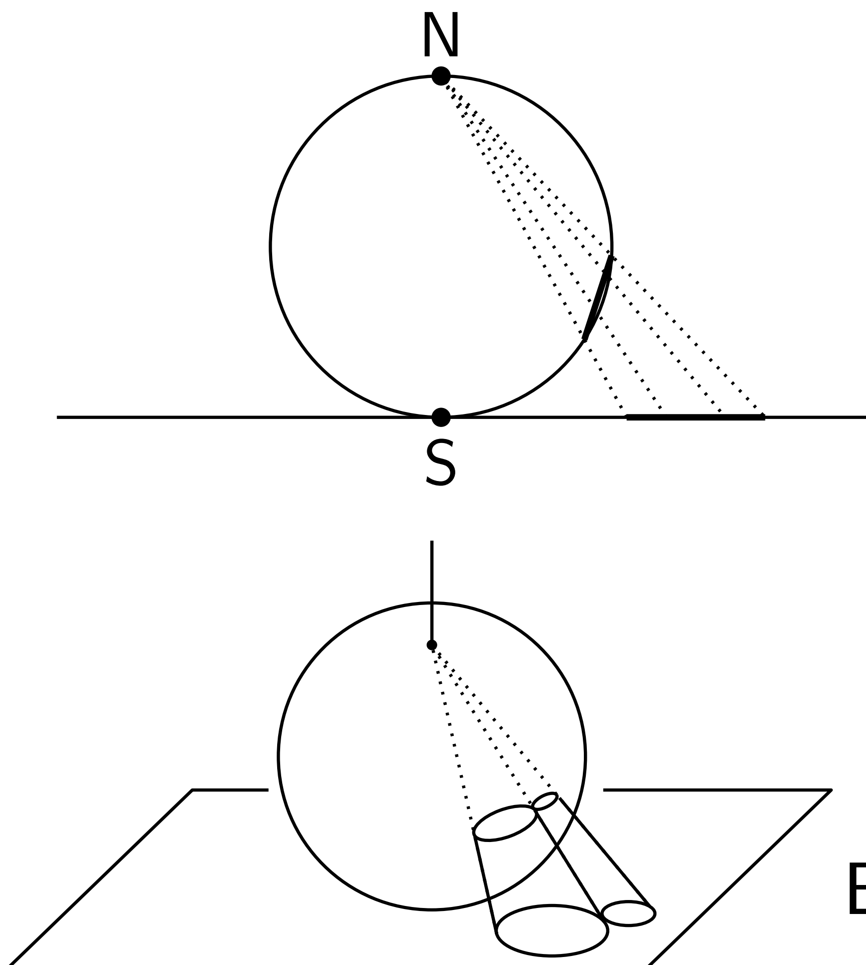


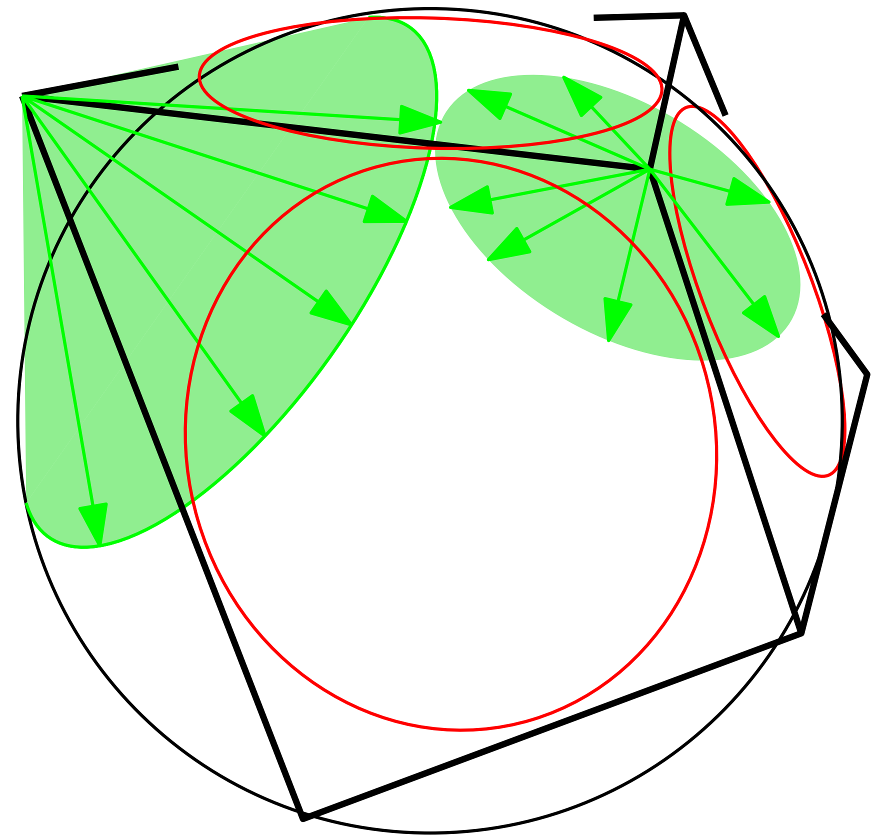
The Koebe–Andreyev–Thurston Circle Packing Theorem (1936):

Every planar graph can be realized as a point contact graph of circular disks.

Simultaneously also the dual graph.

Stereographic Projection





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unique up to Möbius transformations.

Stereographic Projection

Every 3-polytope can be realized with
edges tangent to the unit sphere.

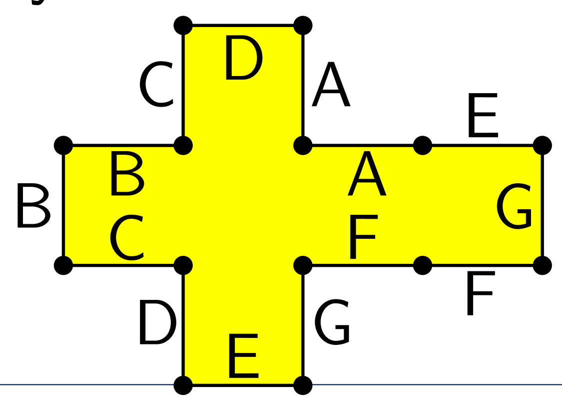
unique up to Möbius transformations.

In addition: barycenter of vertices lies at the sphere center.
[Schramm 1992 (?)]
$\rightarrow$ polytope becomes unique up to reflection.

Extensions of Steinitz' Theorem

- specify the shape of a face [Barnette & Grünbaum 1969]
- choose the edges on the shadow boundary [Barnette 1970]
- respect *all* symmetries of the graph [Mani 1971]
  [follows also from Schramm 1992]
- specify the x-coordinates of vertices (under restrictions)

- with all edge lengths integer? [OPEN]
- specify face areas and directions (but *not* the graph)
  [Minkowski 1897]
- specify the metric on the surface (but *not* the graph)
  [Alexandrov 1936]



Extensions of Steinitz' Theorem

Specifying the x-coordinates of vertices:

- There must be only one local minimum and one local maximum of x-coordinates.

$$\left(\sum_{j\sim i}\omega_{ij}\right)\cdot\mathbf{v}_i=\sum_{j\sim i}\omega_{ij}\mathbf{v}_j$$

IDEA: Use this equation to compute some ω's for given x-coordinates. [Chrobak, Goodrich, Tamassia 1996]

see also [A. Schulz, GD 2009]

A polytope with given x-coordinates exists if

- adjacent vertices have distinct x-coordinates, and
- the minimum and the maximum are incident to a common triangle.

OPEN: Can the last constraint be removed?