Solution of Problem 74 (Volume 41, p. 207 (1994)))

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Two players play the following game. The *dealer* cuts pieces of length $1/2^i$ from a rope of length $s > 1$ and gives them to the *placer*, who puts them on one of the intervals between $k/2^i$ and $(k+1)/2^i$. We present a strategy for the placer for covering the whole unit interval $[0, 1]$ whenever $s \geq 2$. On the other hand, no such strategy exists whenever $s < 4/3$. This is shown by a strategy for the dealer which prevents the placer from winning.

The upper bound. Our winning strategy for the placer, the so-called *method of the third segment*, is a special case of a more general algorithm for covering the unit interval by q-adic subintervals, as opposed to binary subintervals which we consider here. It is described in our paper: "On-line q-adic covering by the method of the n-th segment and its application to on-line covering by cubes", to appear in *Beiträge zur Algebra und Geometrie* **37** (1), 1996. The proof given below is different from the proof in that paper.

The algorithm tries to cover the interval from left to right, but in a slightly modified way. Assume that the contiguous subinterval starting at the left end which is completely covered extends from 0 to b. (Initially $b = 0$.) We call b the *current boundary*. Then the *first segment* of length $1/2^i$ is the interval $[k/2^i, (k+1)/2^i]$ with $k = \lfloor b2^i \rfloor$. This is the leftmost position into which a piece of this length can meaningfully be placed. The second and third segment are the adjacent intervals $[(k+1)/2^i, (k+2)/2^i]$ and $[(k+2)/2^i, (k+3)/2^i]$. By the method of the third segment the placer always tries to put a new piece on the third segment, unless this interval is already completely covered (or lies outside the unit interval). In this case, the placer tries the second segment, and if this also makes no sense because it is totally covered, the piece is put on the first segment.

Lemma 1. *The third segment of length $1/2^i$ may contain at most one piece of length $1/2^i$, but no smaller pieces.*

Proof. The distance from the left endpoint of the third segment to the current b is larger than $1/2^i$. Hence the third segment of any smaller length can never have reached far enough to the right to overlap this segment.

Lemma 2. *Consider the second segment of length $1/2^i$, denoted by Q. The total length of pieces lying in Q is less than $2/2^i$. (We might say, the segment Q is covered with average density less than 2.) Moreover, if Q is not completely covered, then the total length of pieces lying in Q is less than $1/2^i$. (Q is covered with average density less than 1.)*

Proof. We prove this by induction on the length of the segment Q, starting with the smallest piece which has been cut by the dealer, for which the lemma is obvious. The two halves of Q may either be the second and third segments of length $1/2^{i+1}$, or the the third and fourth segments of length $1/2^{i+1}$. The latter case can be subsumed under the former case, because the fourth segment never contains any pieces at all. Henceforth we will assume that the first case holds.

In the case when Q is not completely covered, either the left half or the right half is not completely covered either. We apply the induction hypotheses of the present lemma and Lemma 1 to obtain the bound $1/2^{i+1} + 1/2^{i+1}$ or $2/2^{i+1} + 0$, respectively, which is at most $1/2^i$.

Consider now the case when Q is completely covered. When Q is covered by a piece P of length $1/2^i$, we know that, before P was placed, Q was not completely covered, and we may add the length of P to the above bound, obtaining $1/2^i + 1/2^i \leq 2/2^i$. When Q is not covered by a piece of length $1/2^i$, we may bound the two halves individually by induction and Lemma 1, respectively, and we obtain the bound $2/2^{i+1} + 1/2^{i+1} \leq 2/2^i$.

Note that the current boundary b advances only when a piece is placed on the first segment. The following lemma is crucial for bounding the total length of pieces used.

Lemma 3. *Consider the situation when a piece P is placed on the first segment and the boundary advances from b to $b' < 1$. Then the total length of all pieces overlapping the interval $[b, b']$ is at most $2(b' - b)$.*

Proof. Assume that P has length $1/2^i$, and let $1/2^j \geq 1/2^i$ be the length of the largest piece overlapping the interval $[b, b']$. Thus all pieces that we have to measure have length at most $1/2^j$. By Q we denote the rightmost piece of length $1/2^j$. At the time when Q was placed, it must have been placed on the third segment, and in particular, $Q \neq P$. (Otherwise, the third segment for Q would already be covered by a segment of length at least $1/2^j$, contradicting the choice of Q.) From Lemma 1 we conclude that Q overlaps no smaller pieces, and therefore, the interval occupied by Q is covered with density 1.

Let the first, second, and third segments of length $1/2^j$, at the time before P is placed, be $[a_0, a_1]$, $[a_1, a_2]$, and $[a_2, a_3]$. We have $a_0 \leq b < a_1$, and Q lies either on $[a_1, a_2]$ or $[a_2, a_3]$.

Case 1. Q lies on $[a_2, a_3]$. We have $b' = a_3$. By Lemma 2, the interval $[a_1, a_2]$ is covered with average density at most 2. The same is true for the interval between b and a_1 (before P is placed): The covered part in this interval can be decomposed into a disjoint set of maximal dyadic intervals. Each such interval of length $1/2^n$ is either the second or the third segment of length $1/2^n$ (otherwise it could not have been covered) and therefore Lemma 1 or 2 can be applied. Therefore the length of P plus the total length of all pieces contained in the interval $[b, b']$ is at most

$$1/2^i + 2(a_1 - b) + 2(a_2 - a_1) + (a_3 - a_2) = 1/2^i + 2(b' - b) - 1/2^j \leq 2(b' - b).$$

Case 2. Q lies on $[a_1, a_2]$. By Lemma 1 and the assumption that Q is the rightmost segment, we know that $[a_2, a_3]$ is covered by no segment, and thus we have $b' = a_2$. Arguing similarly as above, and taking into account that $[a_1, a_2]$ is covered only by Q, we obtain the bound

$$1/2^i + 2(a_1 - b) + (a_2 - a_1) = 1/2^i + 2(b' - b) - 1/2^j \leq 2(b' - b).$$

With these lemmas we can now prove the following theorem.

Theorem. *When the total length of all pieces cut by the dealer is at least 2, the method of the third segment guarantees that the whole unit interval is covered.*

Proof. The total length of all pieces lying between 0 and the current boundary b is at most $2b$: No piece is ever placed to the left of the current boundary, and whenever the current boundary b advances, the new pieces to the left of b are accounted for by Lemma 3.

On the other hand, the total length of all pieces lying between the current boundary b and 1 is at most $2(1 - b)$: The covered part to the right of b can be partitioned into a disjoint set of maximal dyadic intervals. Each such interval of length $1/2^i$ is either the second or the third segment of length $1/2^i$ and therefore Lemma 1 or 2 can be applied.

The lower bound. The strategy for the dealer which proves the lower bound 4/3 is given explicitly in Figure 1. Every entry in the table is identified by a number between 1 and 53. The lower part shows a possible *situation* of a partially covered unit interval. The number x_i on the right side of the i-th entry means that the dealer has a winning strategy in this situation if the remaining length of the rope is smaller than x_i. This strategy consists of cutting a piece whose length is given on the left side. Each box above the picture of the unit interval indicates a possible position where the placer can put the piece, together with the number of the successor situations to which this placement leads. So the dealer can simply refer to the respective entry to decide what to do next.

For example, in situation 3, the dealer cuts a piece of length 1/8, which may lead to situations 12, 7, or 13 (or again 3, if the placer wants to play stupid). One can check that the remaining rope, $x_3 - 1/8$, is not bigger than x_{12}, x_7, and x_{13}. The white boxes (in this case, the last four eighths) indicate the choices where equality holds: $x_3 - 1/8 = x_{13}$. They are the "optimal" choices for the placer against this strategy for the dealer.

When the two subintervals $[0, 1/2]$ and $[1/2, 1]$ in a situation are exchanged, this is of course irrelevant for the game: Both the dealer and the placer can adapt their strategies to such a change. The same is true when the two halves of any dyadic subinterval are exchanged. Therefore only one of many equivalent situations has an entry in the table. For example, when the placer in situation 3 puts the piece in the rightmost eighth, this leads to a situation which is not listed explicitly, but it can be changed to situation 13 by interchanging the last two eighths, and then the last two quarters. Note however that situations 4 and 5, for example, are not equivalent in this sense: the second and third quarters cannot be exchanged because they are not parts of the same half.

When one half of the unit interval is completely covered, we may ignore this half and concentrate on the remaining half. In this case we scale the interesting part to unit size, for example, when we place a piece on the second quarter in situation 32. This is indicated by a star preceding the successor number, *6. The inequality which has to be checked in this case is $x_{32} - 1/4 \leq x_6/2$.

Certain situations are replaced by *dominating* situations in which some additional parts are covered. In terms of the rules of the game, this means that the dealer may choose to cut a piece from another, different rope and put it anywhere in the unit interval, at no cost to the placer. For example, the placer's rightmost choice in situation 13 leads to situation 20 only when the leftmost free sixteenth is covered in addition. Of course this modification of the rules can help only the placer, but it reduces the necessary size of the table. Such cases are indicated by successor numbers in a slanted font. When the dominating situation is reached by another choice of the placer in the same situation, this is indicated by an arrow pointing to that choice, such as in situation 4.

The figure includes only situations which can be reached if the dealer follows the given strategy (and applies the domination rules). For example, the situation where only one sixteenth of the whole interval is covered, will never occur.

Since the dealer always cuts at least a constant fraction from the remaining rope (in fact, more than 3/64), the dealer will eventually cut the whole rope.

We finally remark without proof that the bound of 4/3 is the optimal bound when the dealer is restricted to cut pieces of length at least 1/16 of the smallest remaining dyadic subinterval which is not completely covered. (This means, 1/16 of the whole interval which is shown in the pictures of Figure 1.) By allowing also pieces of half this length, a slightly better bound of 41/30 instead of 4/3 can be shown with the help of a computer.

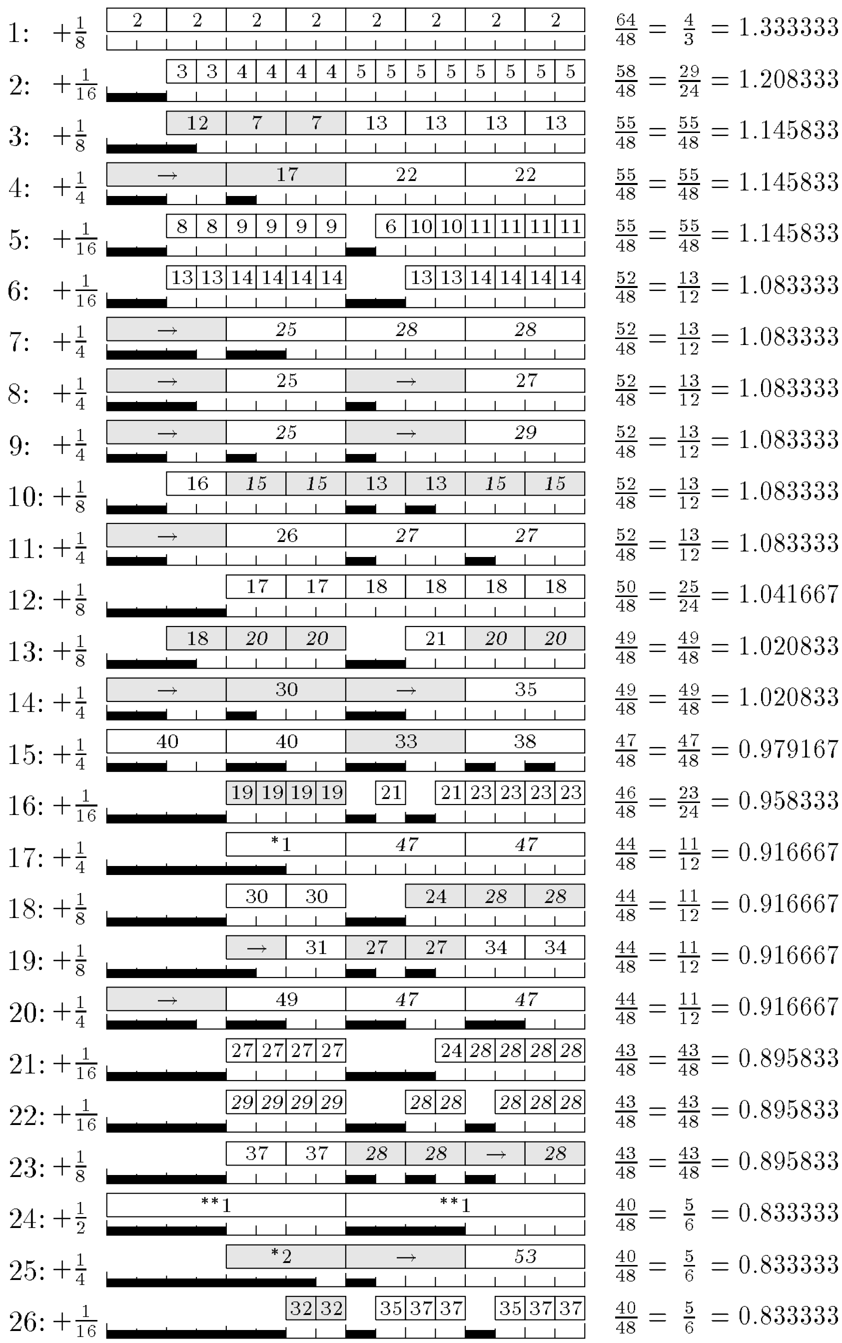

Figure 1 (first part). The strategy for proving the lower bound.
The rightmost column gives the "value" x_i of each situation i.

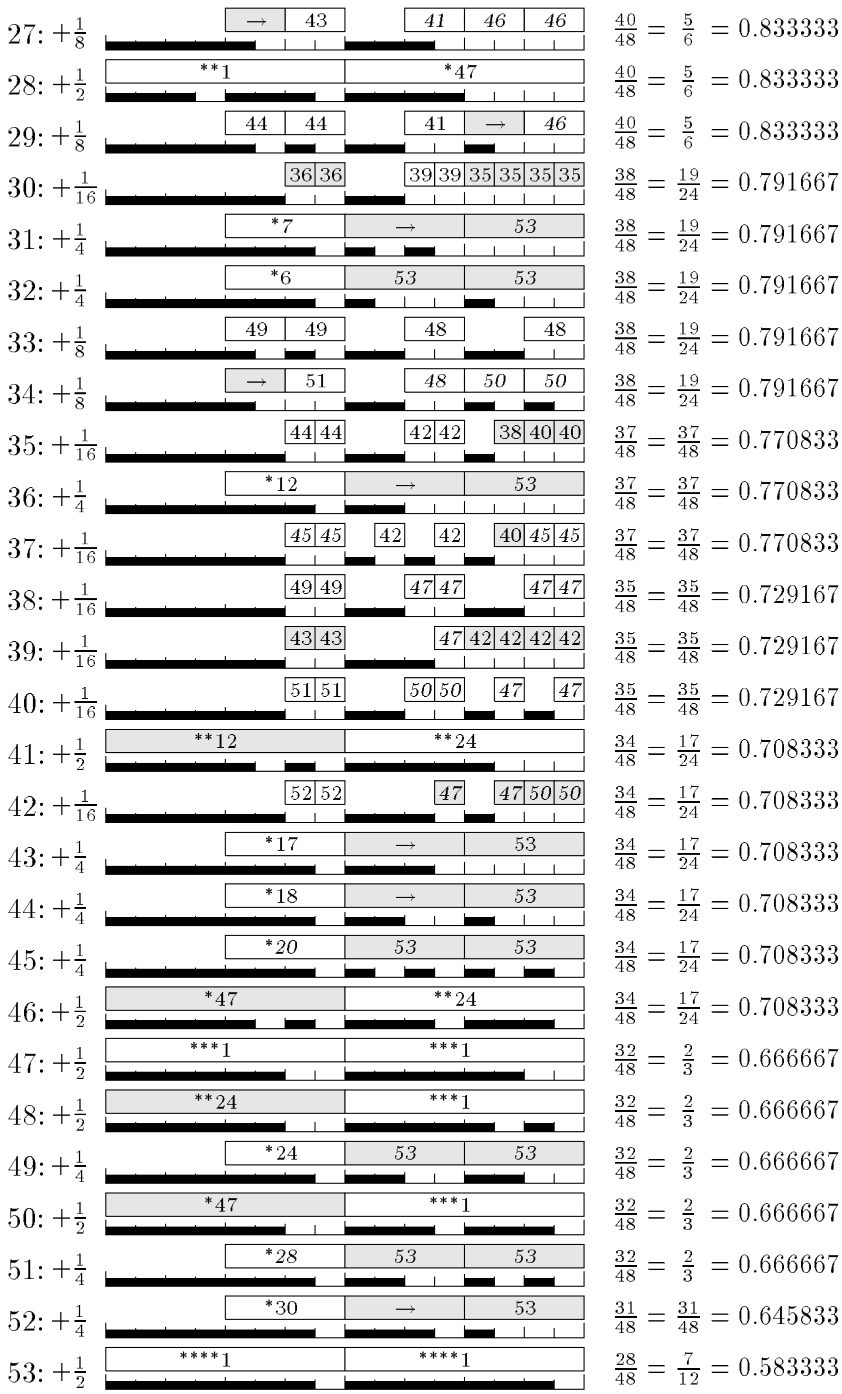

Figure 1 (continued).