Flip Graphs of Bounded-Degree Triangulations

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Freie Universität Berlin, Institut für Informatik

joint work with

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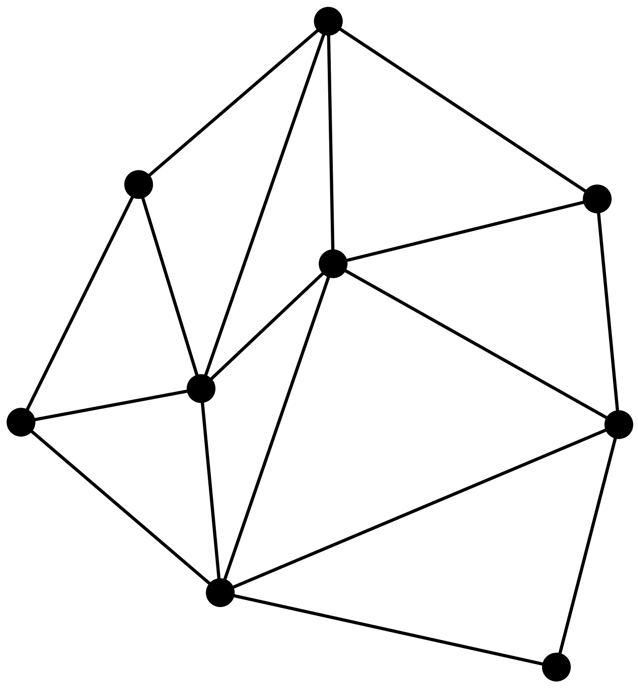
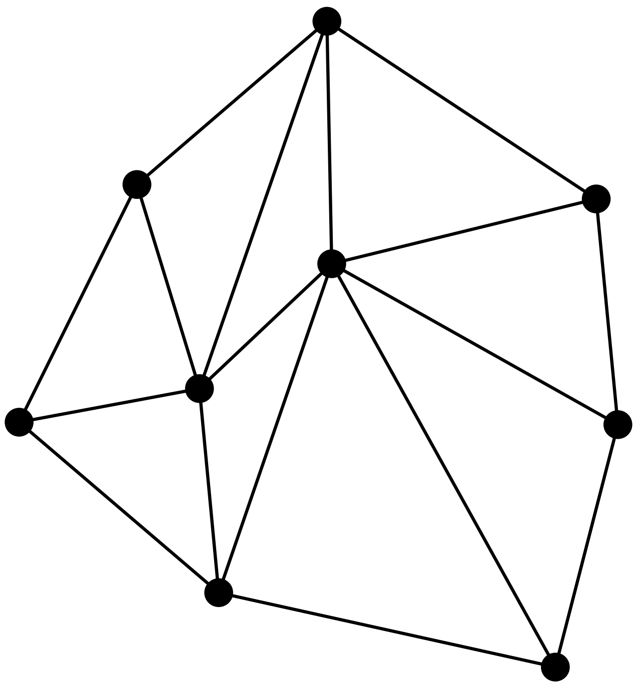
Pedro Ramos, André Schulz, Bettina Speckmann

Edge Flips in Triangulations





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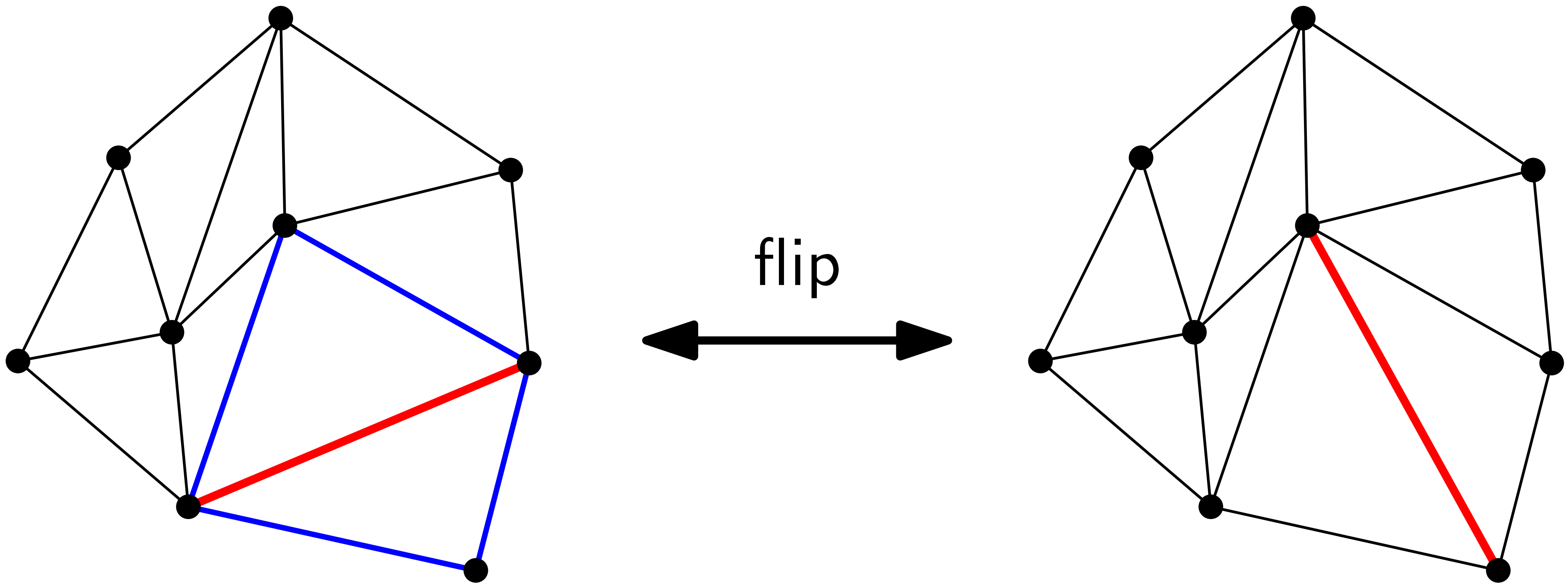
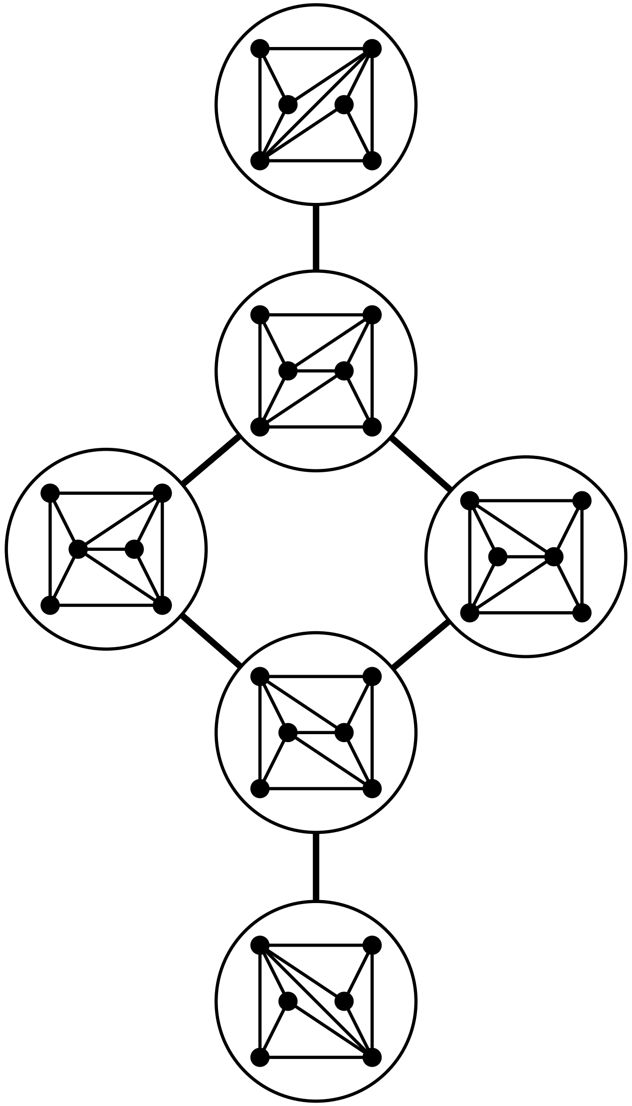


The Flip Graph



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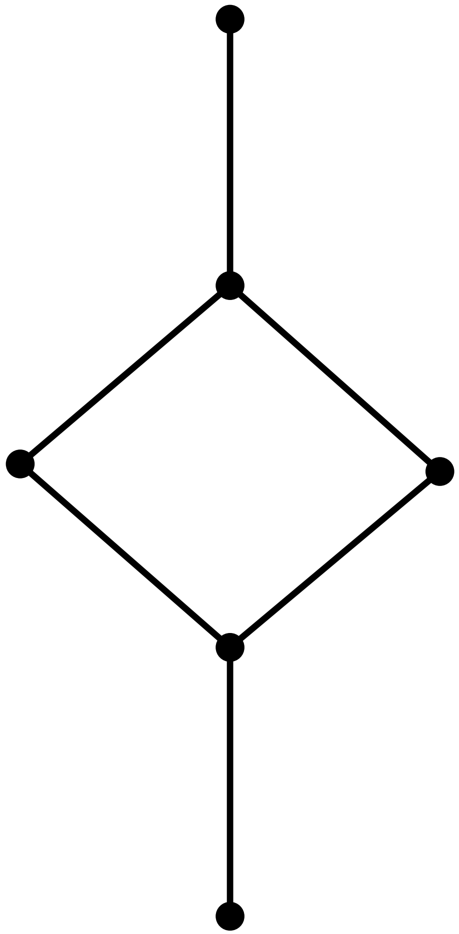

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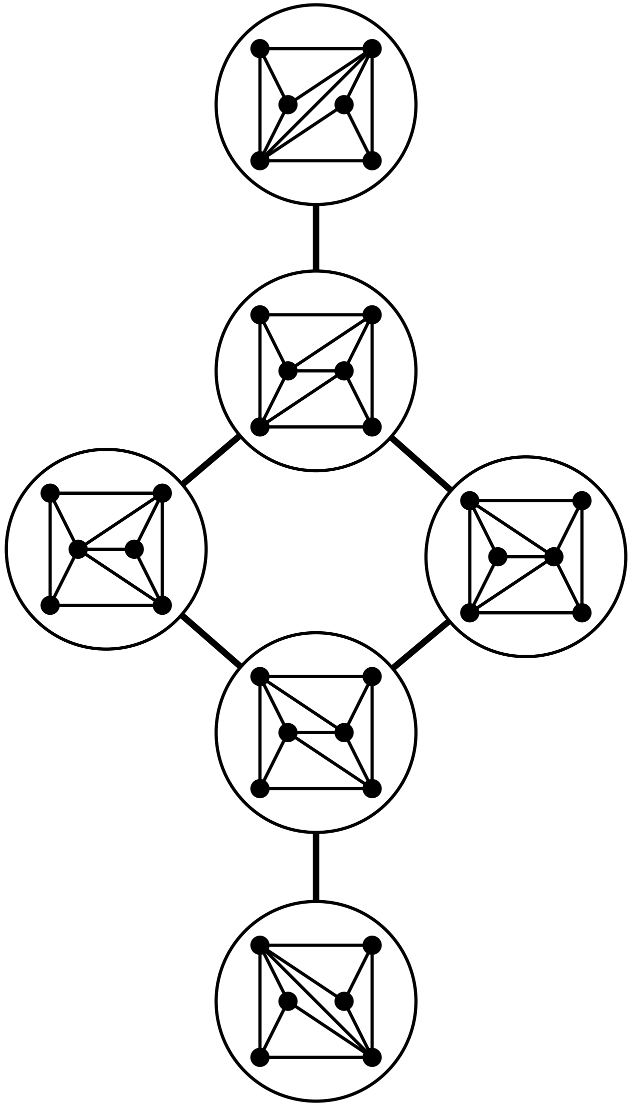

- connected.
- diameter

  $O(n^2)$

  $O(n)$ in convex position

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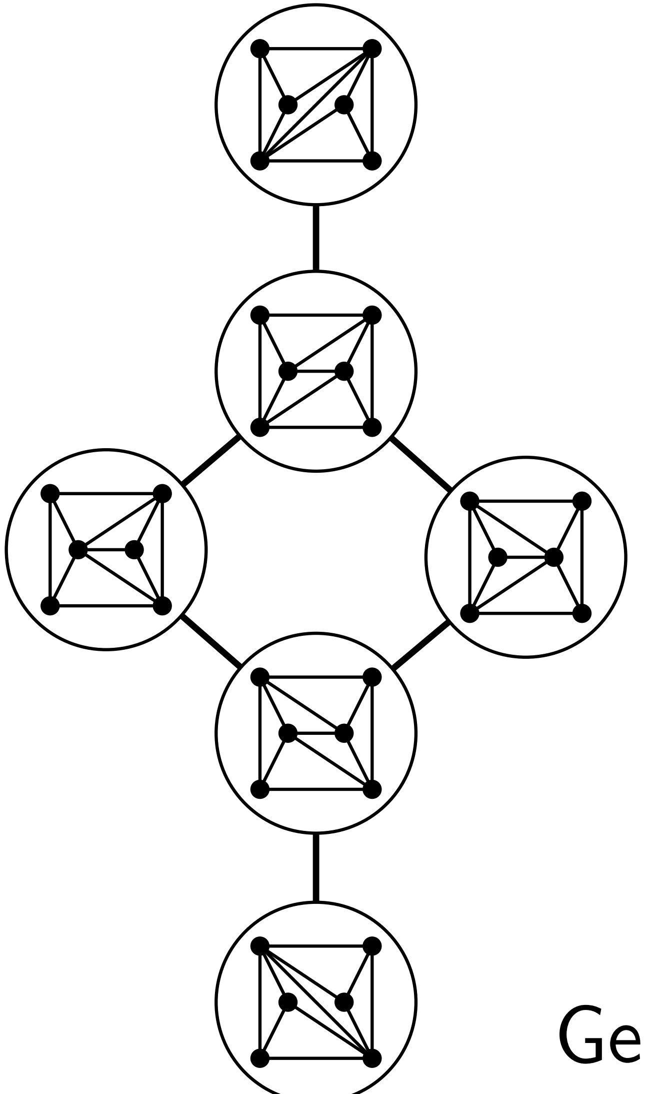

- connected.
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Subgraphs of the flip graph:
maximum degree $\leq k$ (small)

- connected?
- diameter?

General question: Local transformations between combinatorial structures

Results

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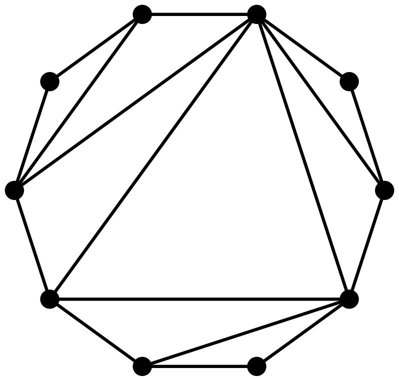


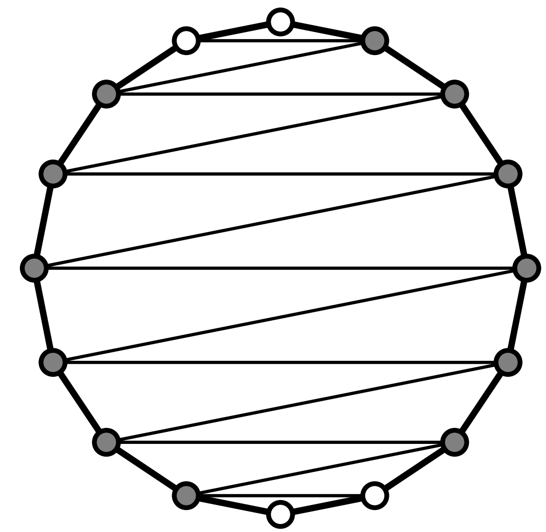
For points *in convex position*, the flip graph of triangulations with degree $\leq k$ is

- **disconnected**, for $k = 4, 5, 6$
- **connected**, for $k \geq 7$. The diameter is $O(n^2)$.

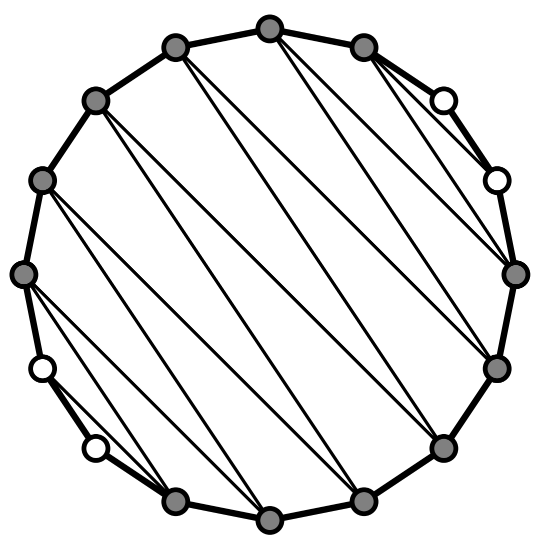
For points in general position, the flip graph of triangulations with degree $\leq k$ can be **disconnected**, for any k.

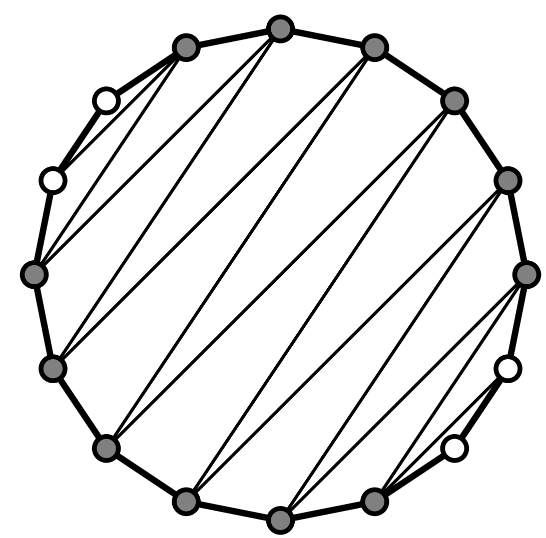
Points in Convex Position

maximum degree $k \leq 4$:
zigzag triangulations



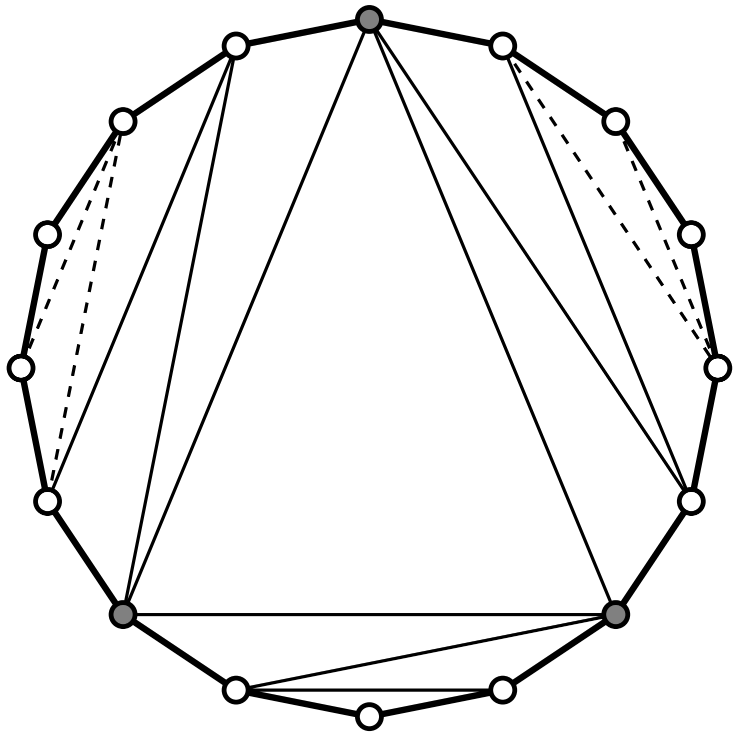
No edge can be flipped without creating a degree-5 vertex.



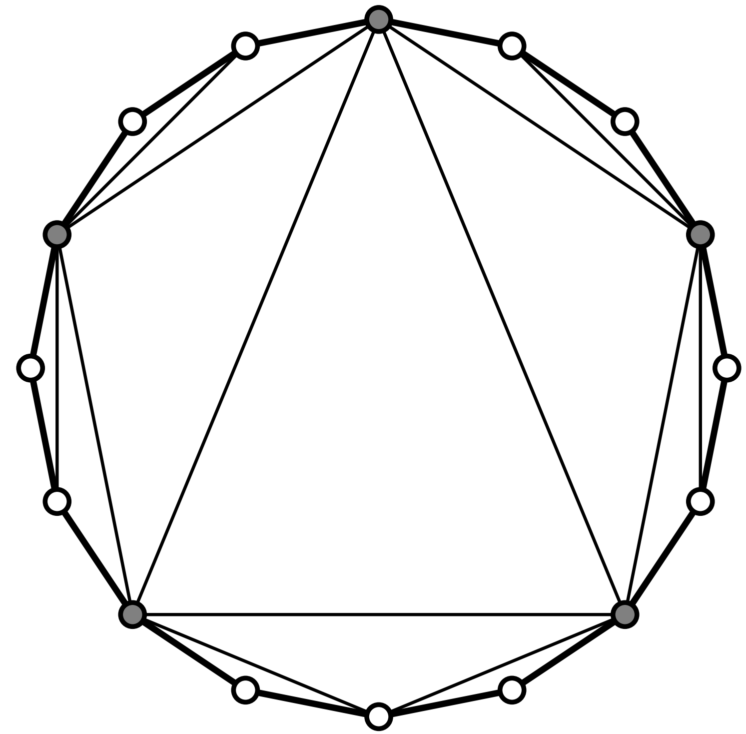


Points in Convex Position

maximum degree $k \leq 5$:



maximum degree $k \leq 6$:



Convex Position, $k \geq 7$

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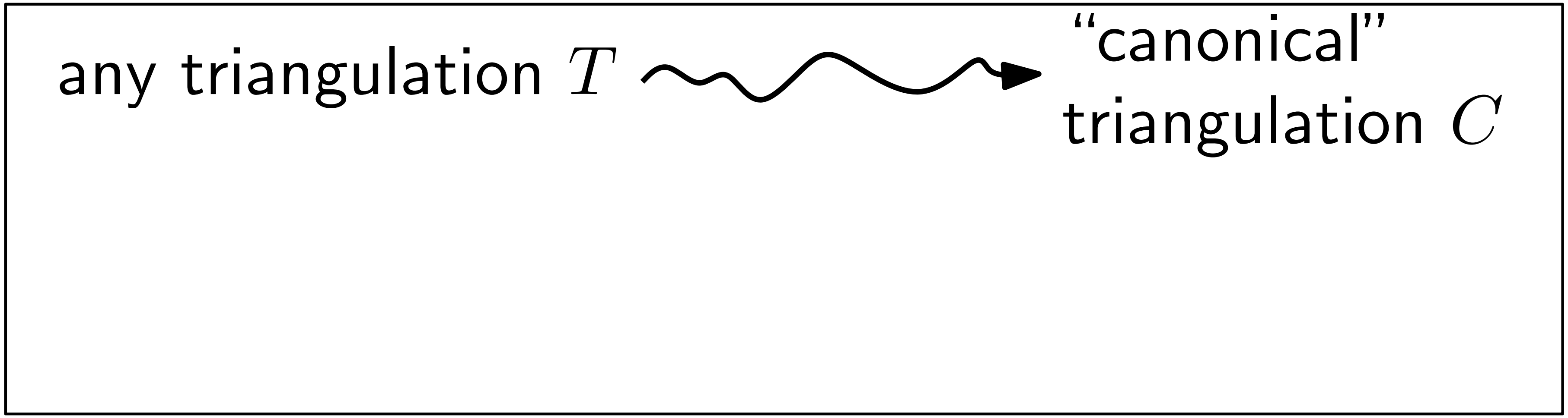
Theorem. Let $k \geq 7$.
Any two triangulations of a set of n points with maximum degree $\leq k$ can be transformed into each other by a sequence of $O(n^2)$ flips, without exceeding vertex degree k.

Proof strategy:



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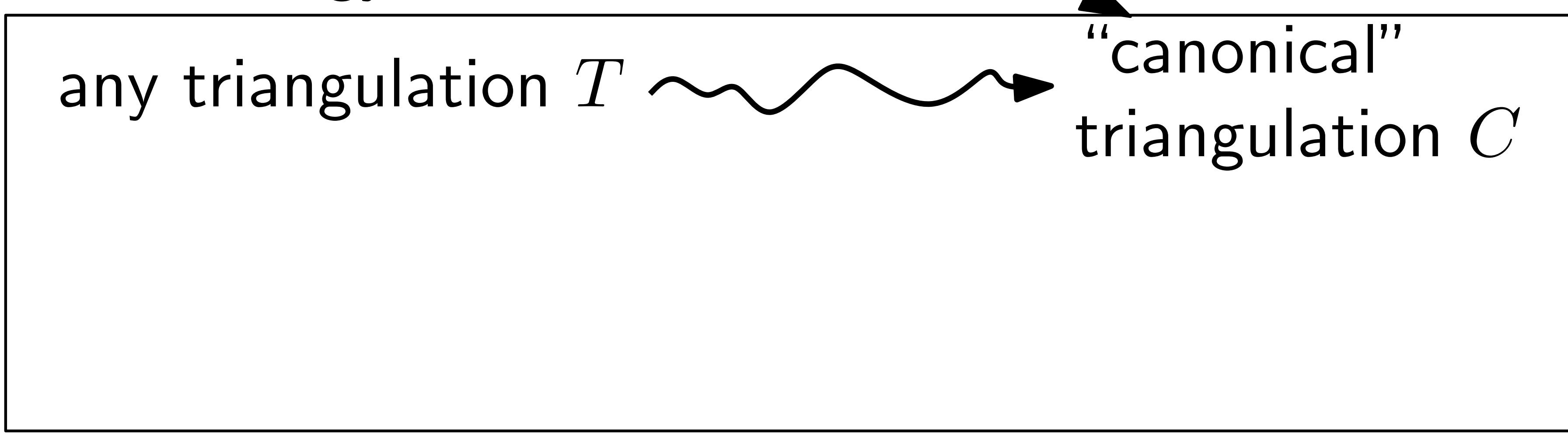

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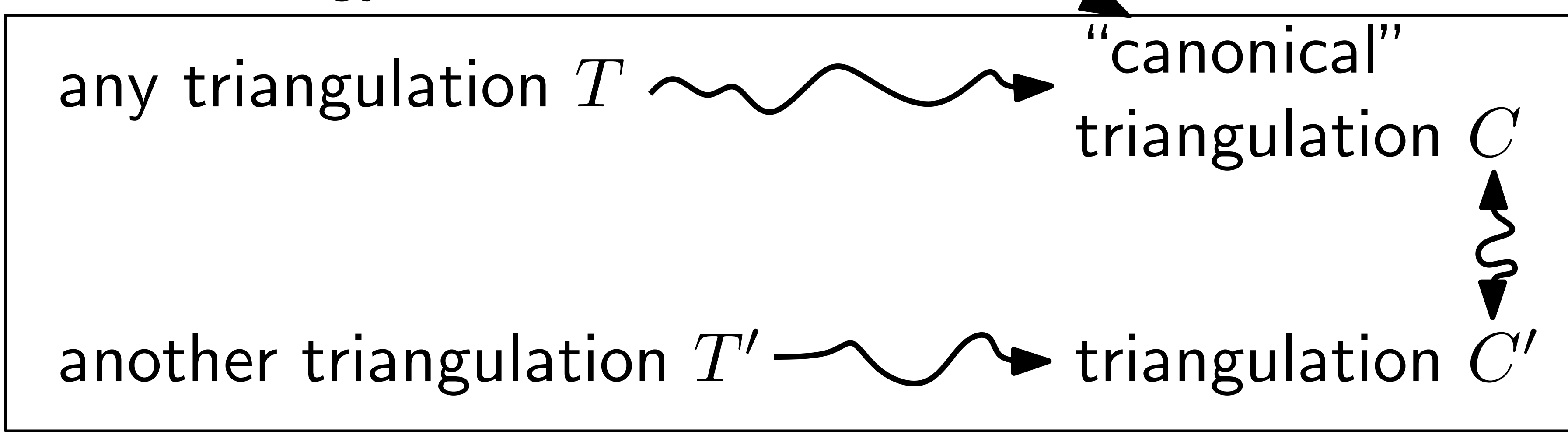

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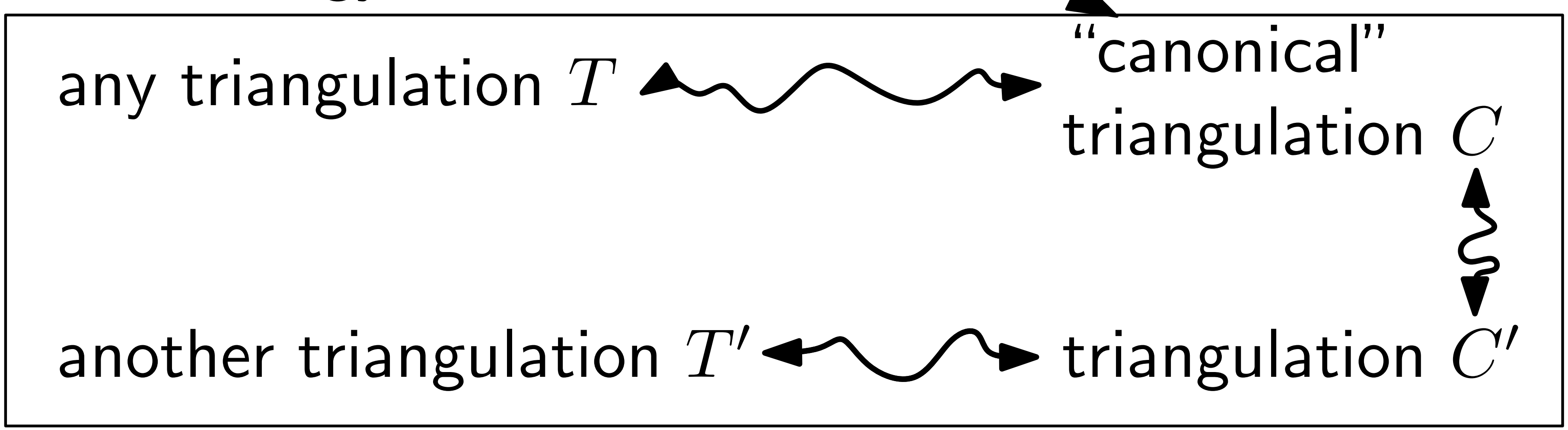
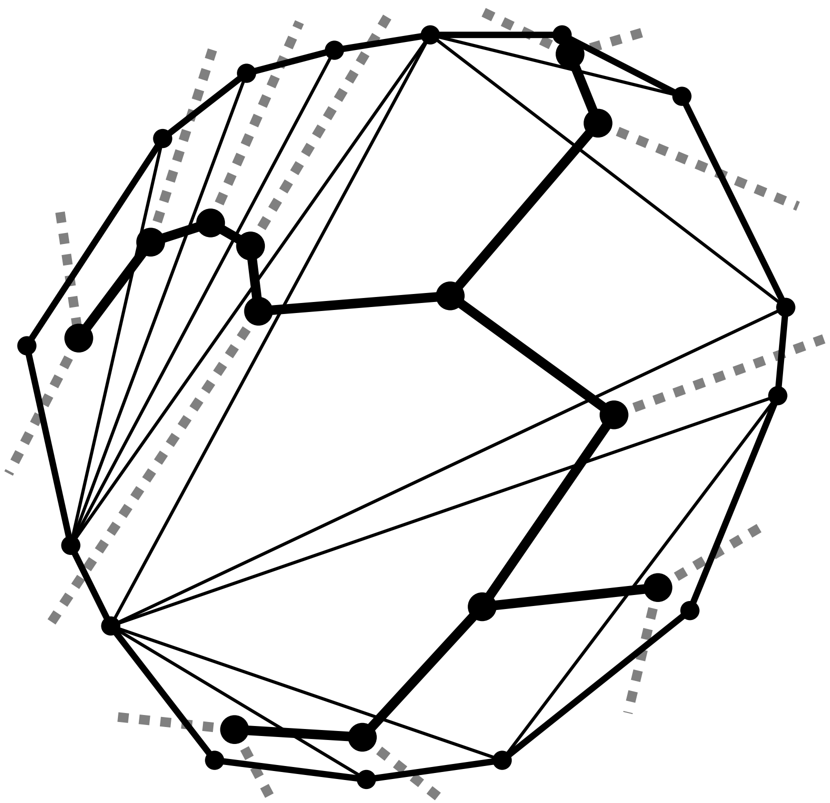


The Dual Graph



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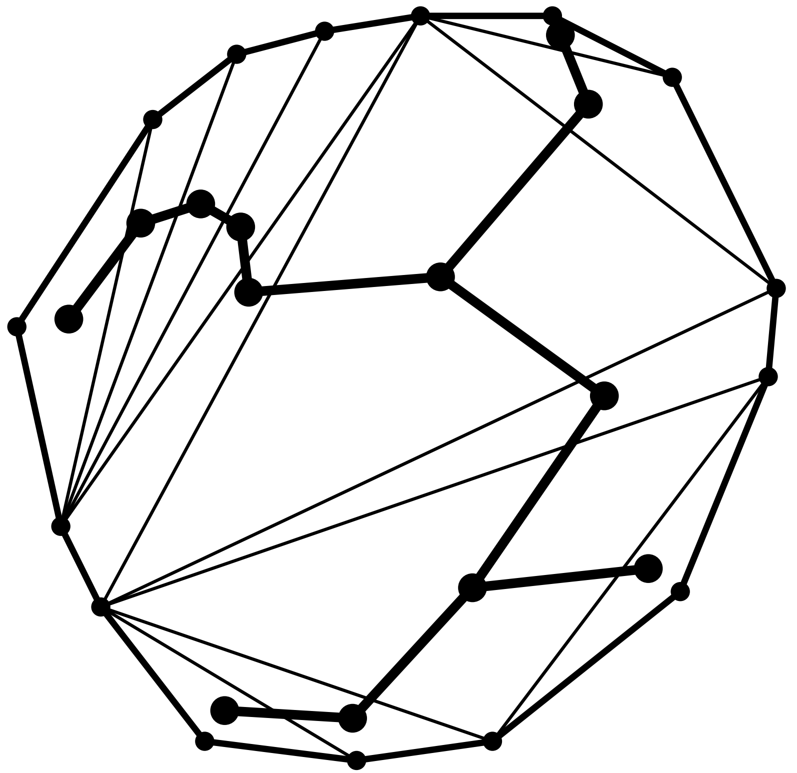

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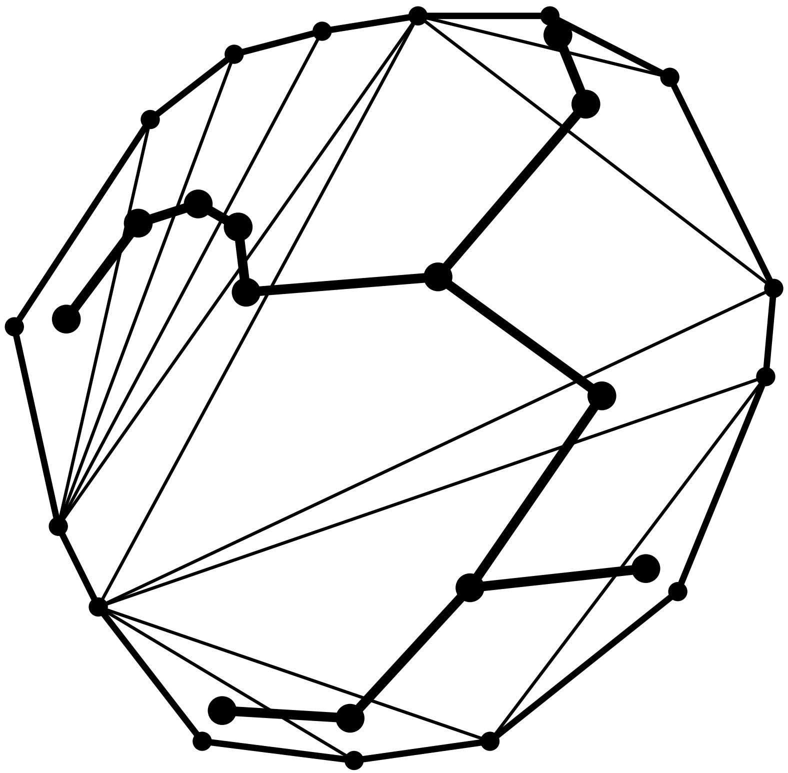
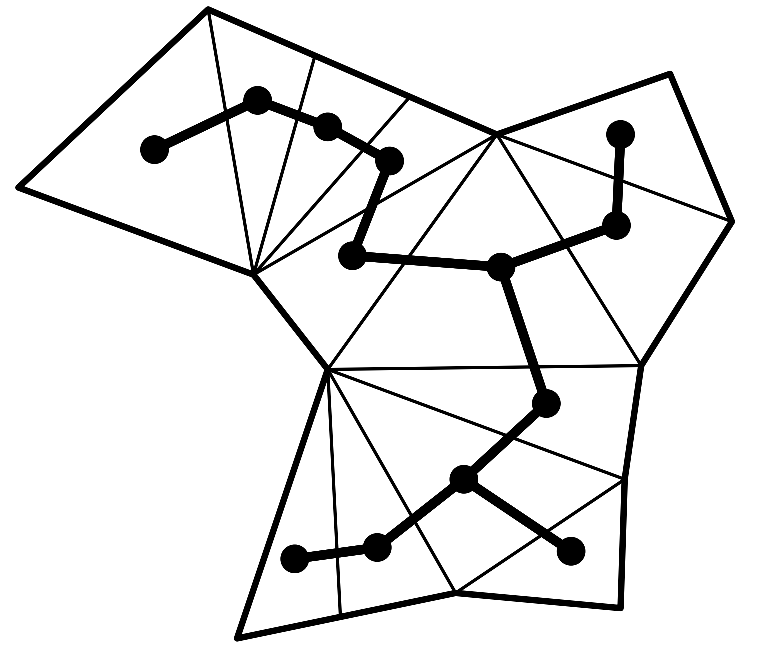


schematlic drawing



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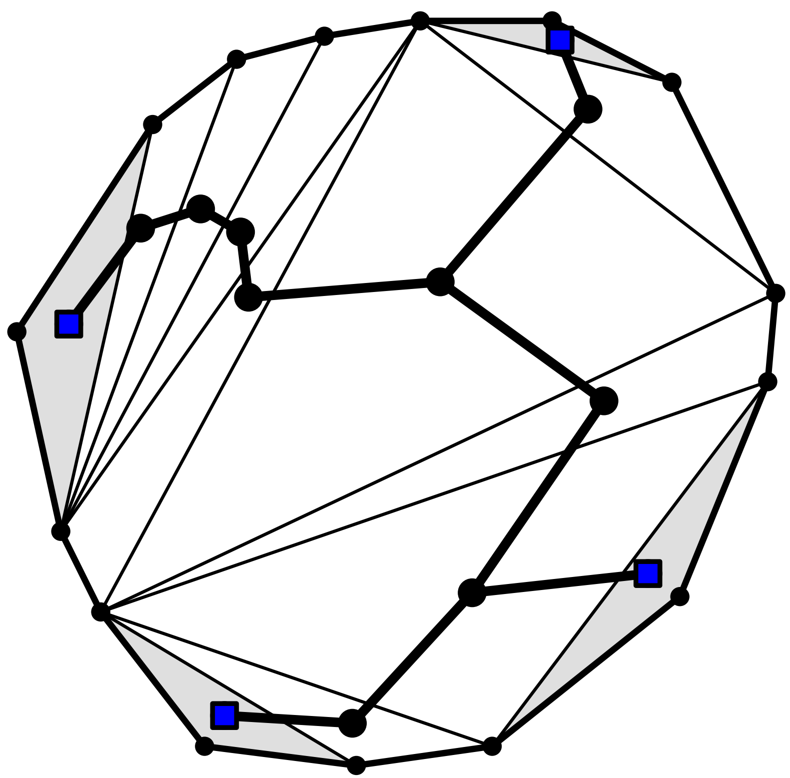
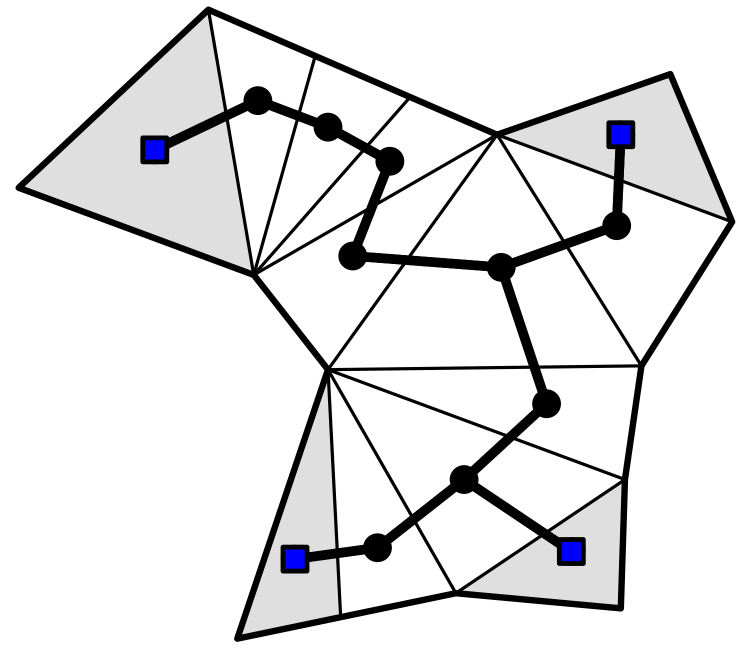


ears (degree 1, leaves)

schematlic drawing



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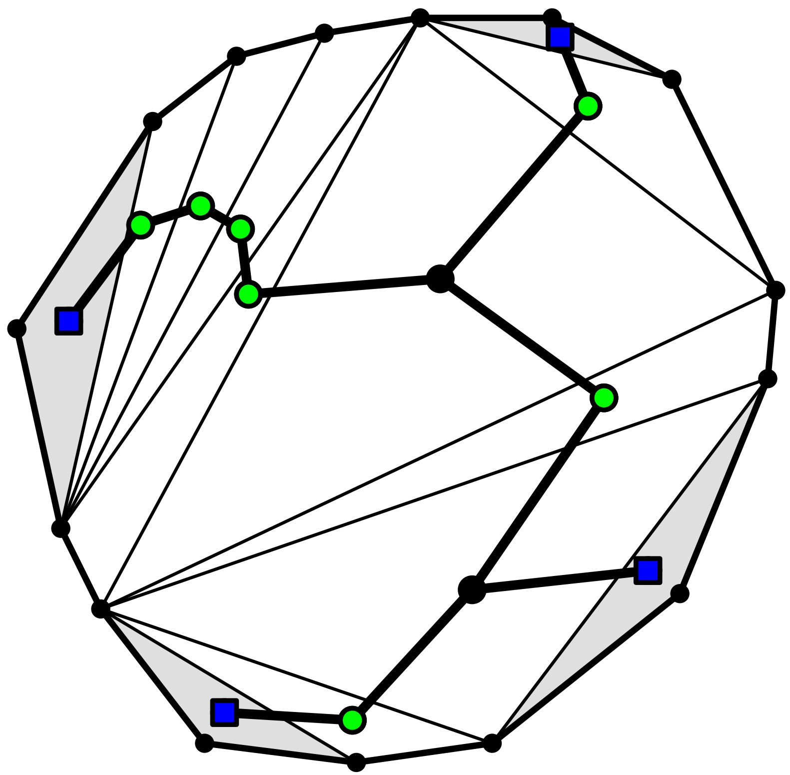

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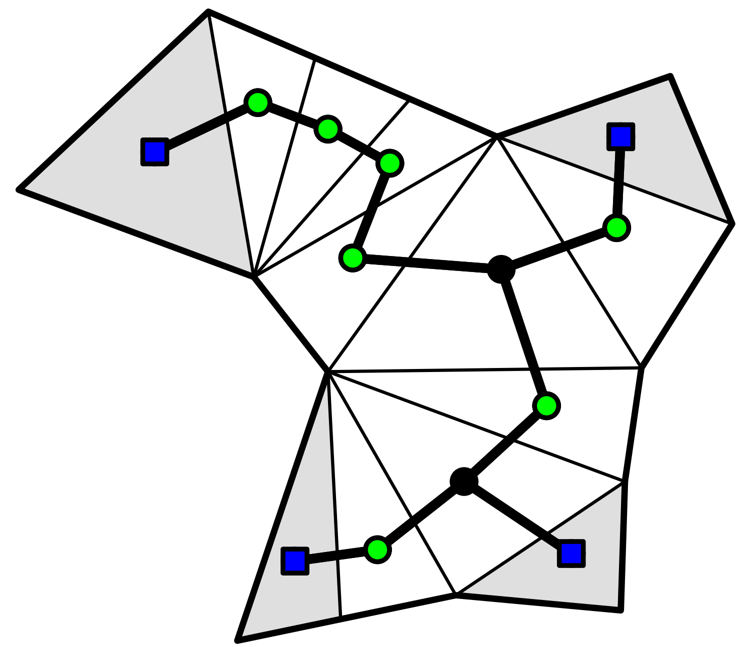
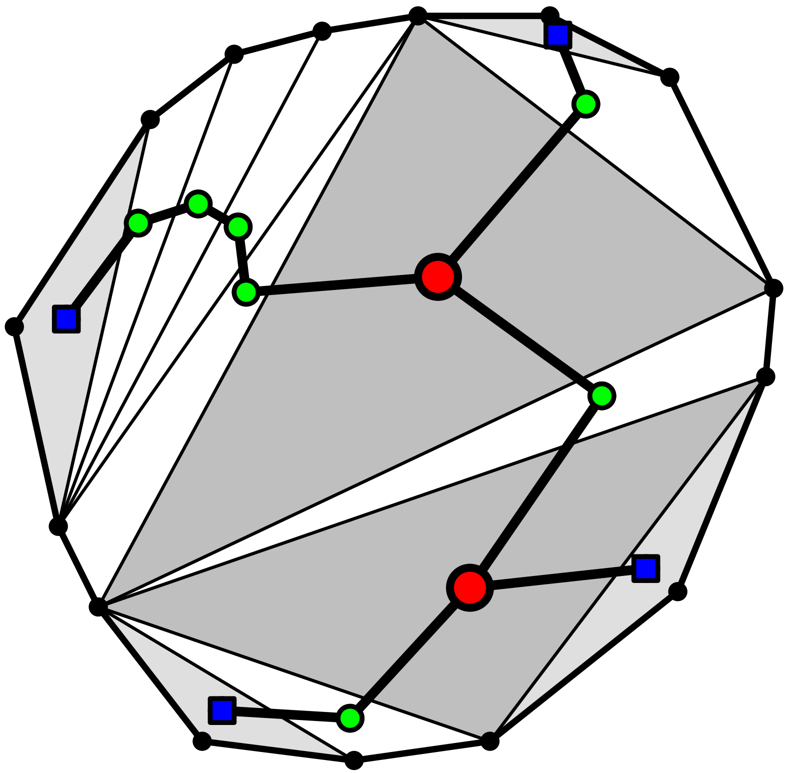


ears (degree 1, leaves)

path triangles (degree 2)

The Dual Graph



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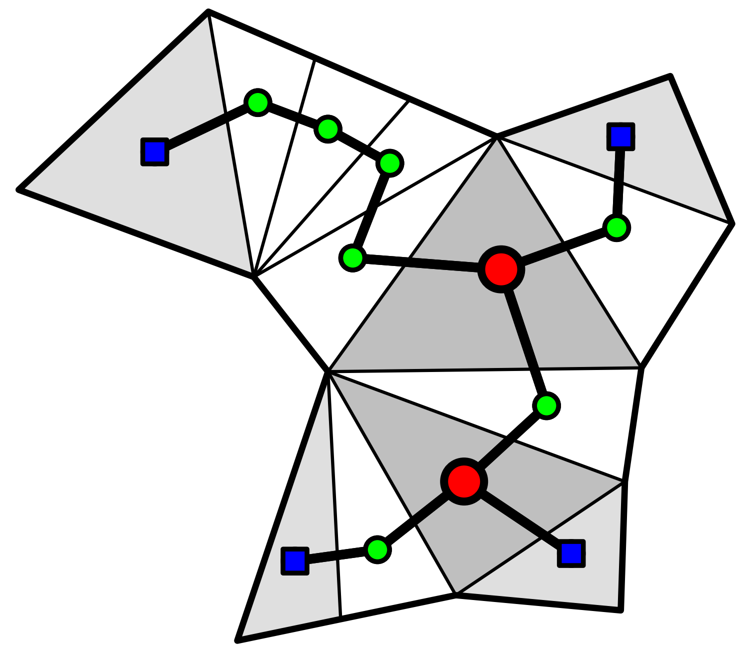

ears (degree 1, leaves)

path triangles (degree 2)

inner triangles (degree 3, branching vertices)

Paths in the Dual Tree

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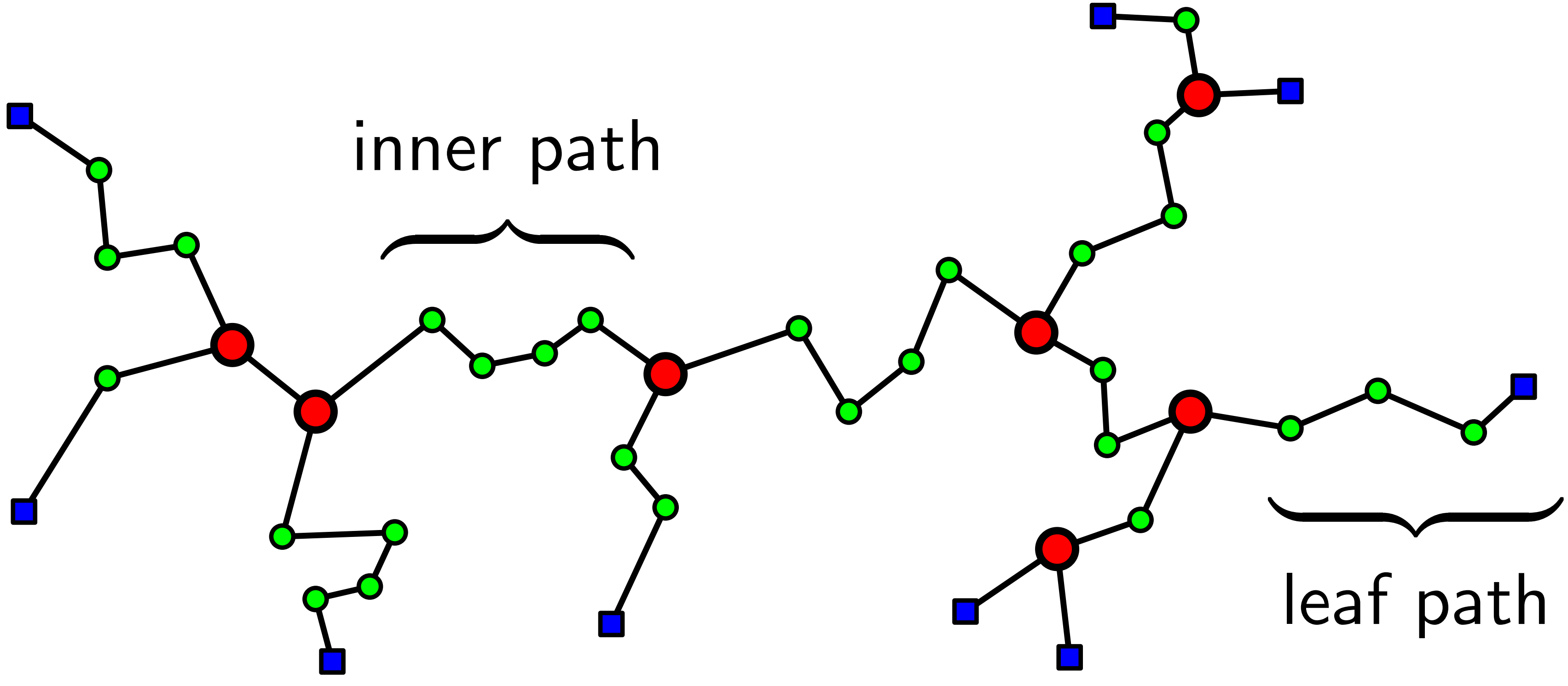

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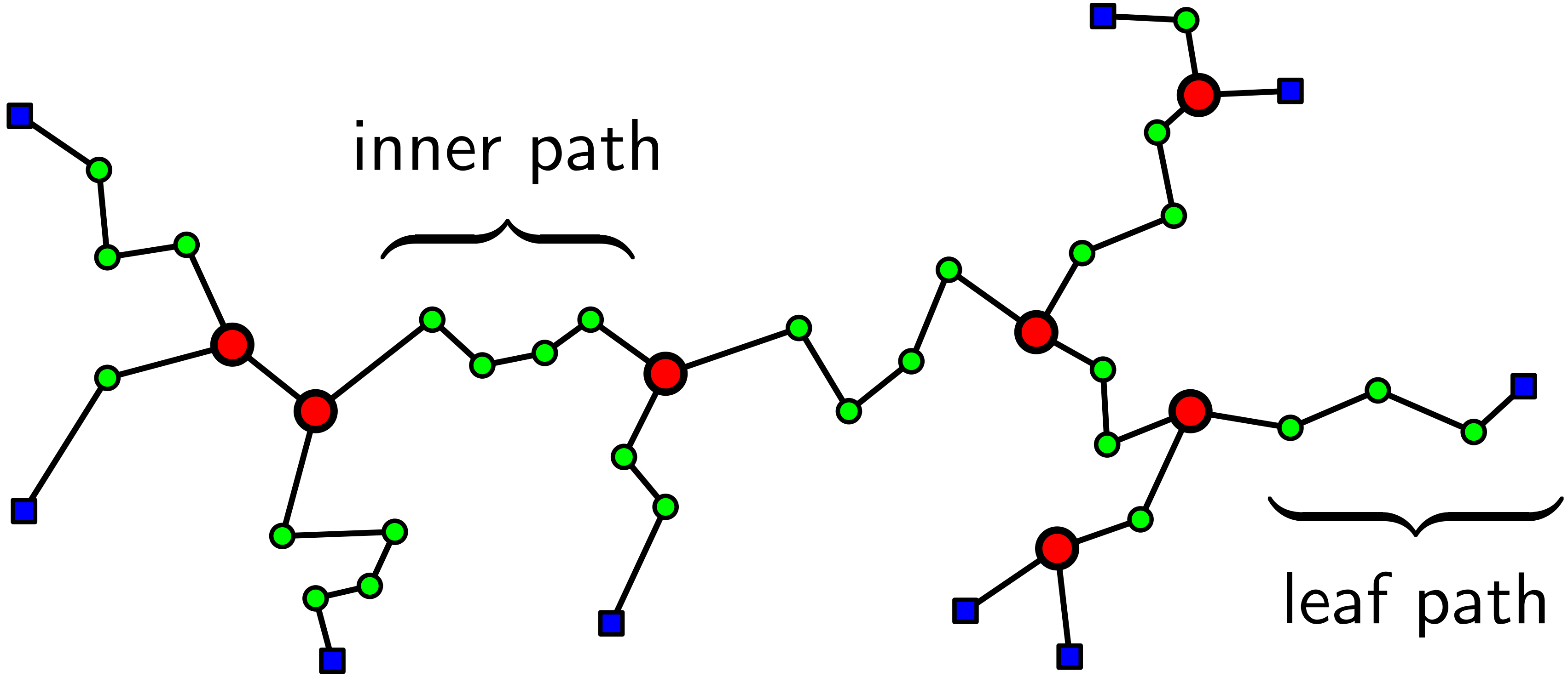
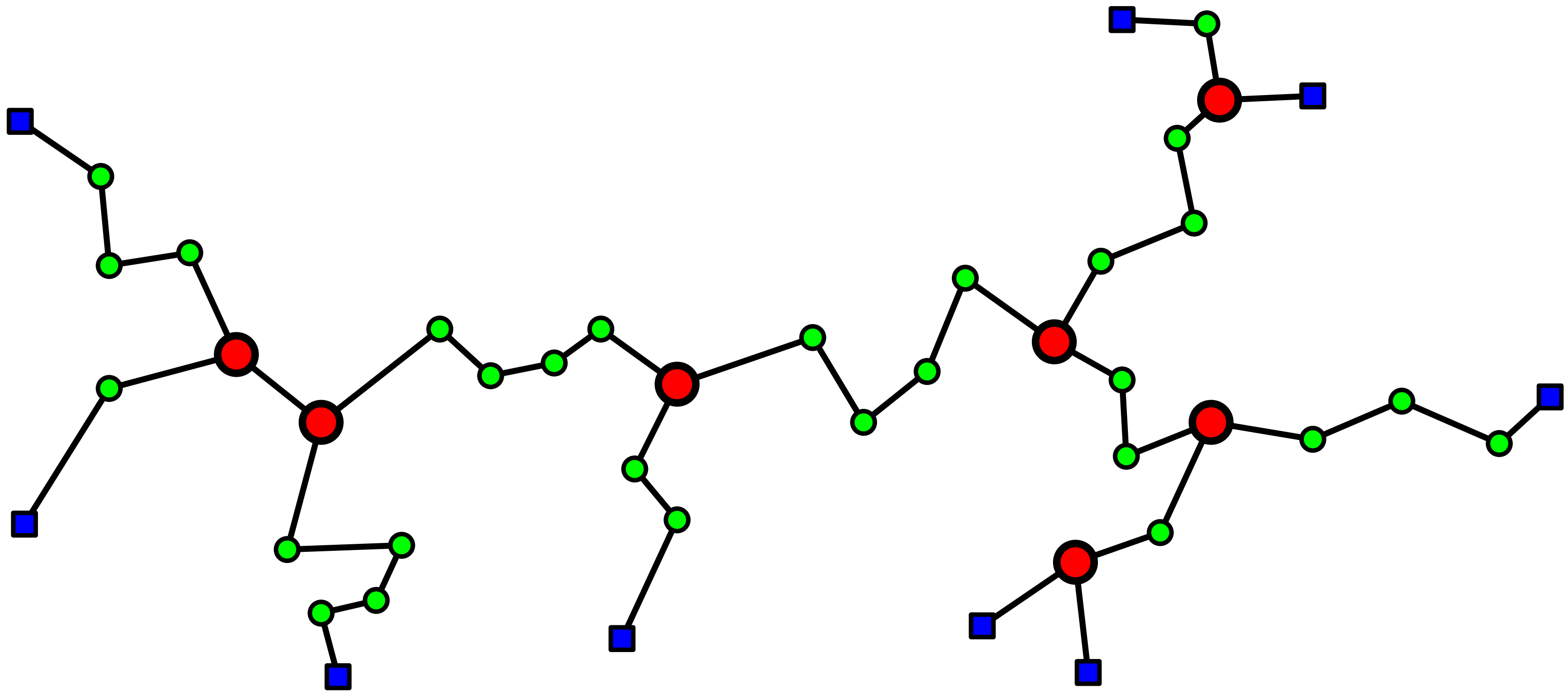


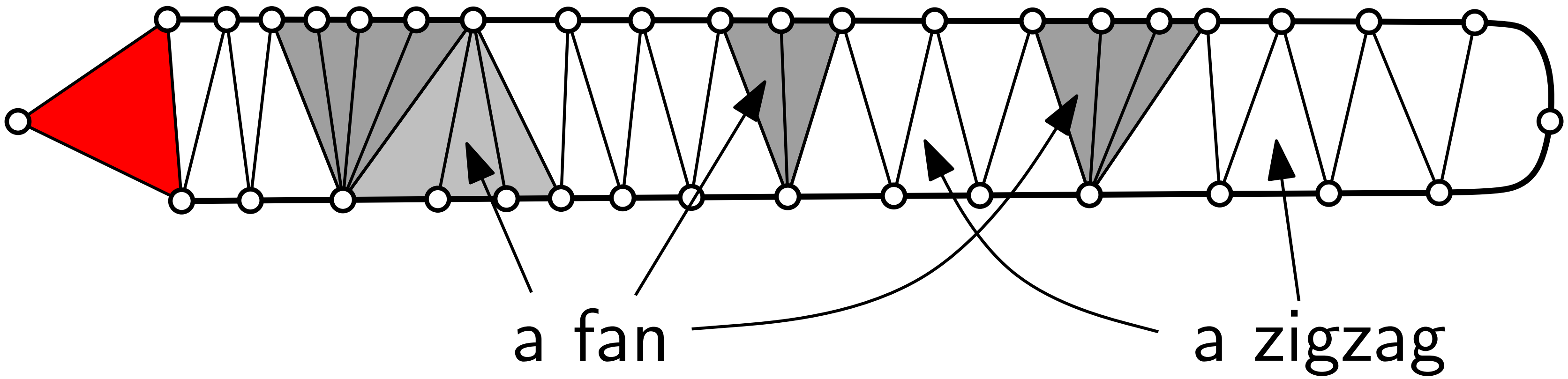
goal: zigzag

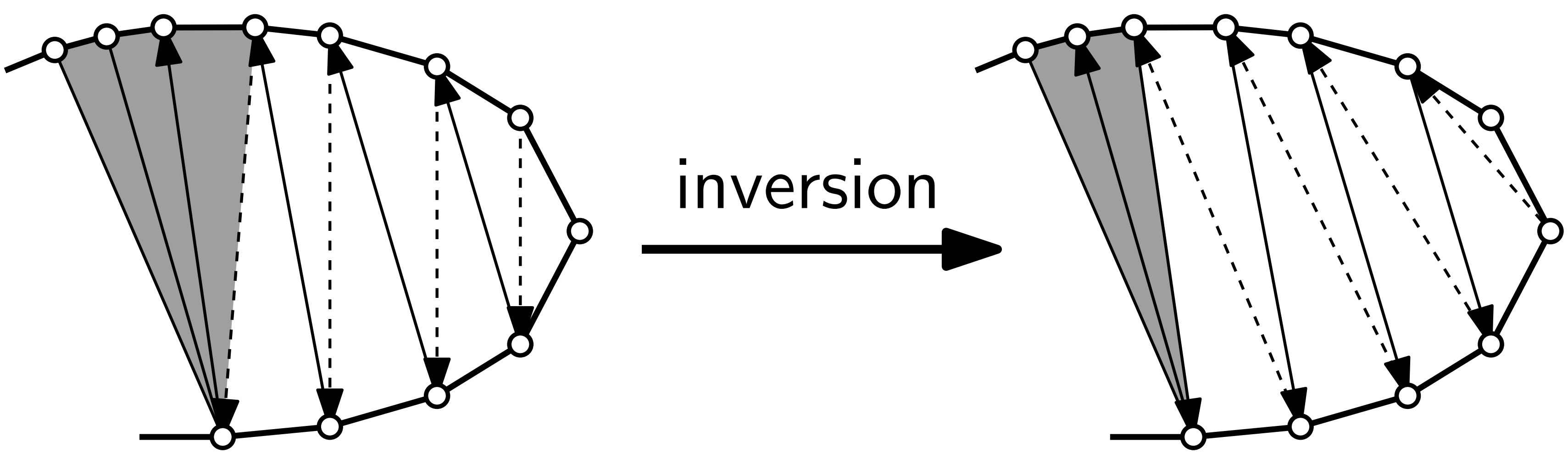
Overall Strategy



- Turn leaf paths into zigzags $O(n^2)$ flips
- Eliminate inner triangles $O(n^2)$ flips
- Rotate final zigzag $O(n^2)$ flips

Turn Leaf Path into Zigzag





$O(n)$ flips for extending the zigzag by $1 \to O(n^2)$ total

Zigzag Rotation

STRATEGY: Always turn leaf paths into zigzags.

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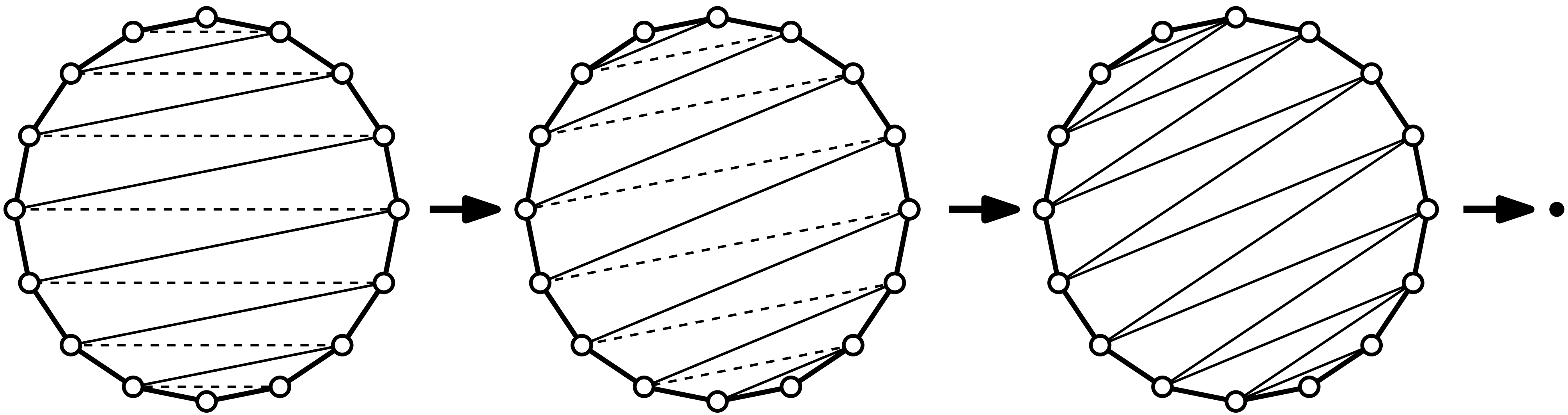
can also rotate final zigzag:



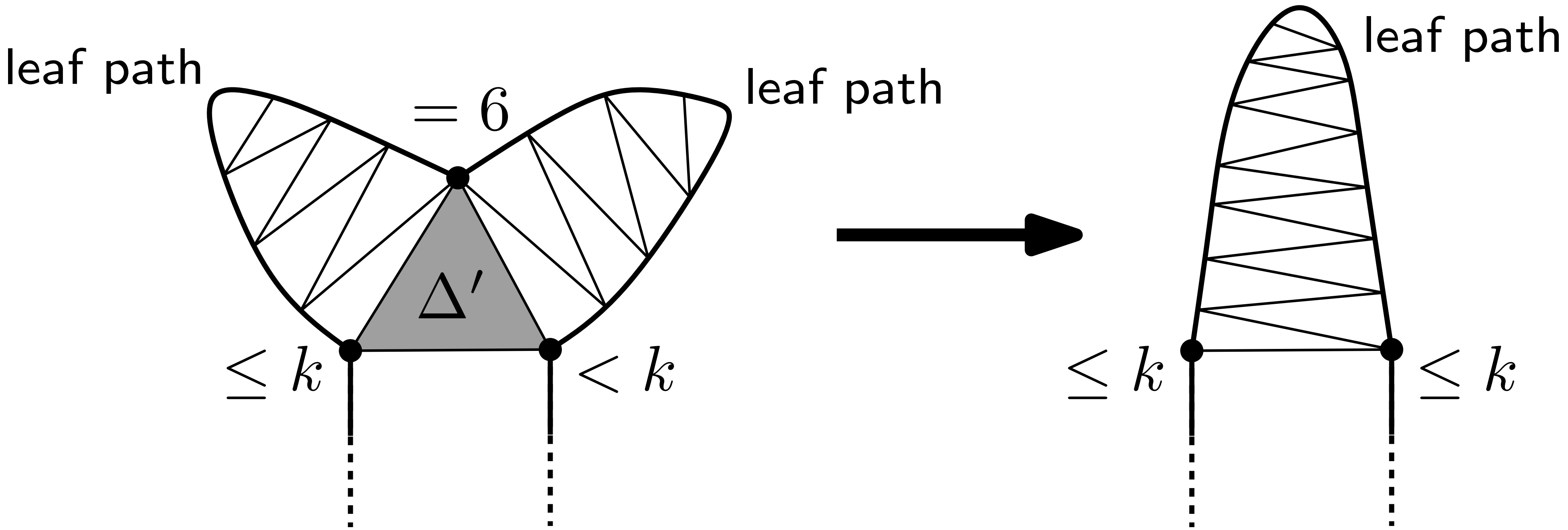
$O(n^2)$ flips in total. (can even be done in $O(n)$ flips)

Create a Zigzag Triangulation

Goal: dual tree $D \rightarrow$ a zigzag path

Strategy:
Process the tree from the leaves towards the center.

Find a *good merge triangle* and eliminate it:



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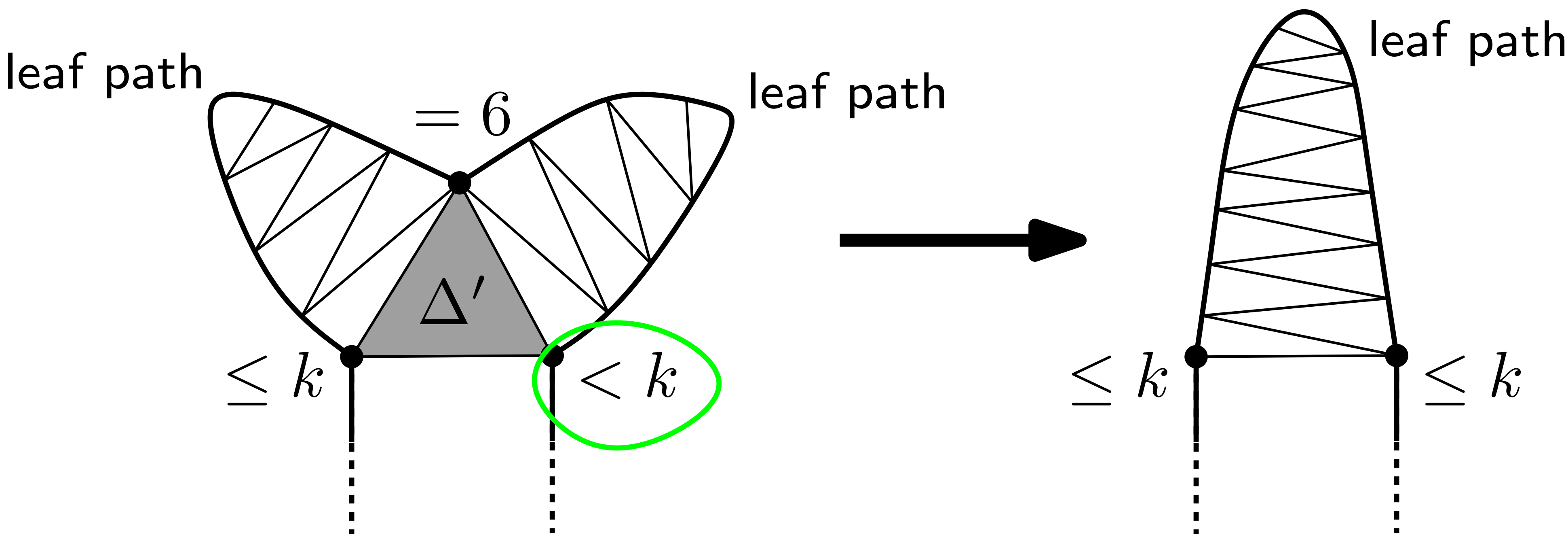
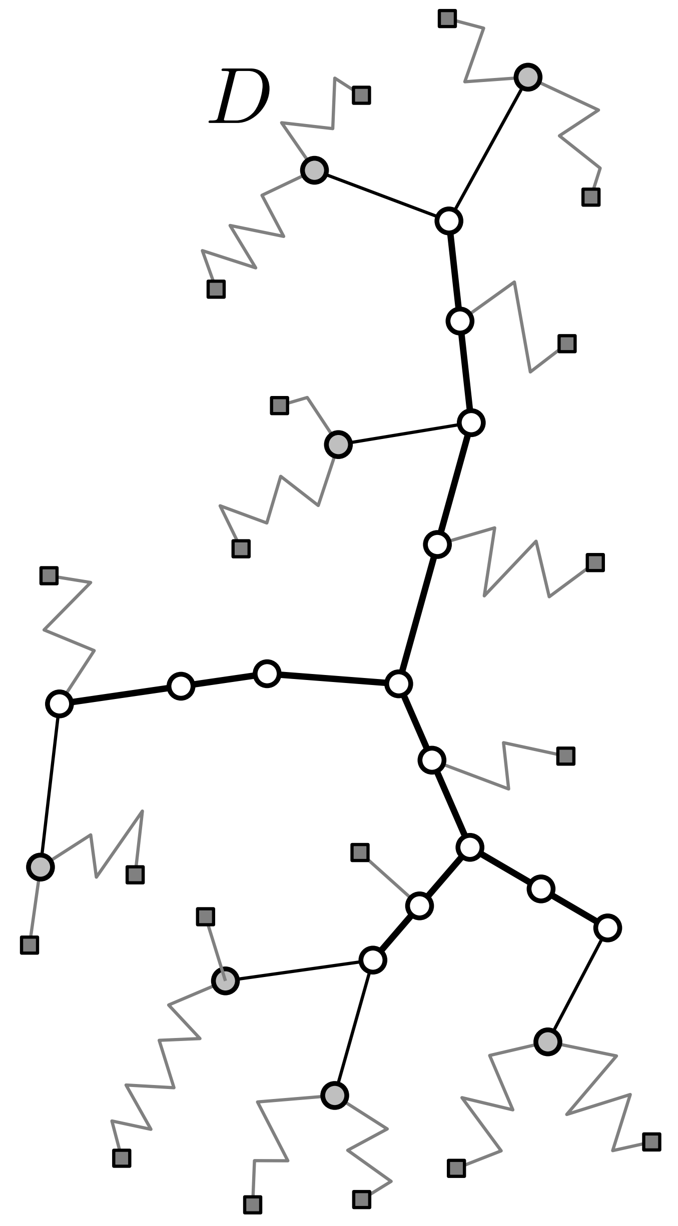


Find a Good Merge Triangle Δ'

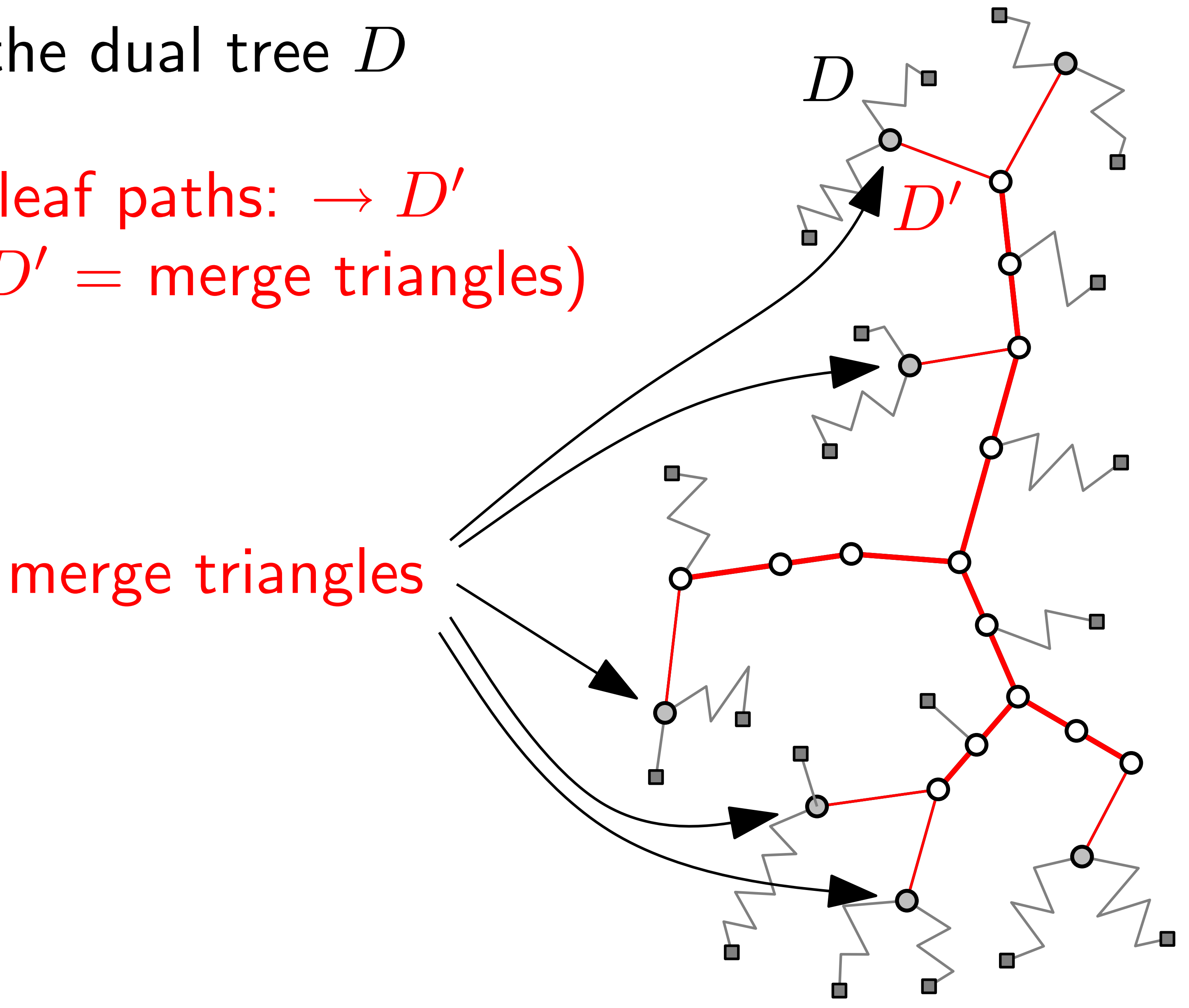
start with the dual tree D



Find a Good Merge Triangle Δ'

start with the dual tree D

remove all leaf paths: $\rightarrow D'$
(leaves of D' = merge triangles)



Find a Good Merge Triangle Δ'

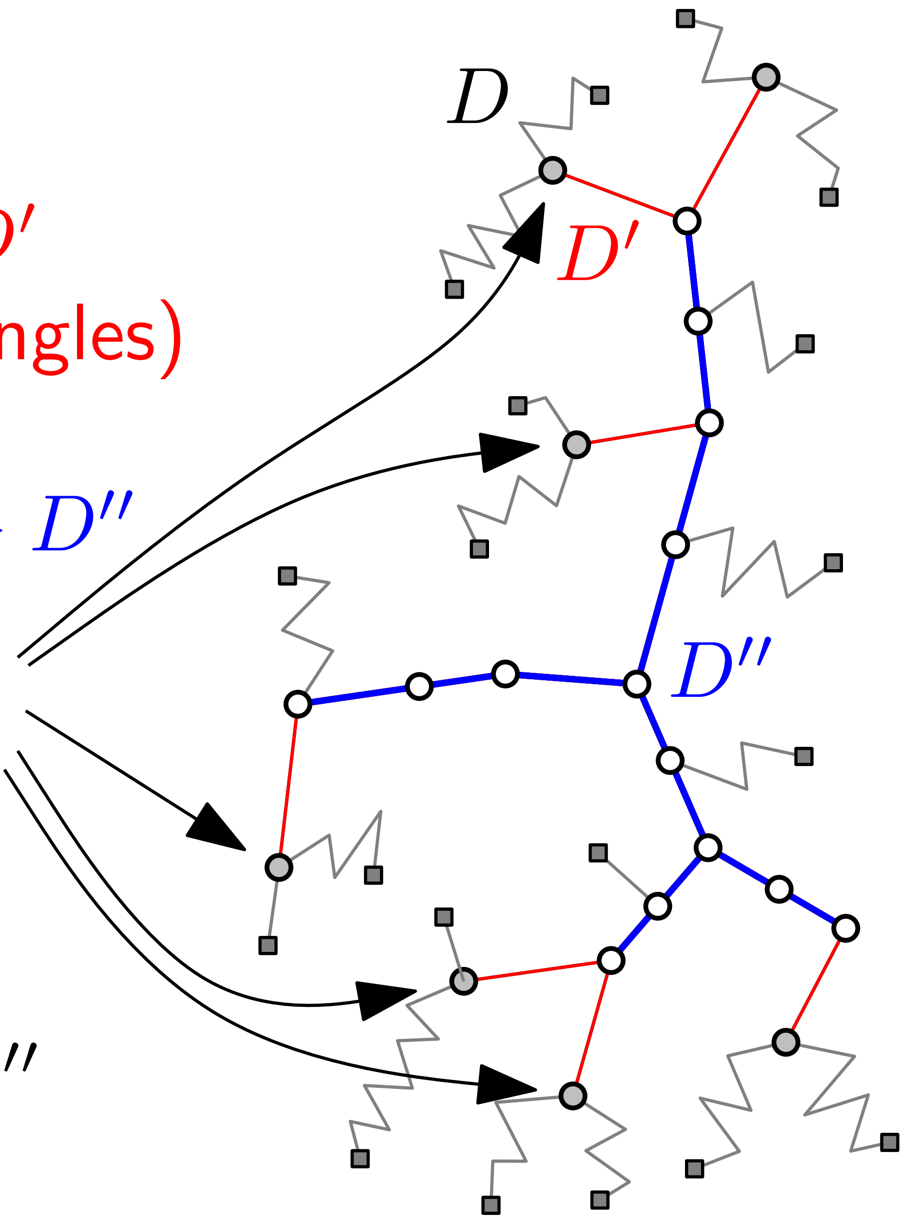
start with the dual tree D

remove all leaf paths: $\rightarrow D'$
(leaves of D' = merge triangles)

remove all leaves of D': $\rightarrow D''$

merge triangles

take a merge triangle Δ'
adjacent to a leaf Δ'' of D''



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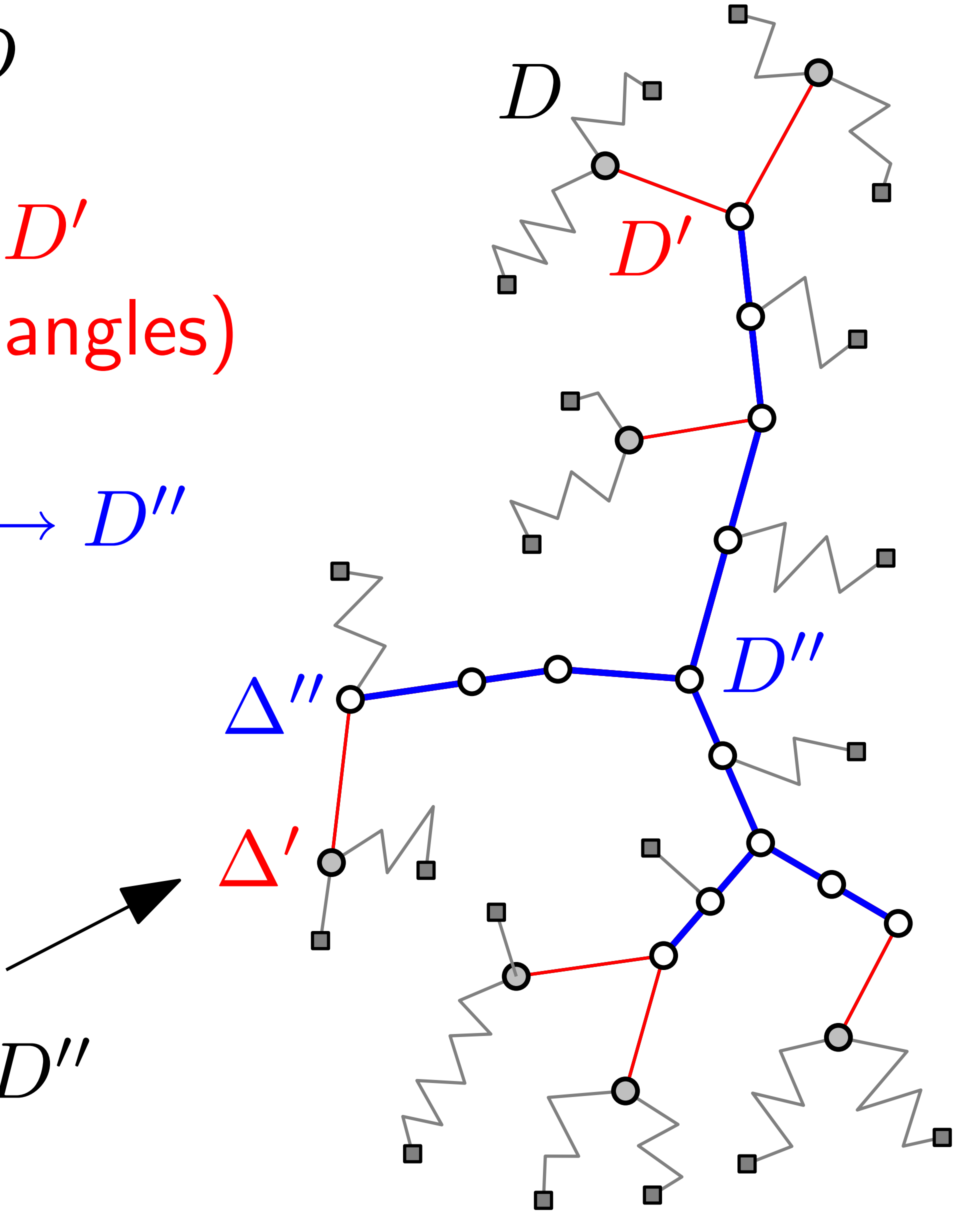

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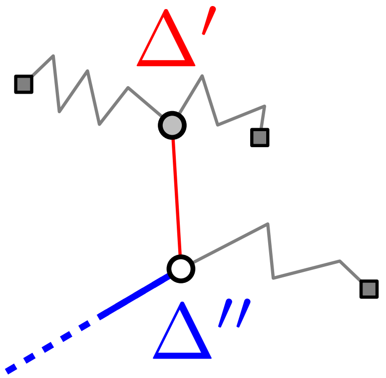

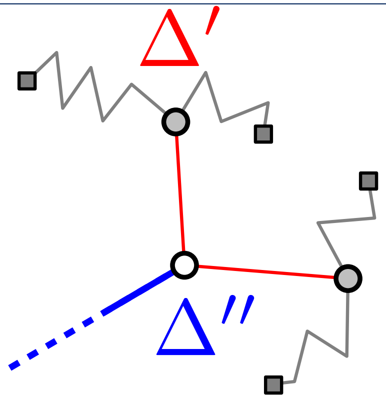

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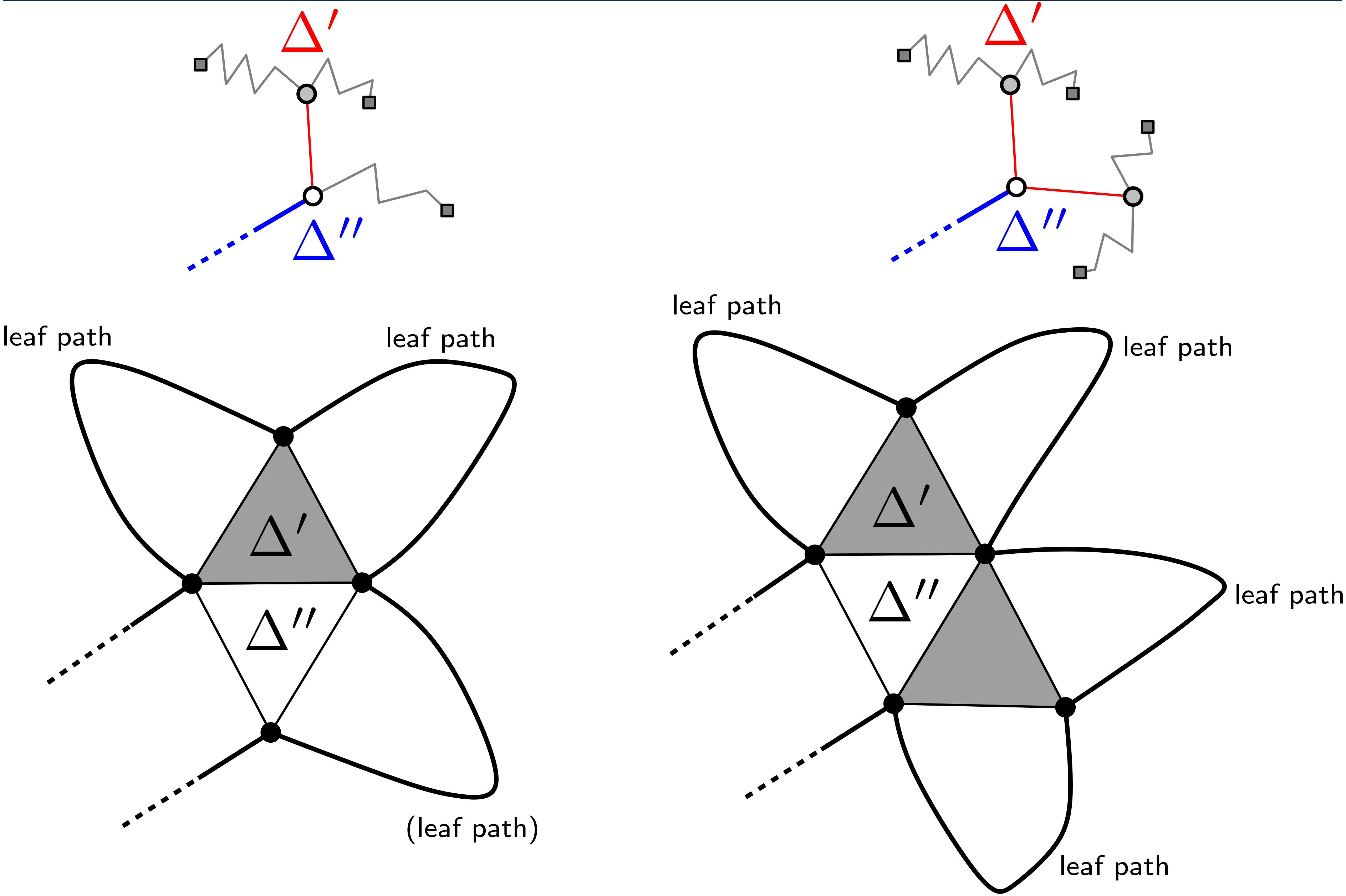

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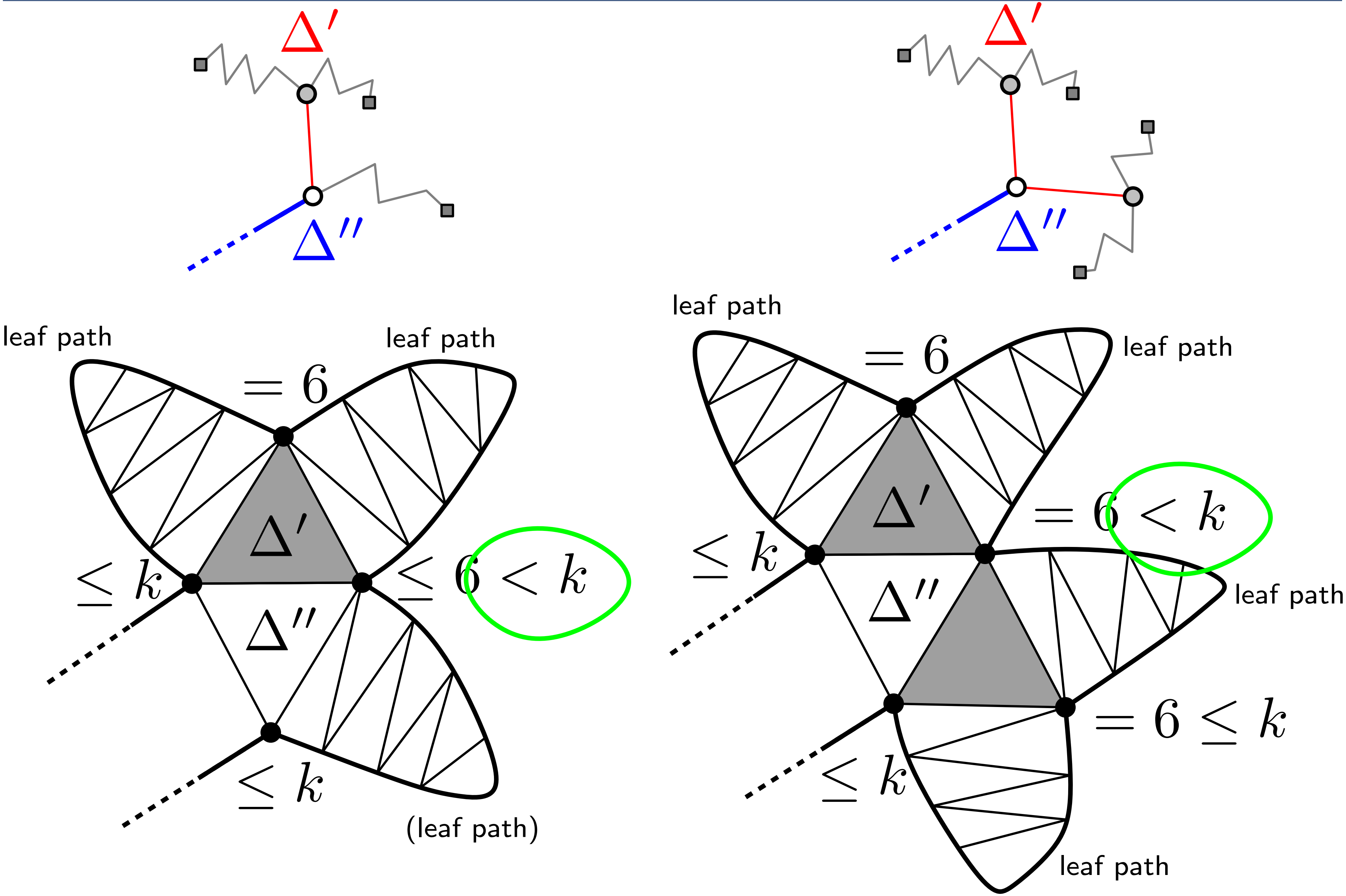

Eliminate a Merge Triangle

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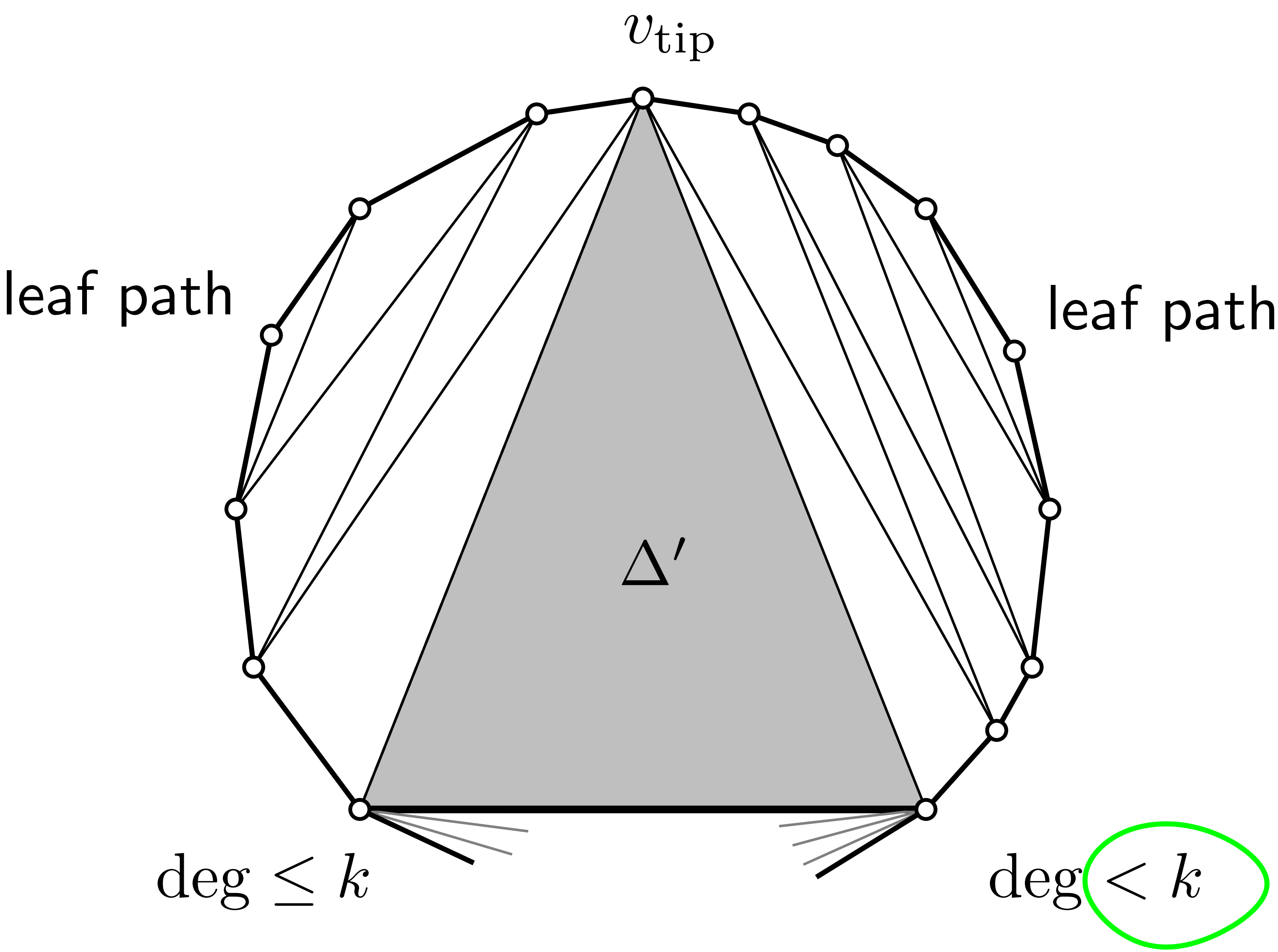

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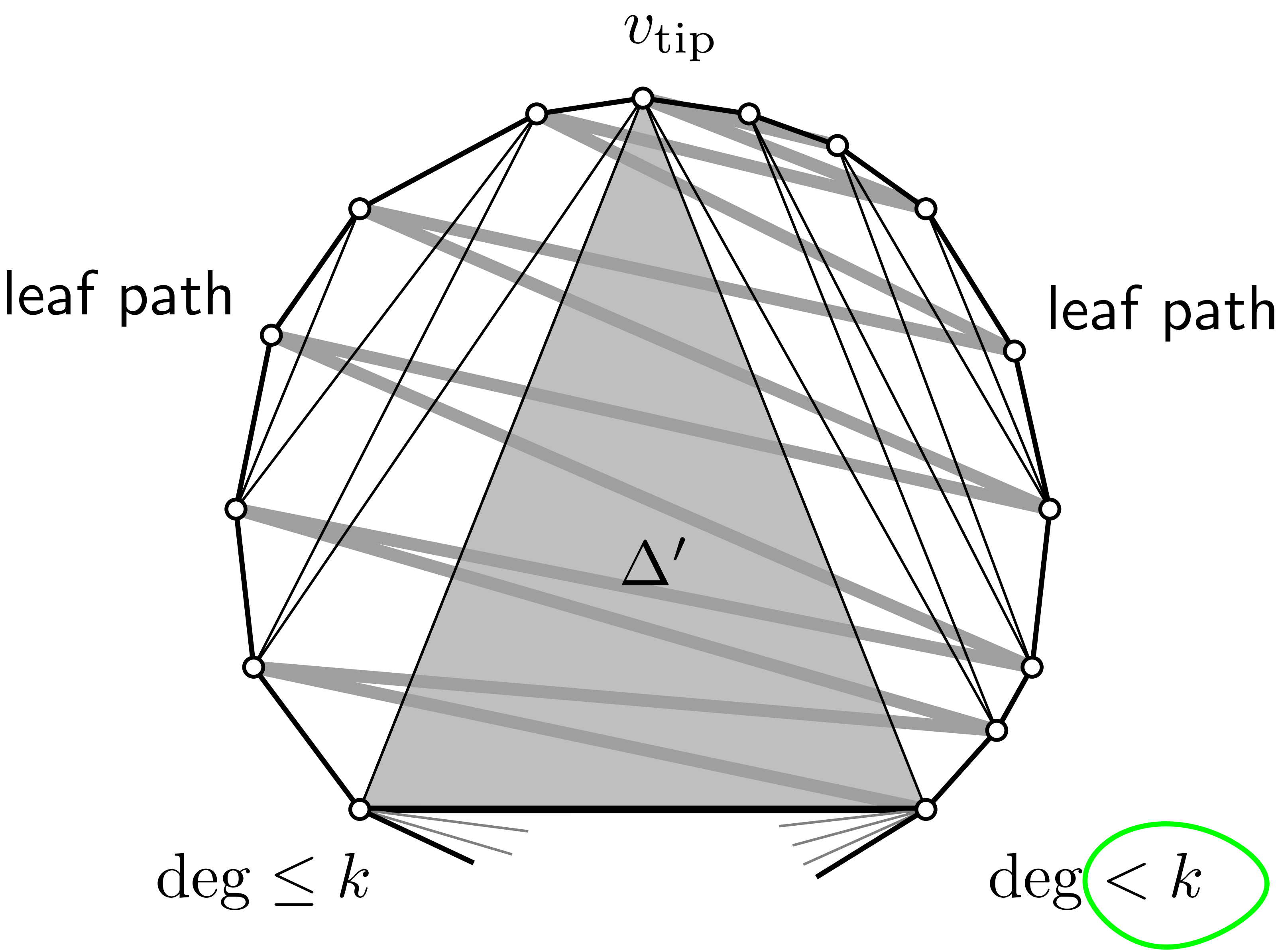

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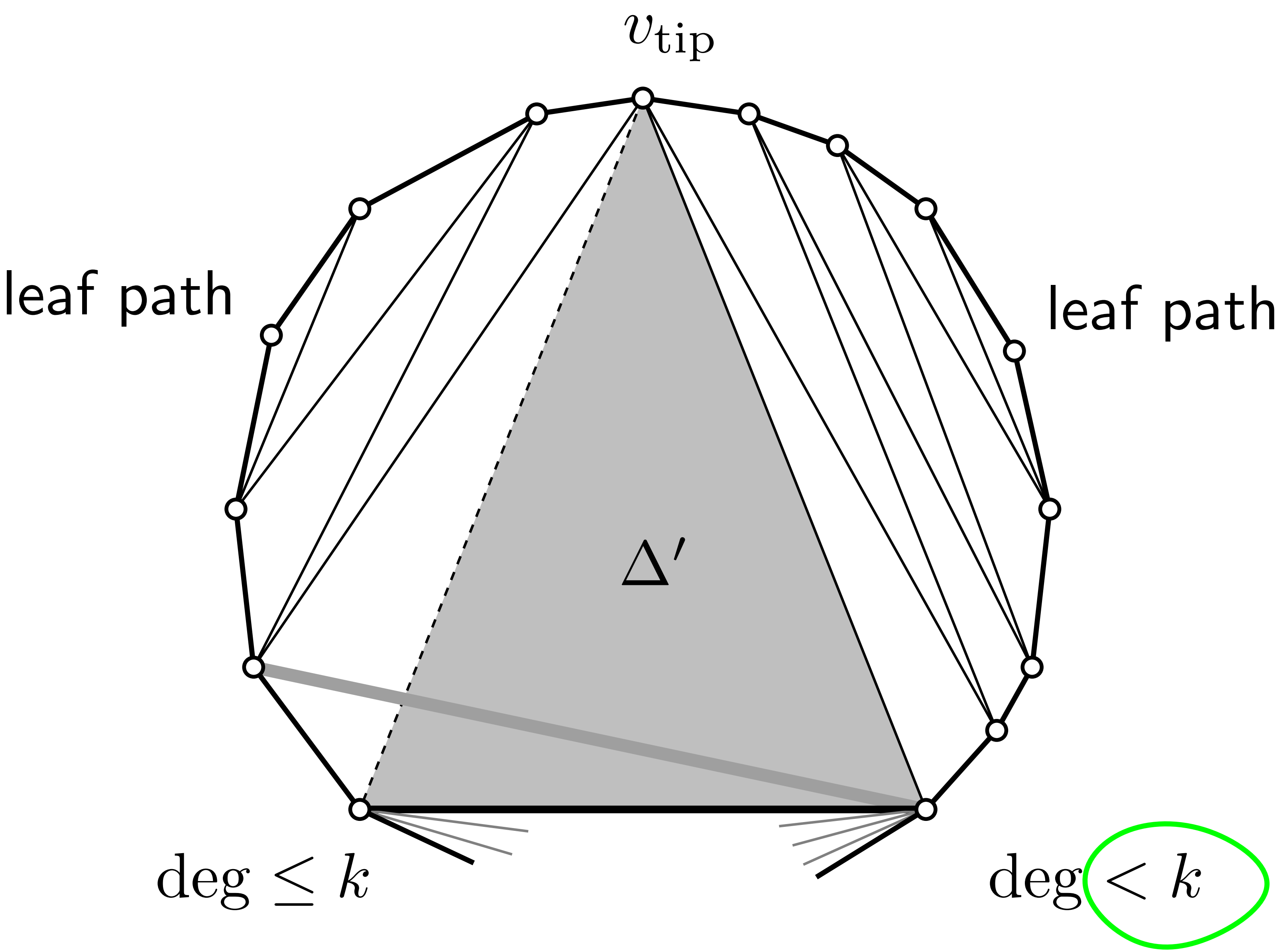

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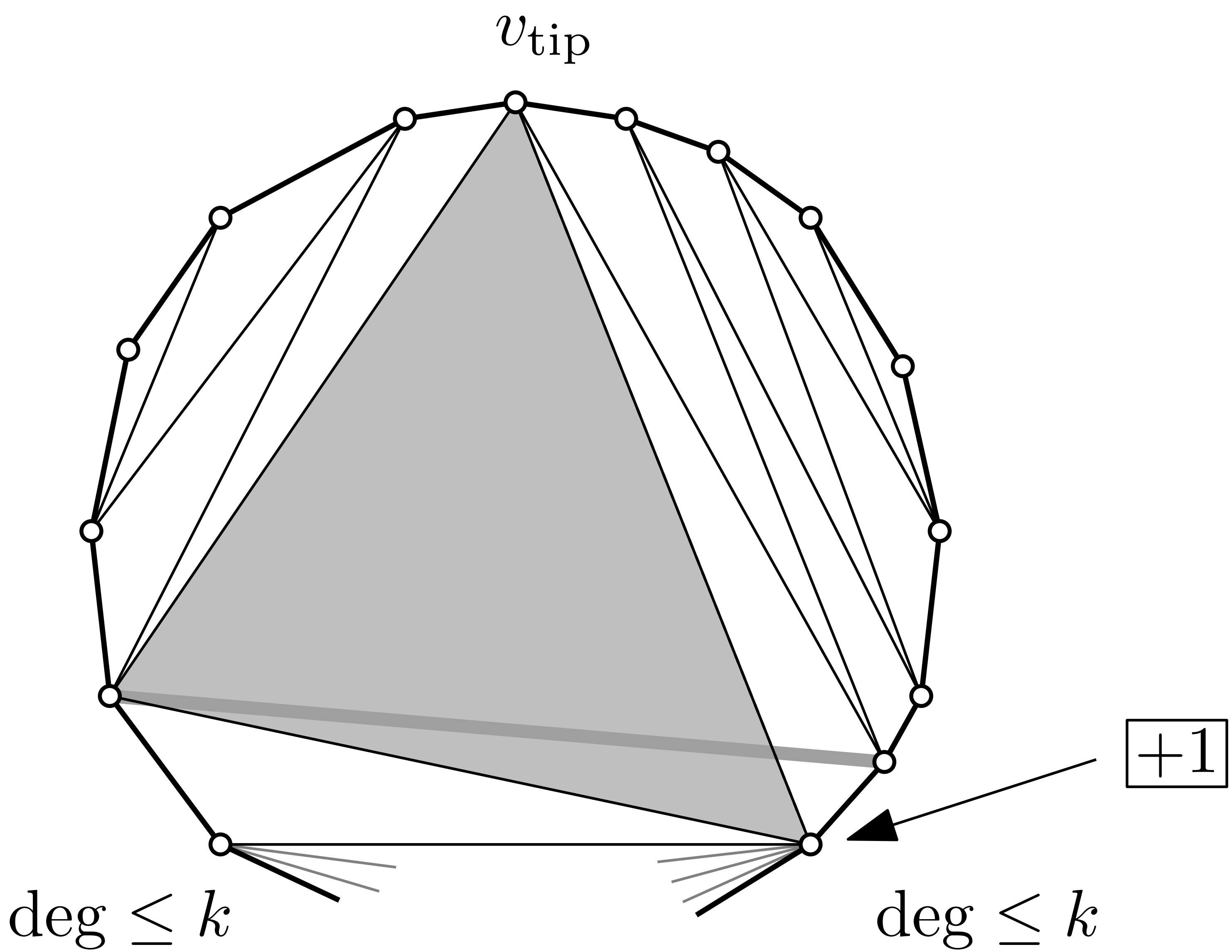

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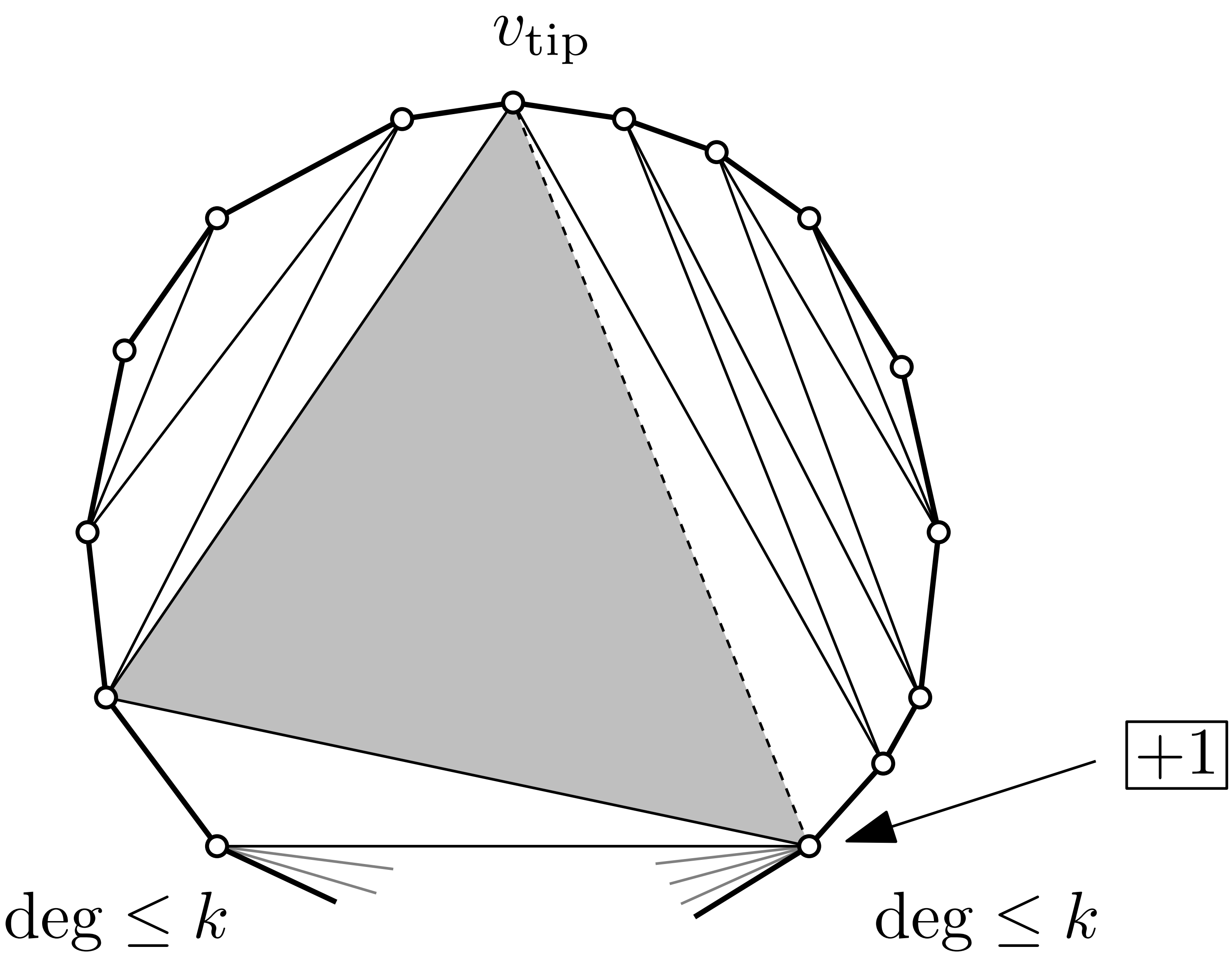

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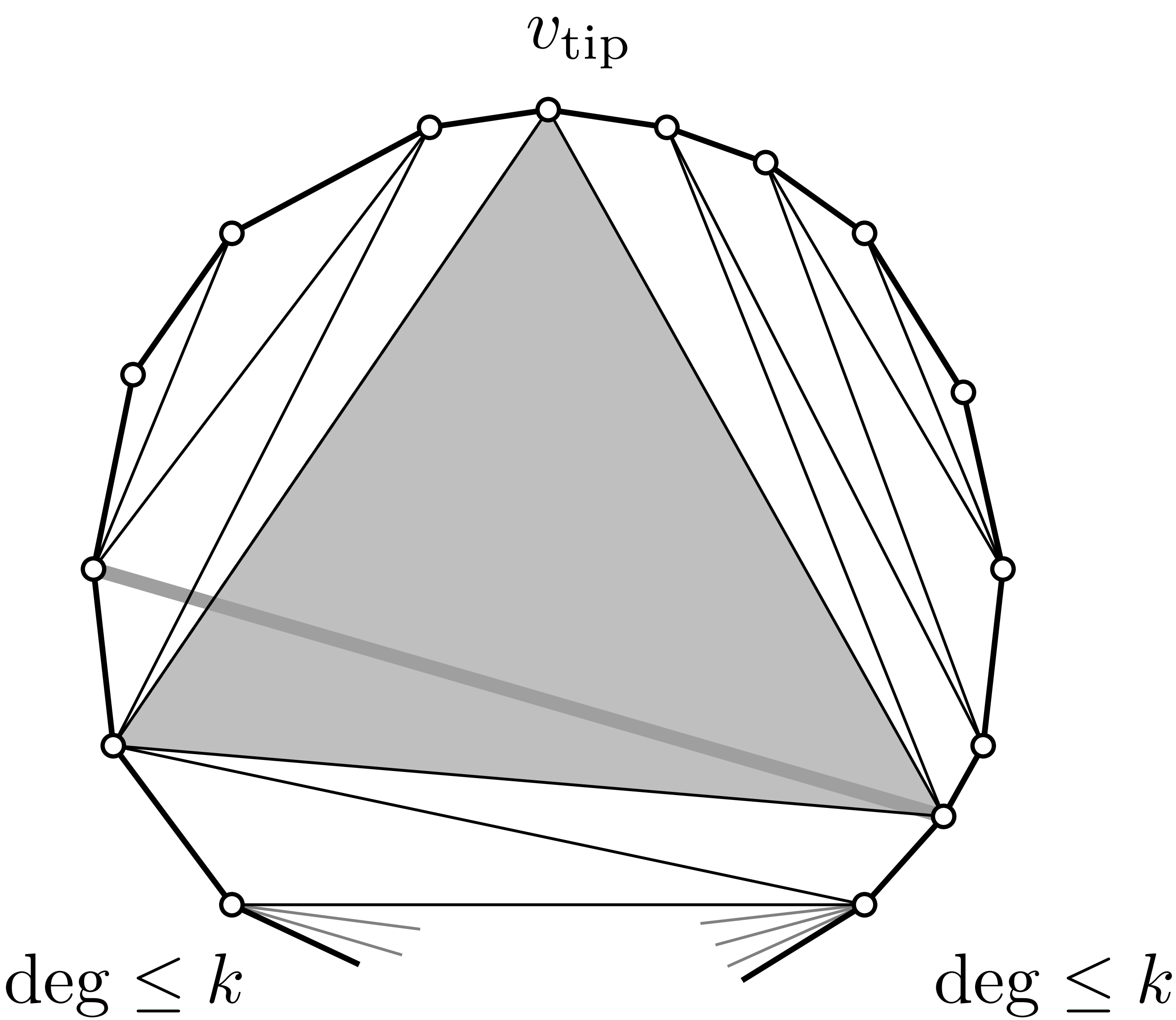

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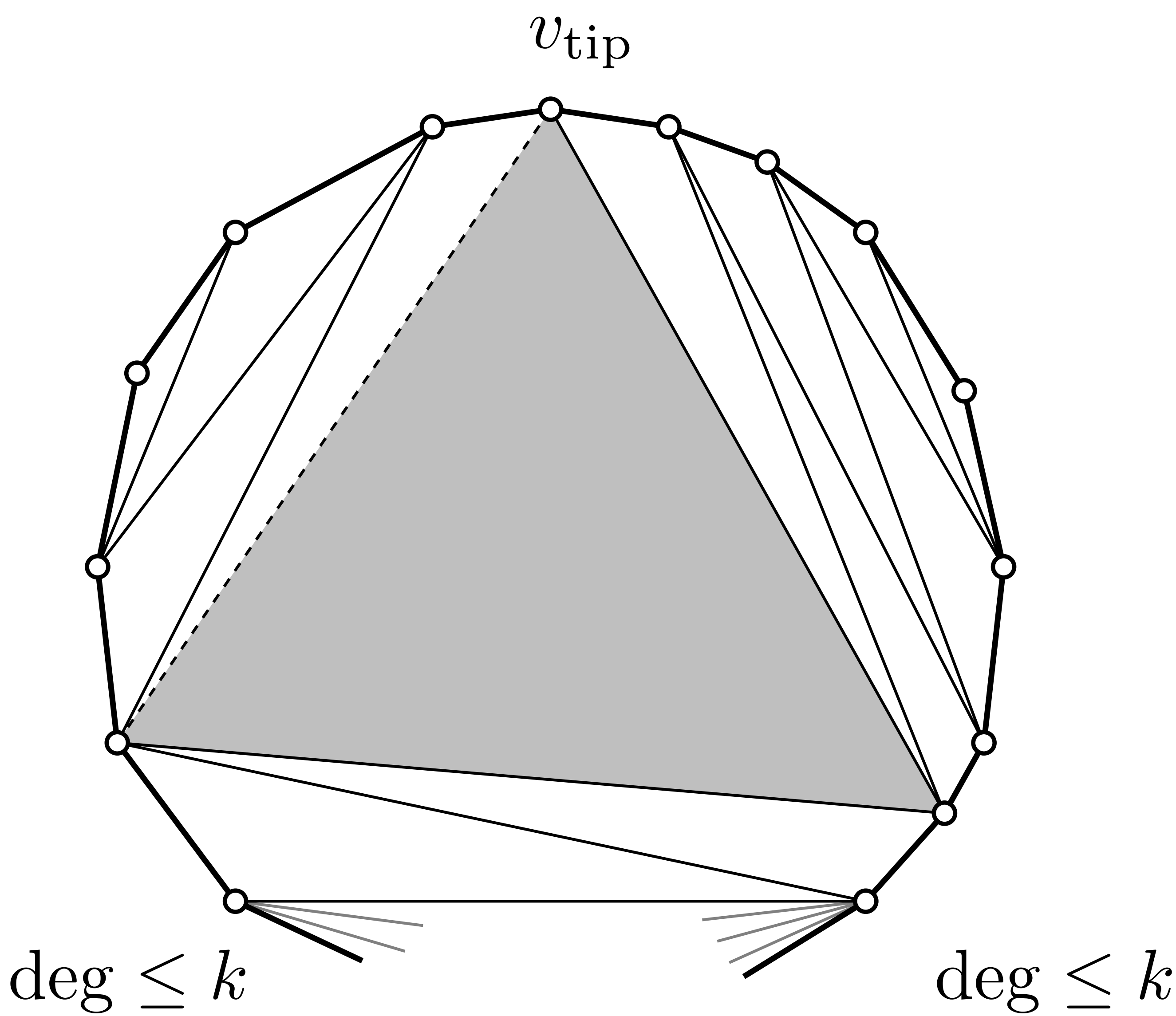

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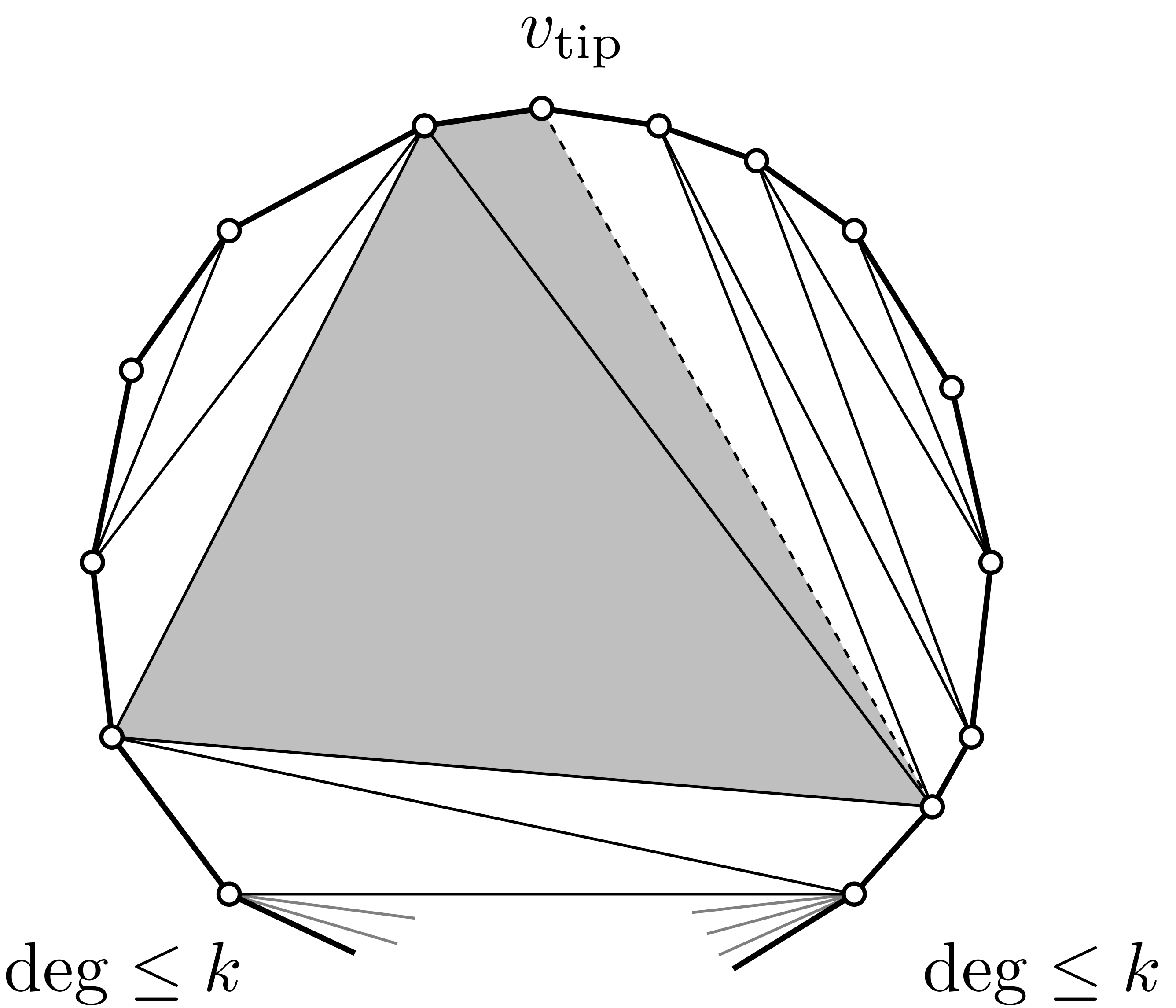

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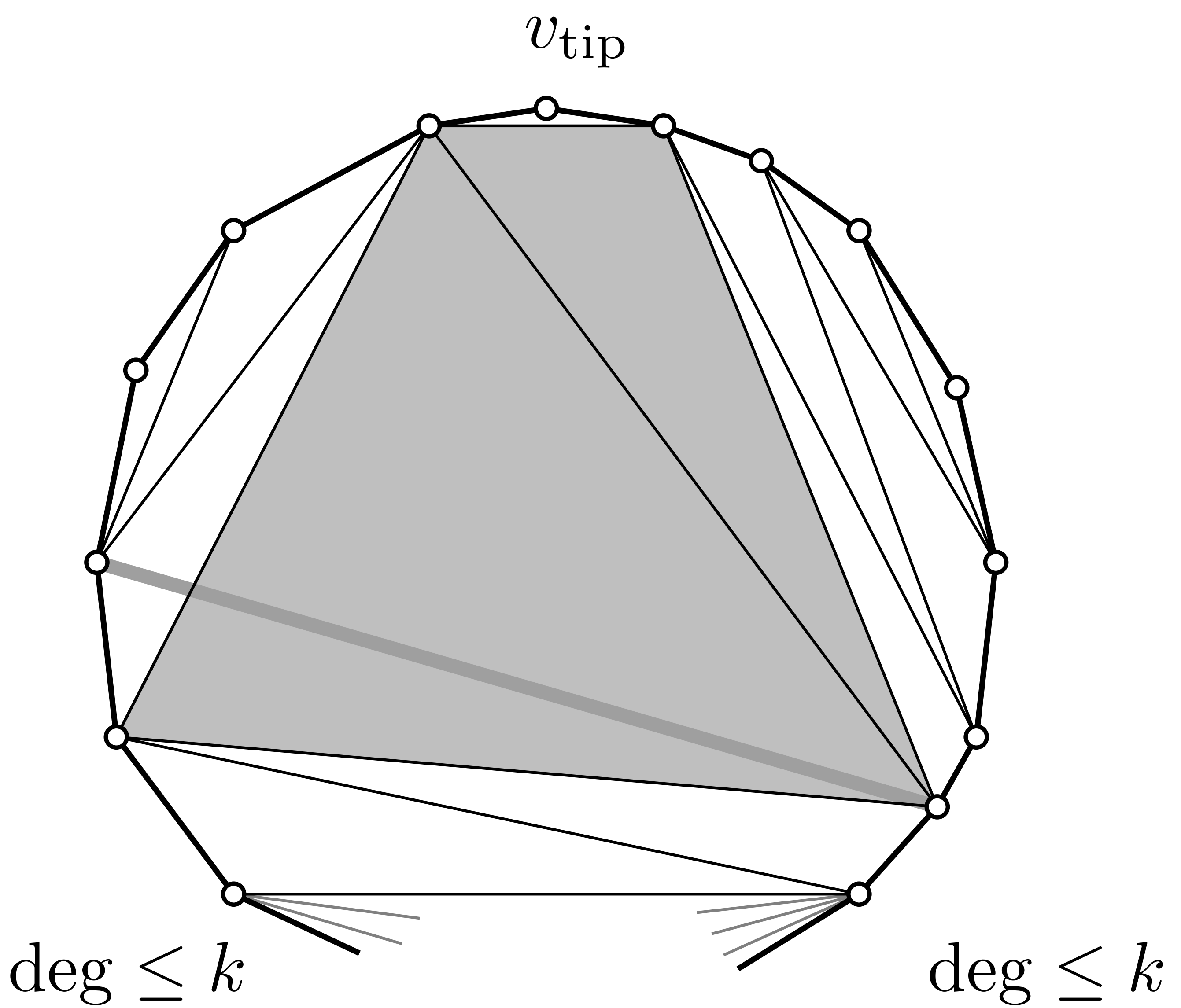

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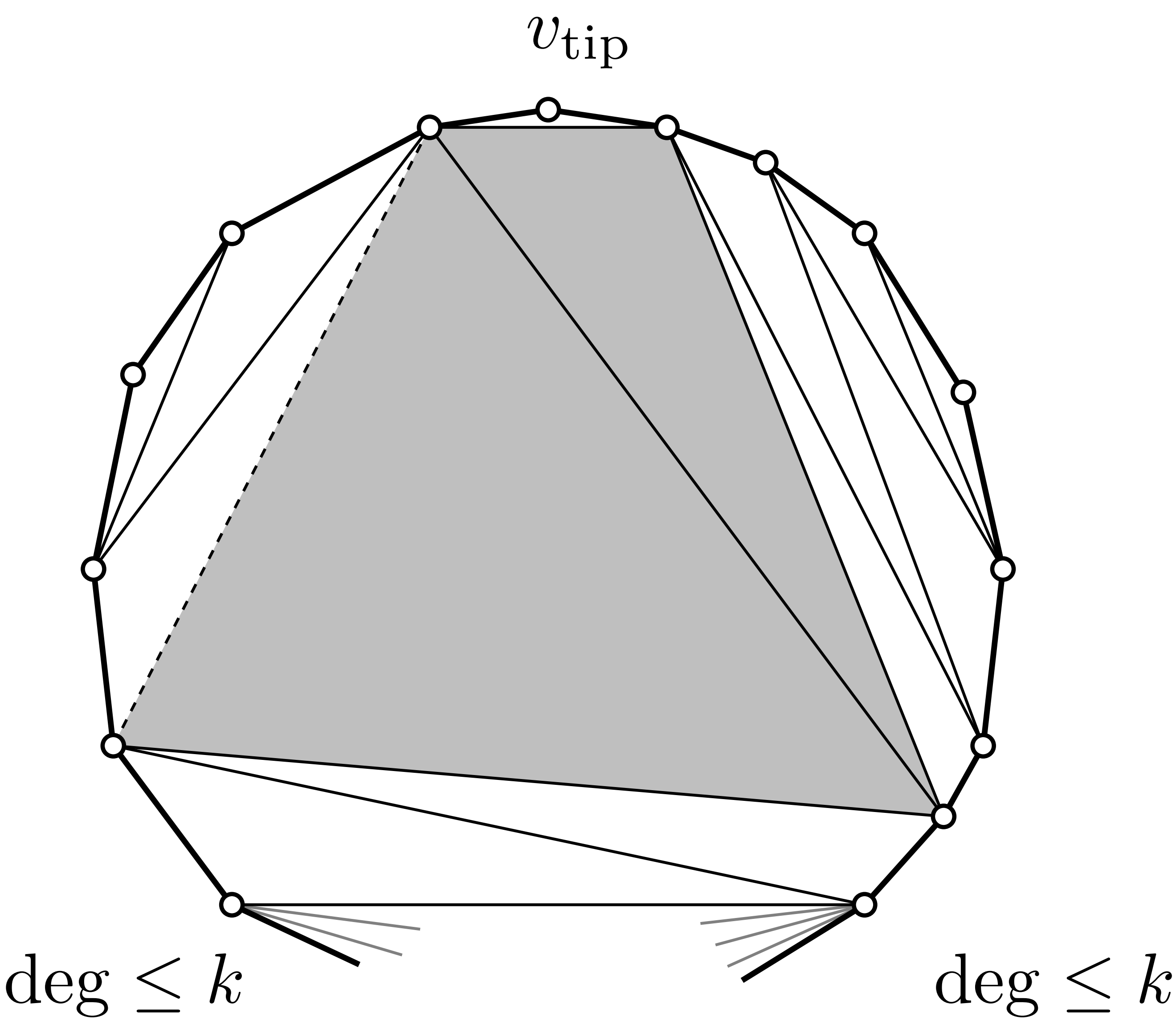

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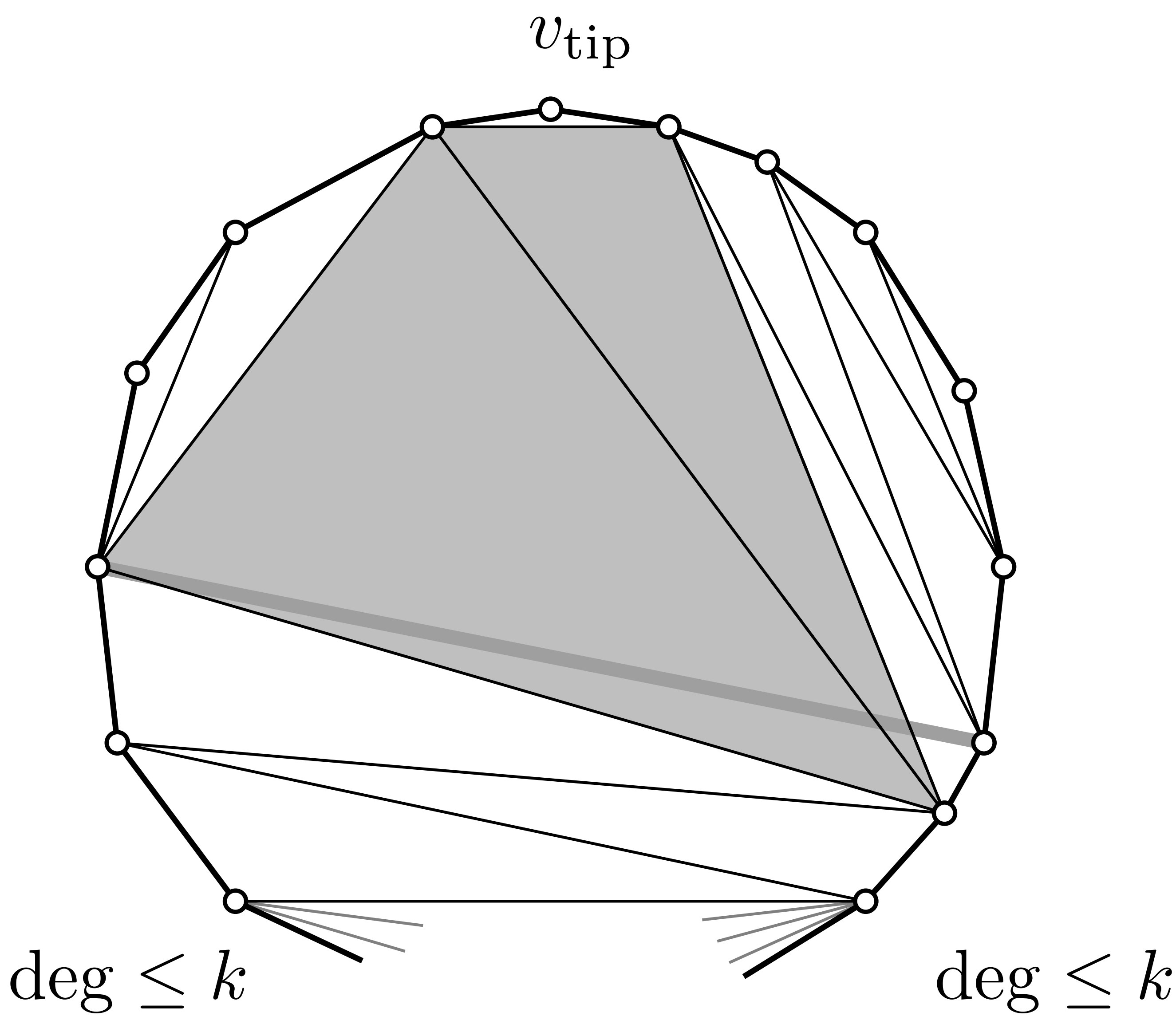

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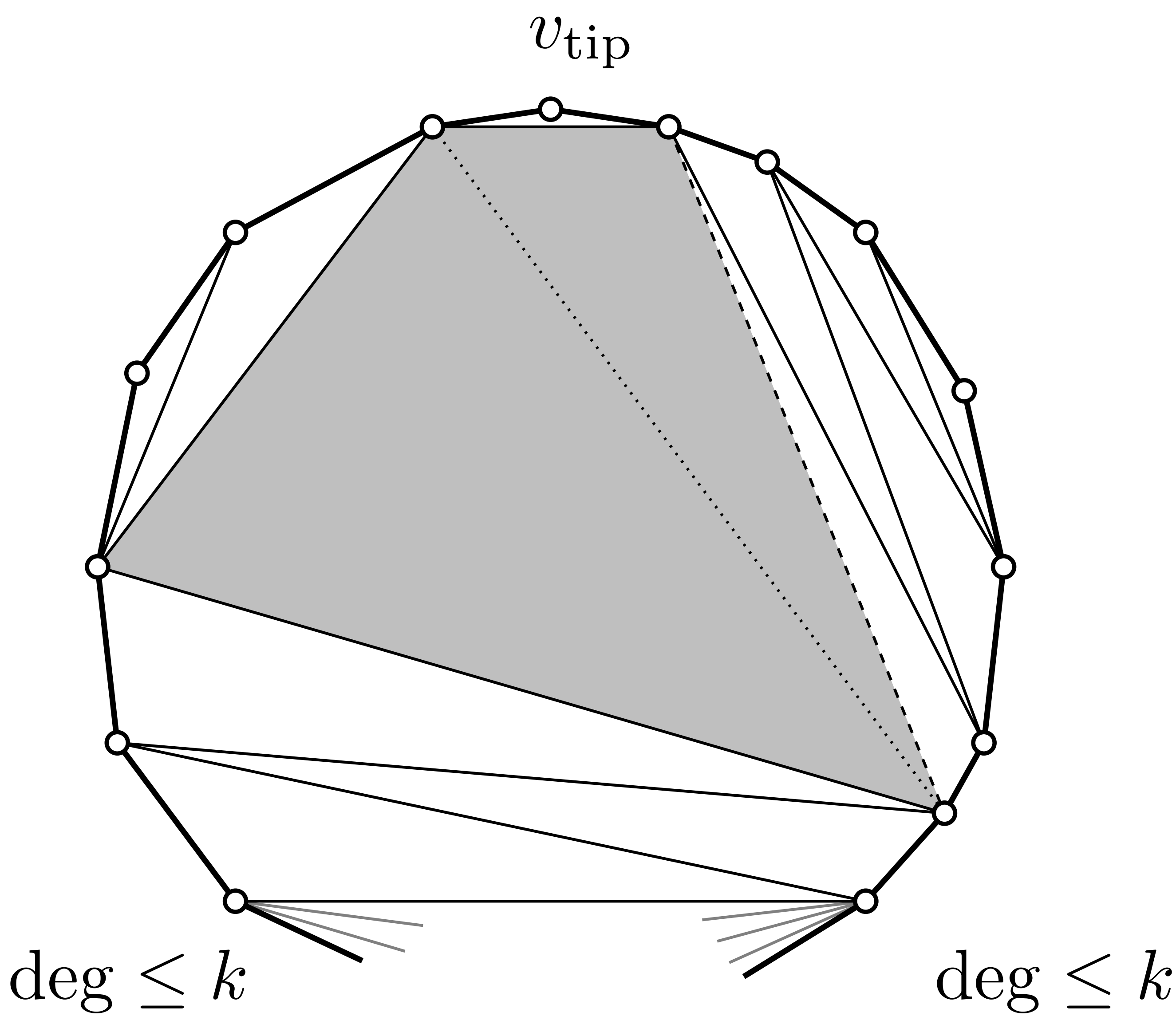

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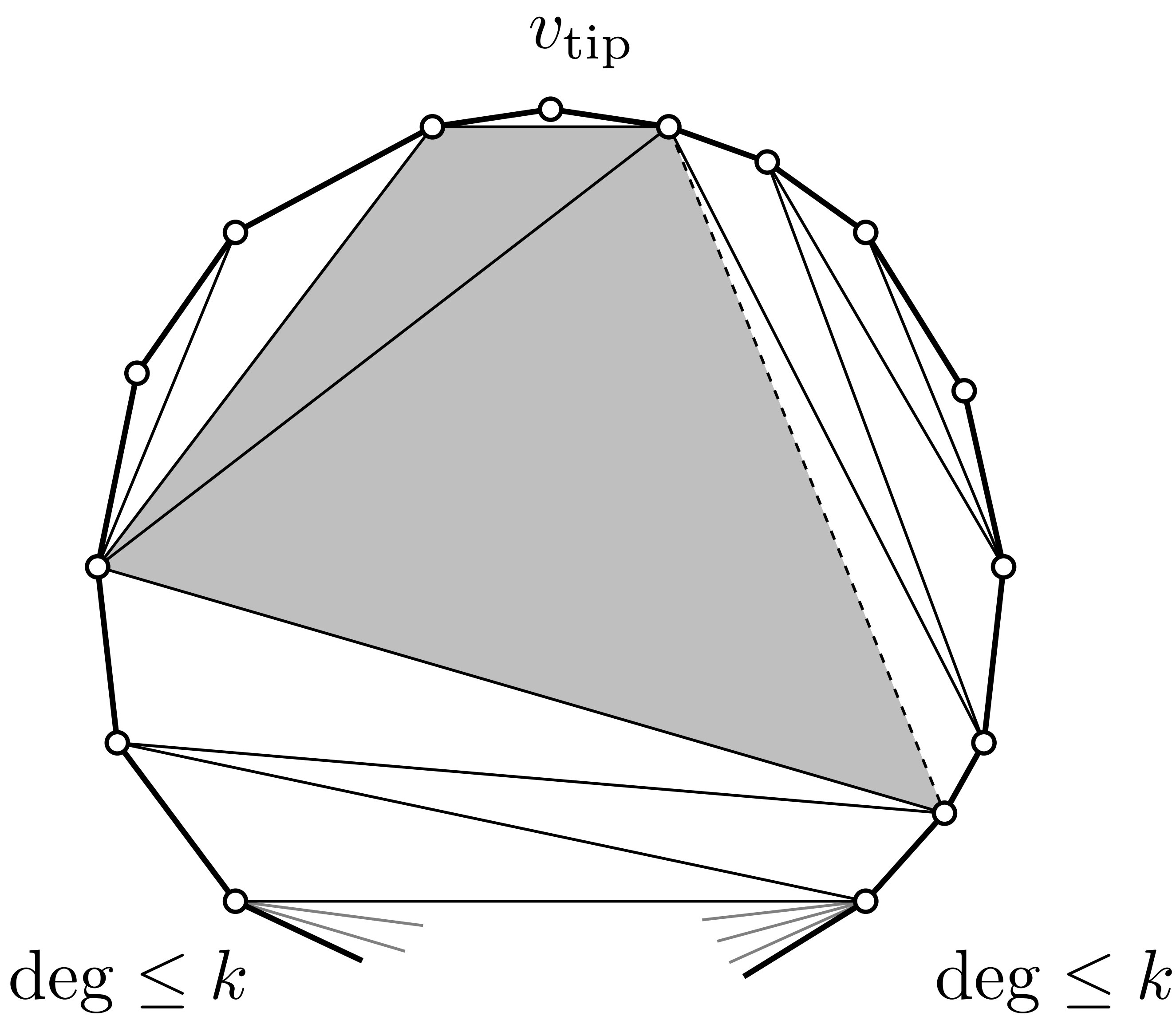

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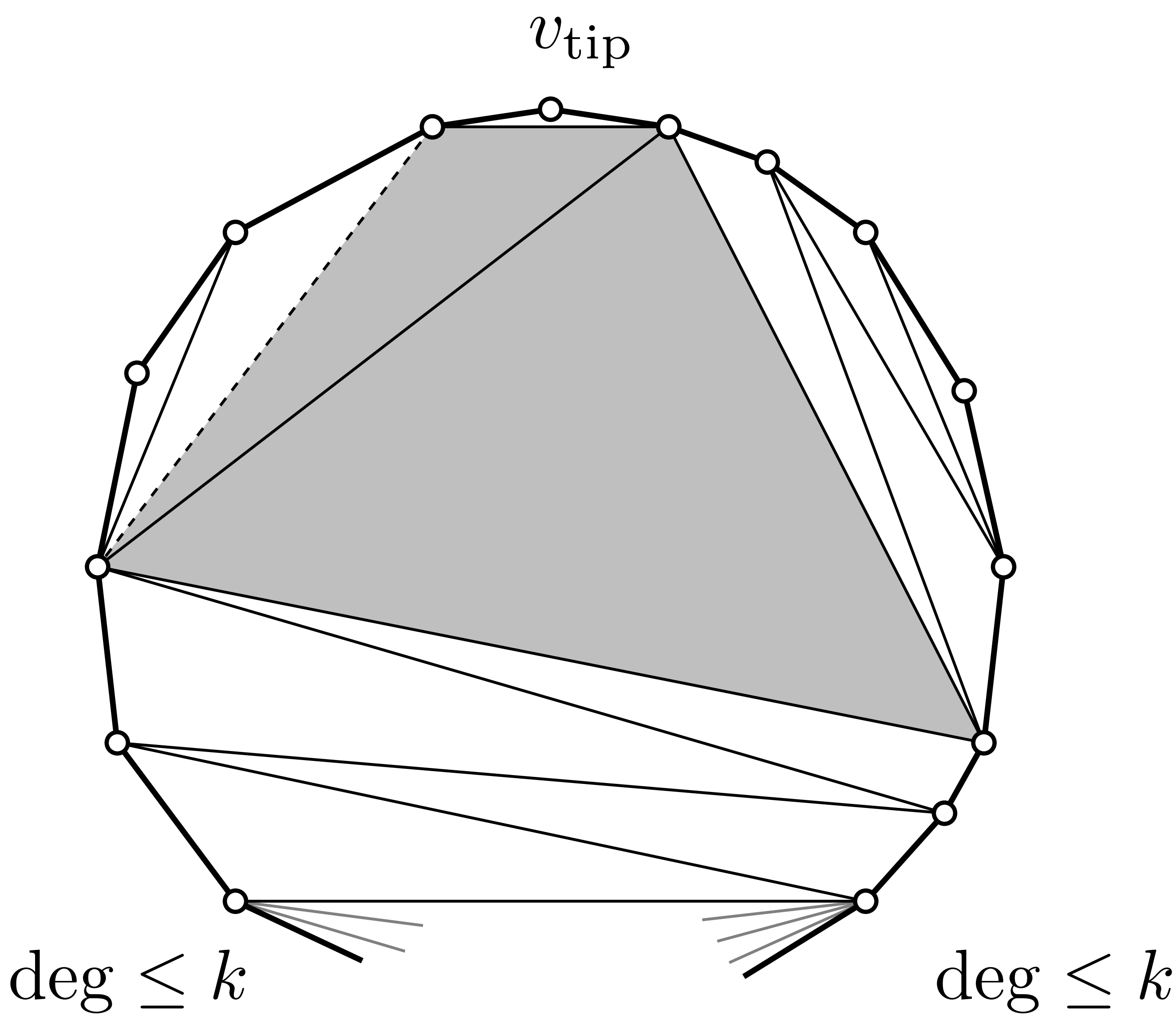

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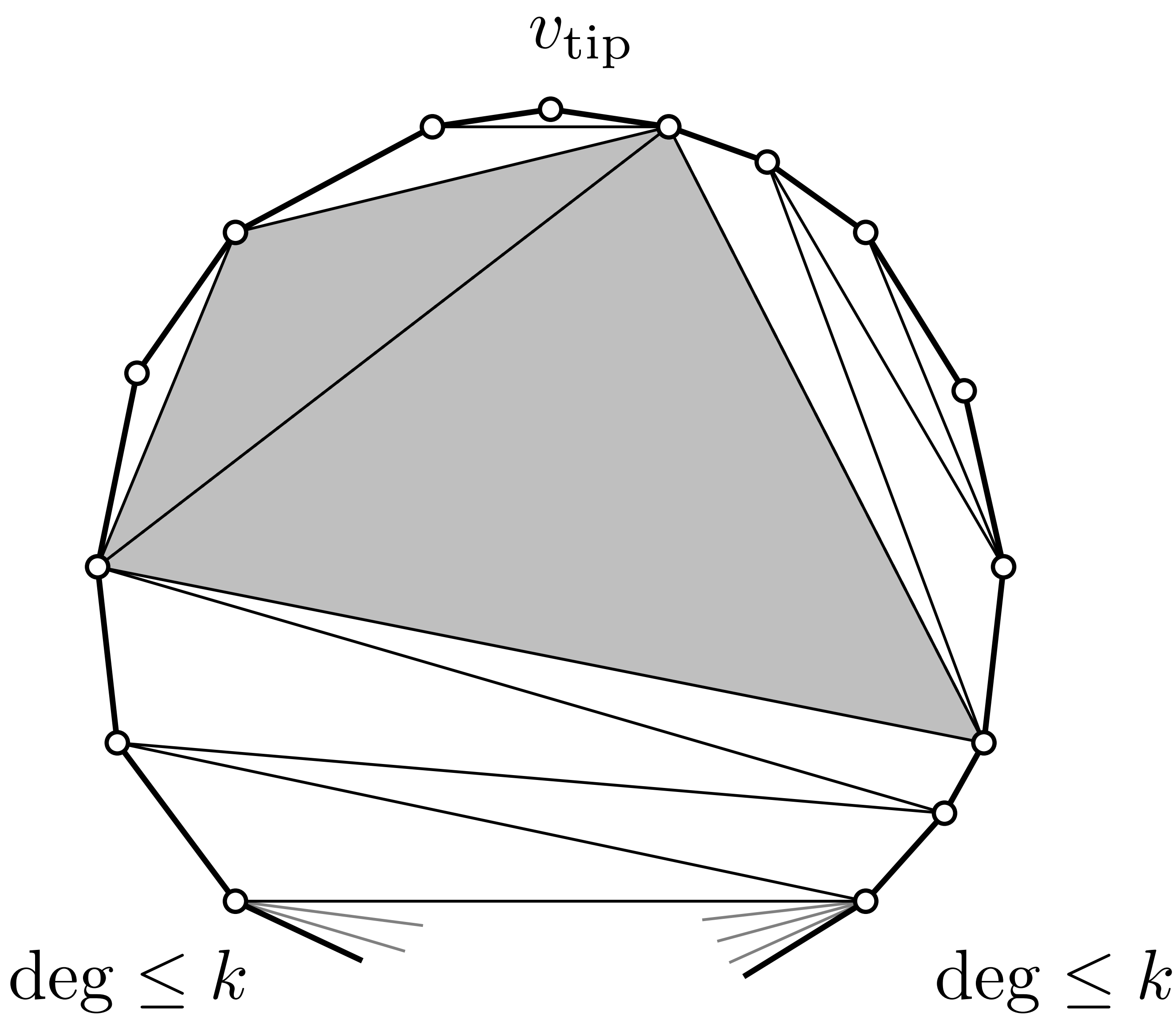

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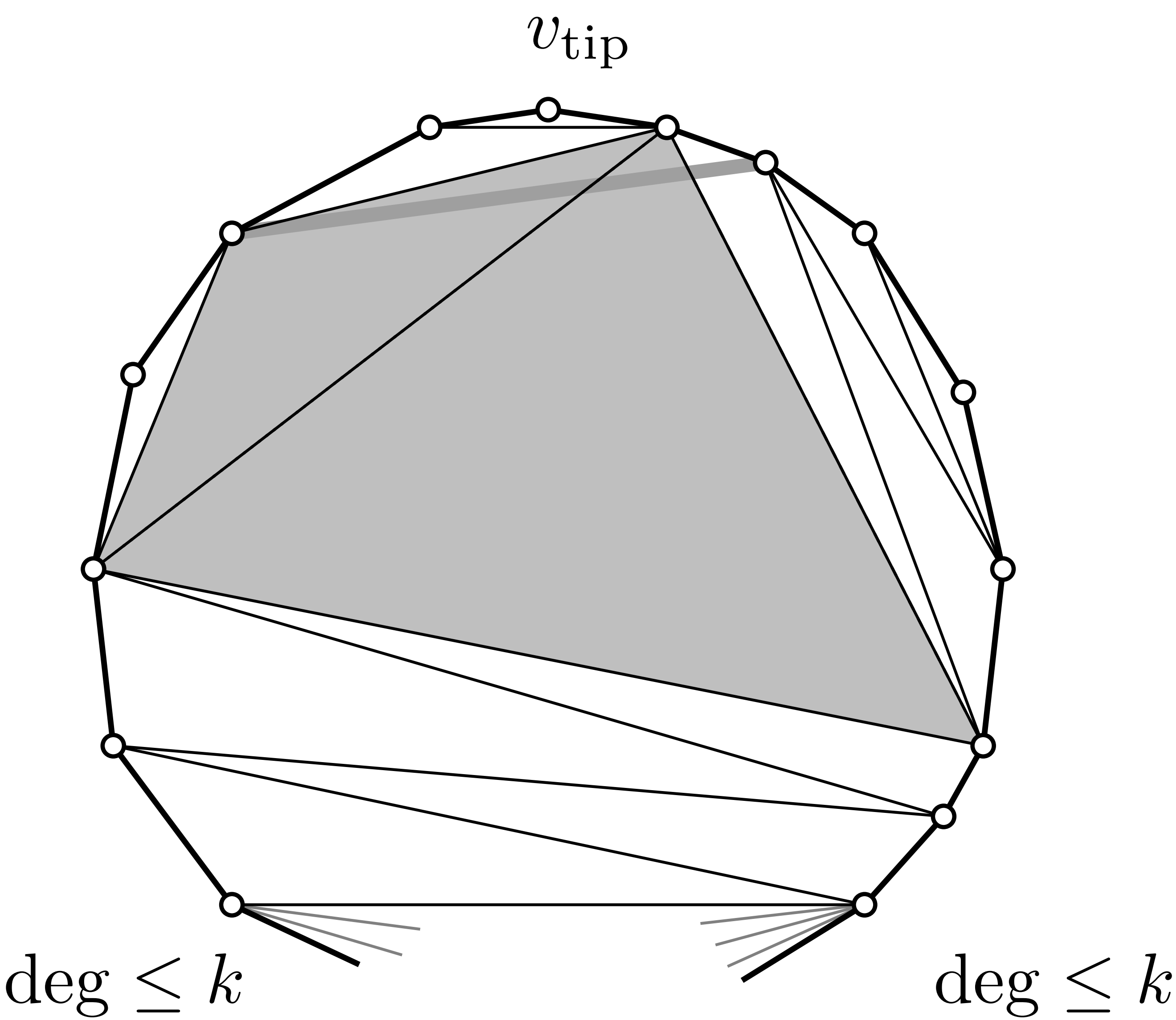

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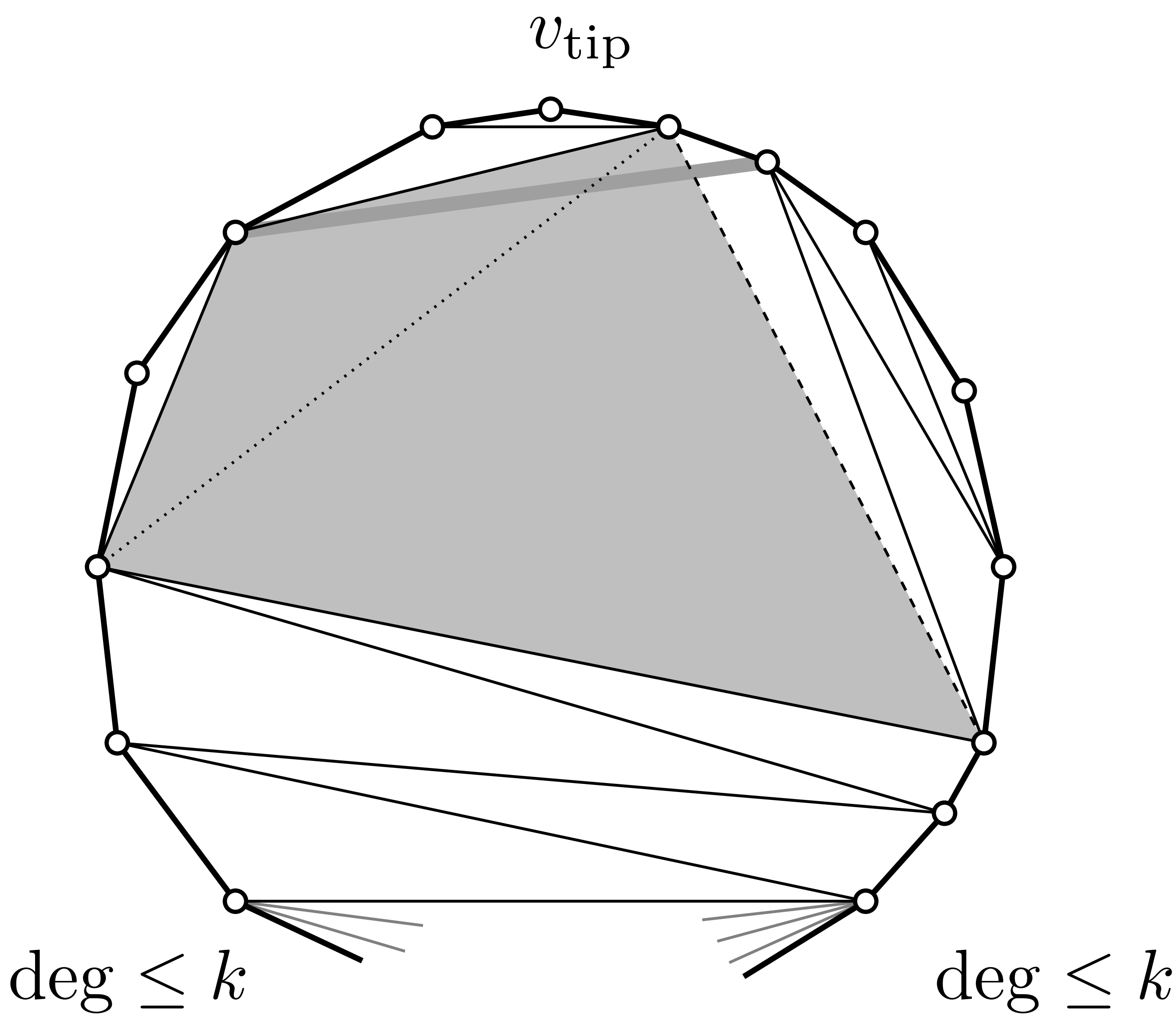

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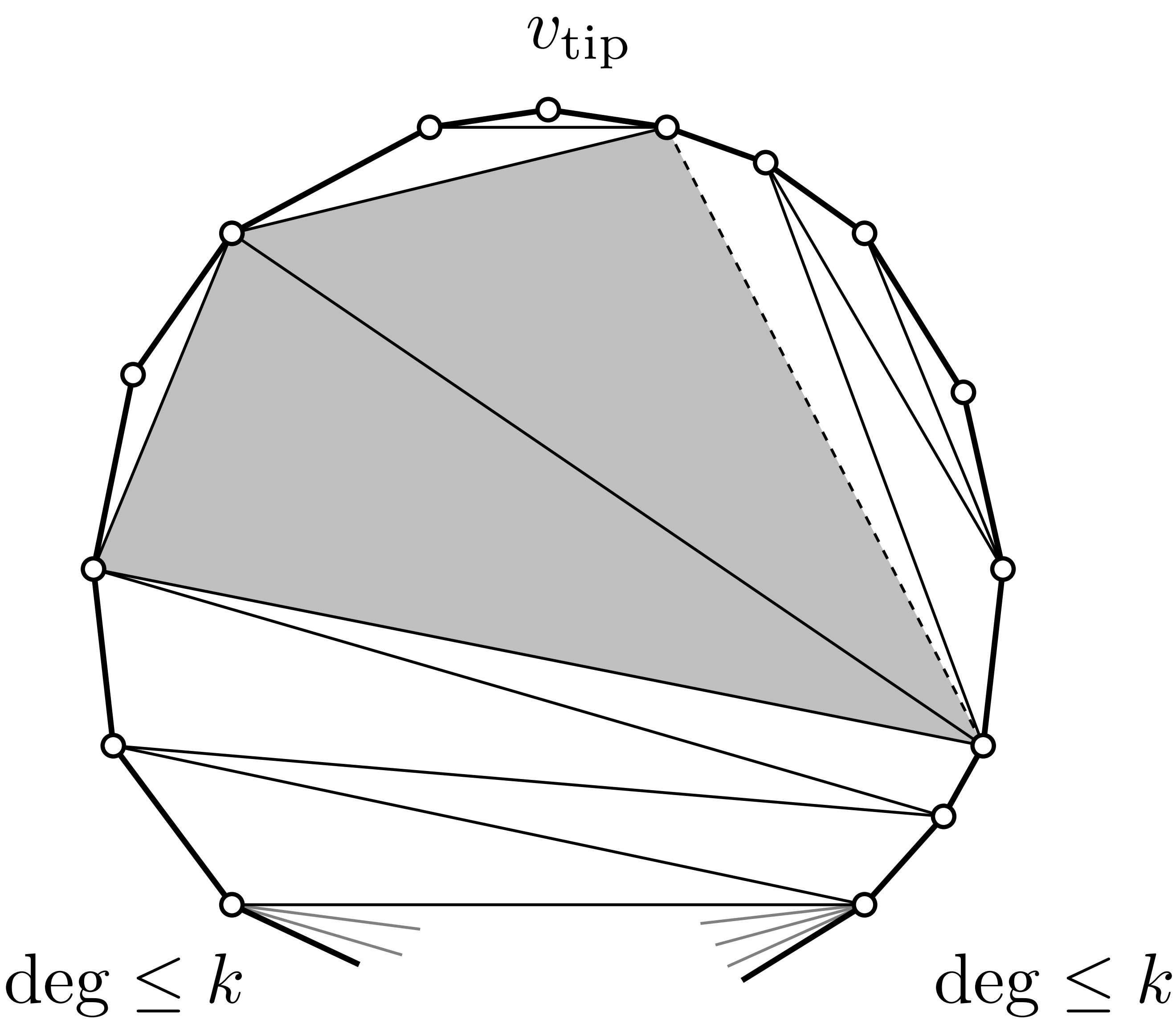

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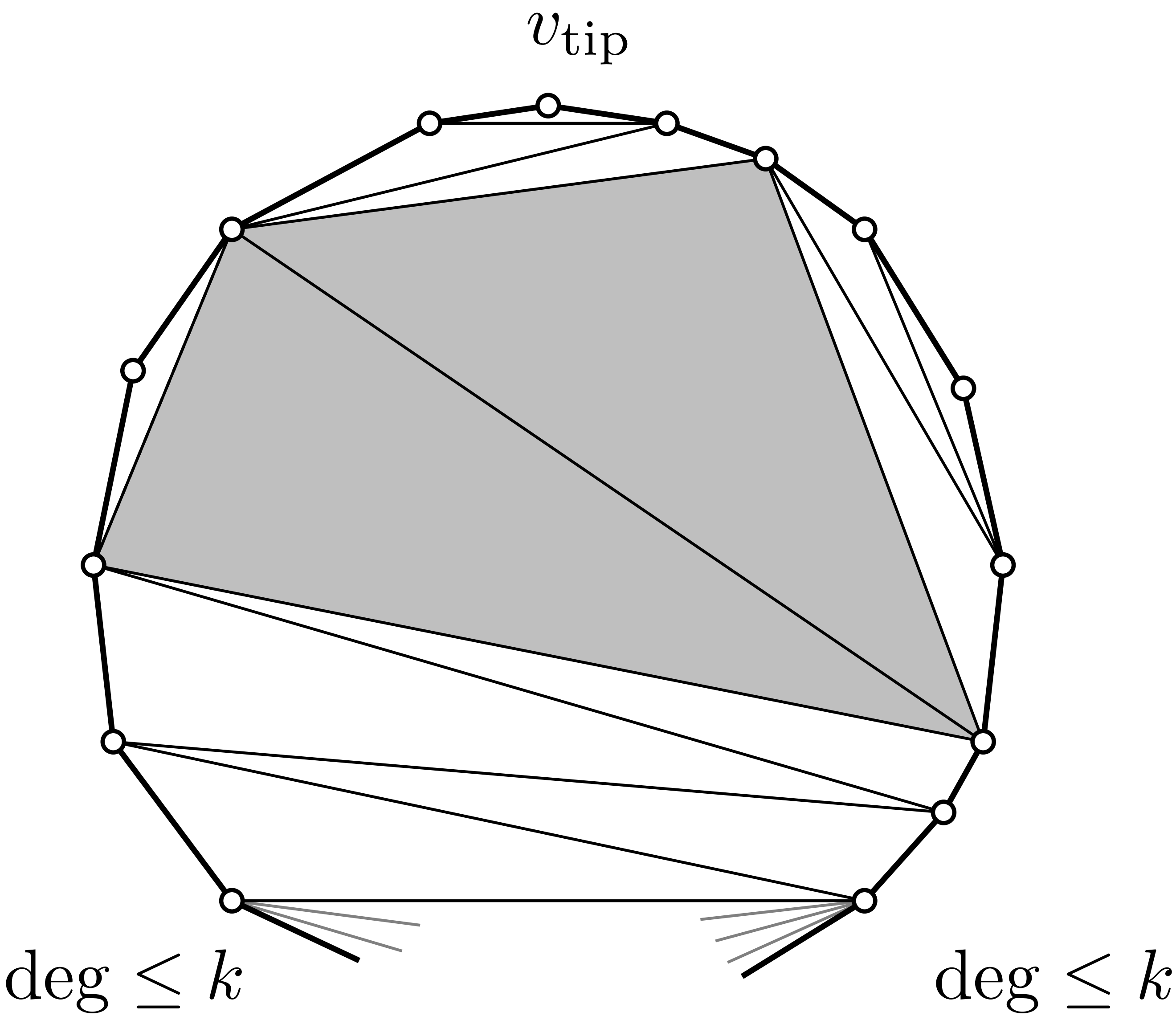

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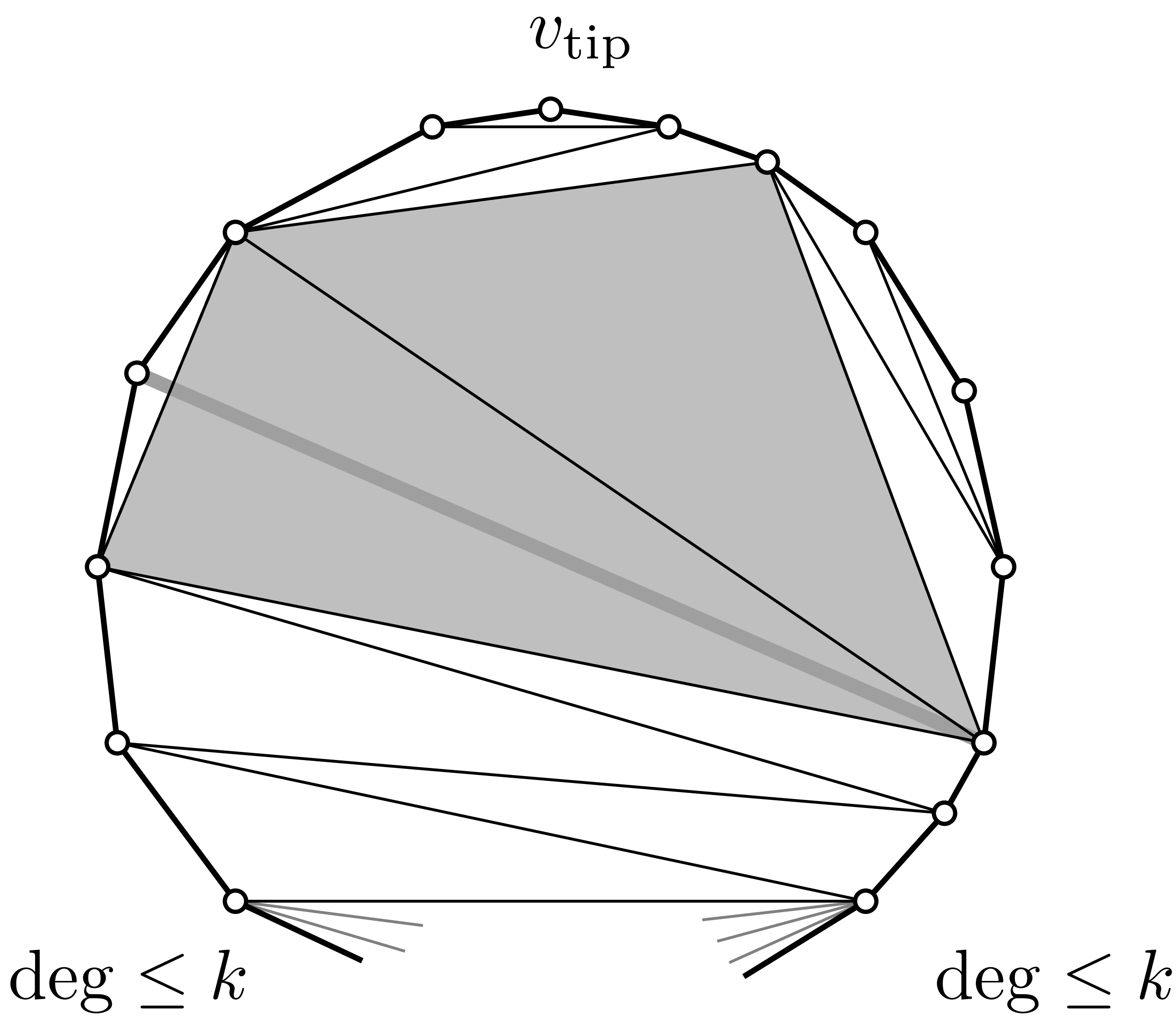

Eliminate a Merge Triangle

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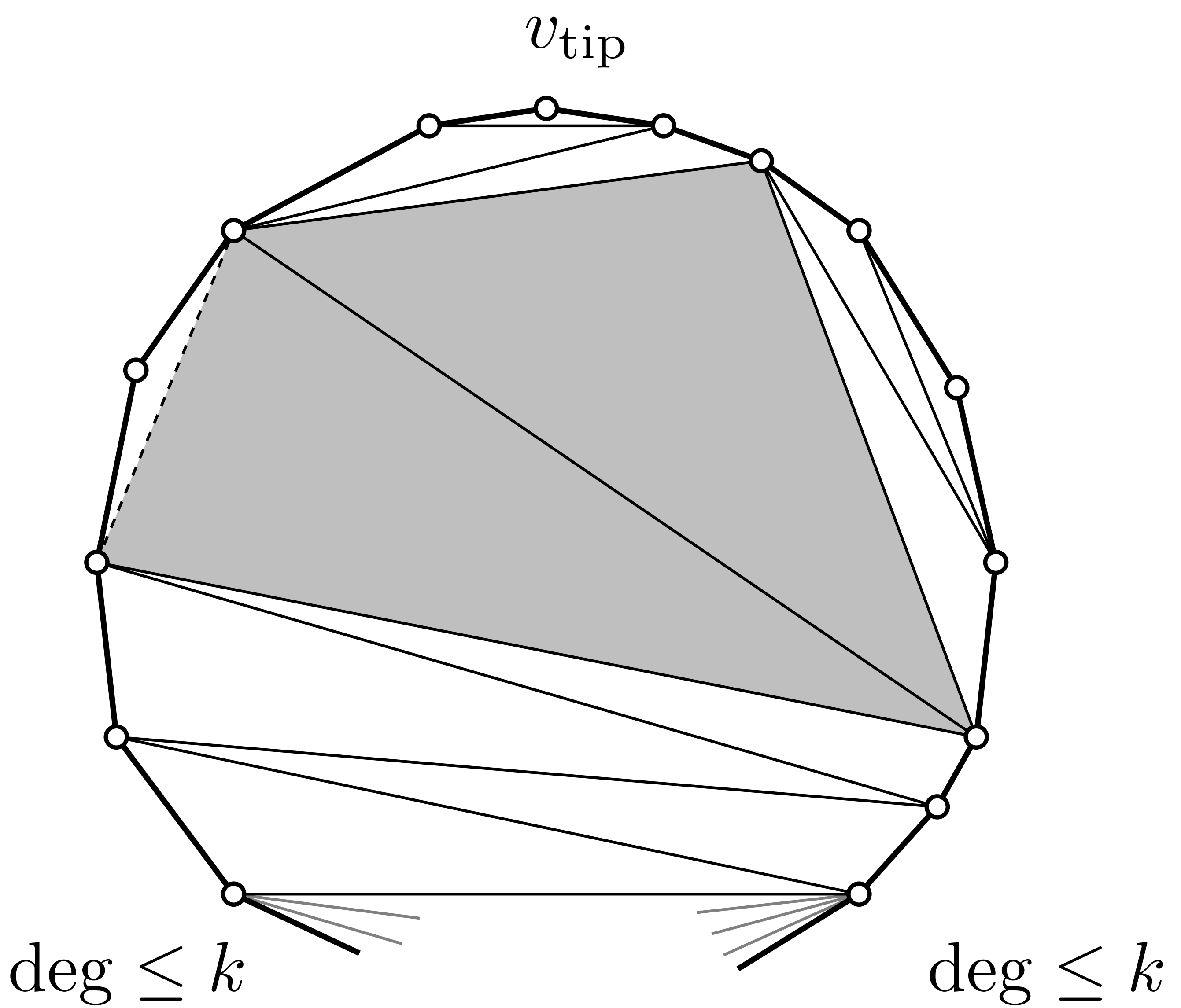

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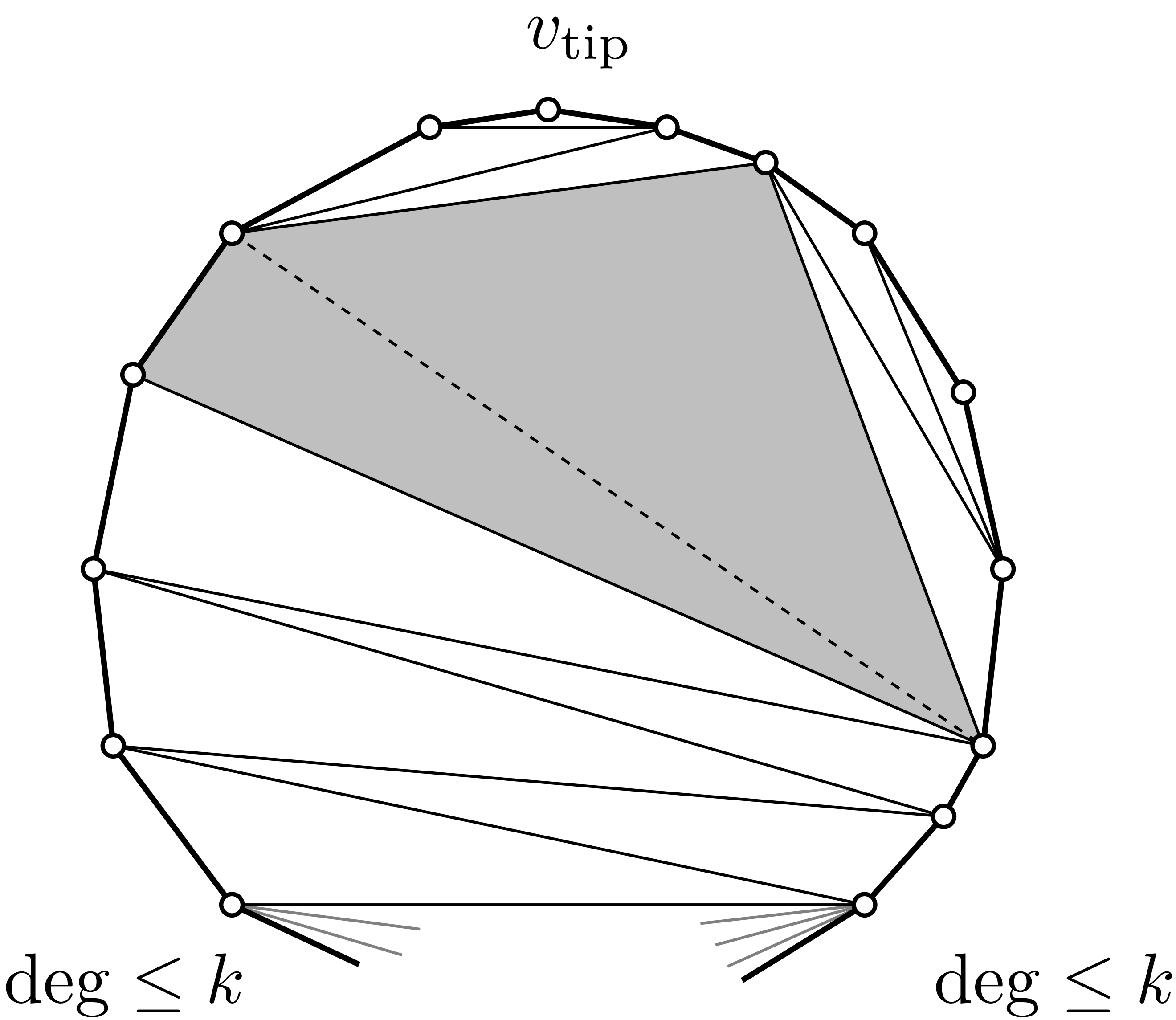

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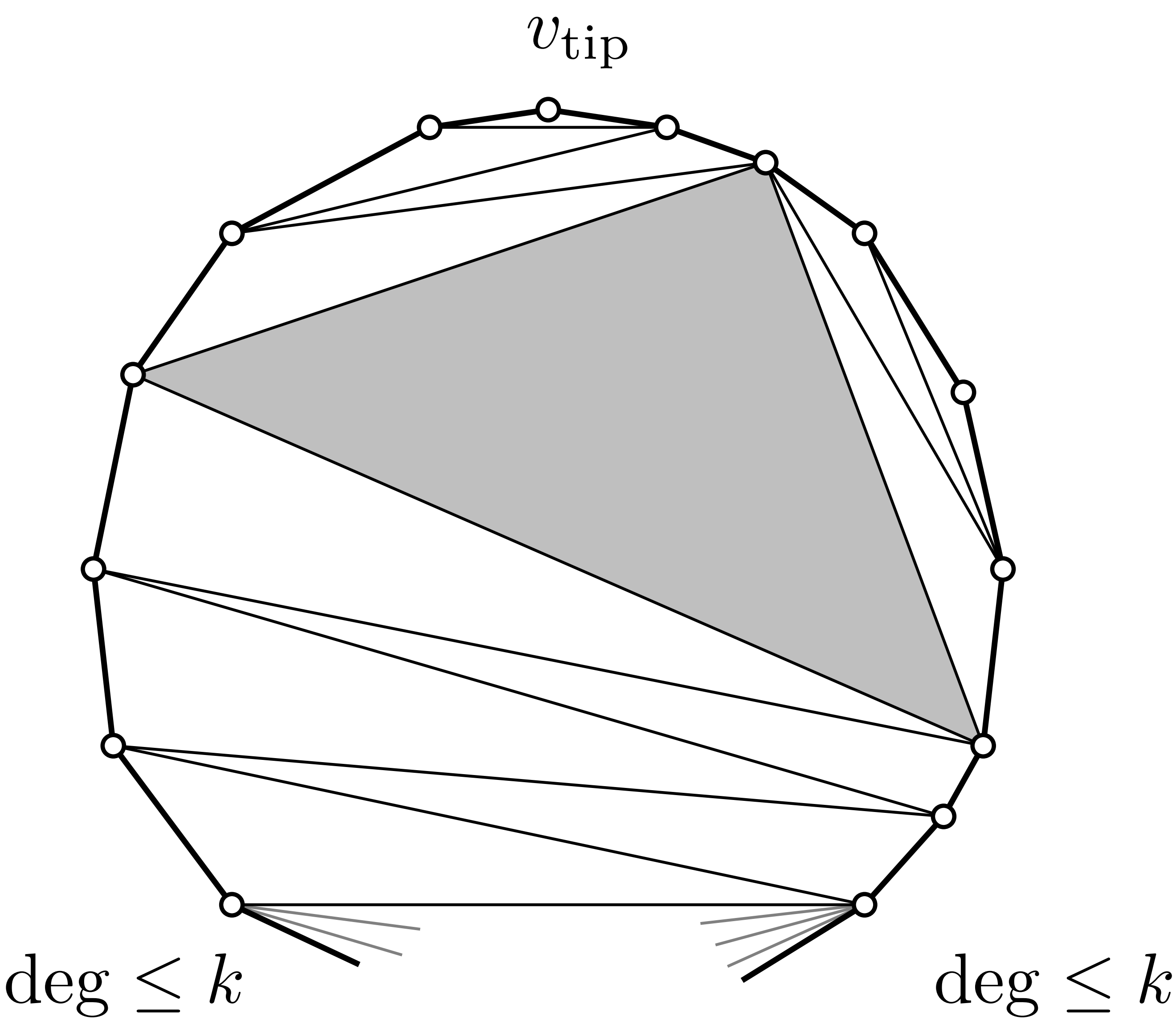

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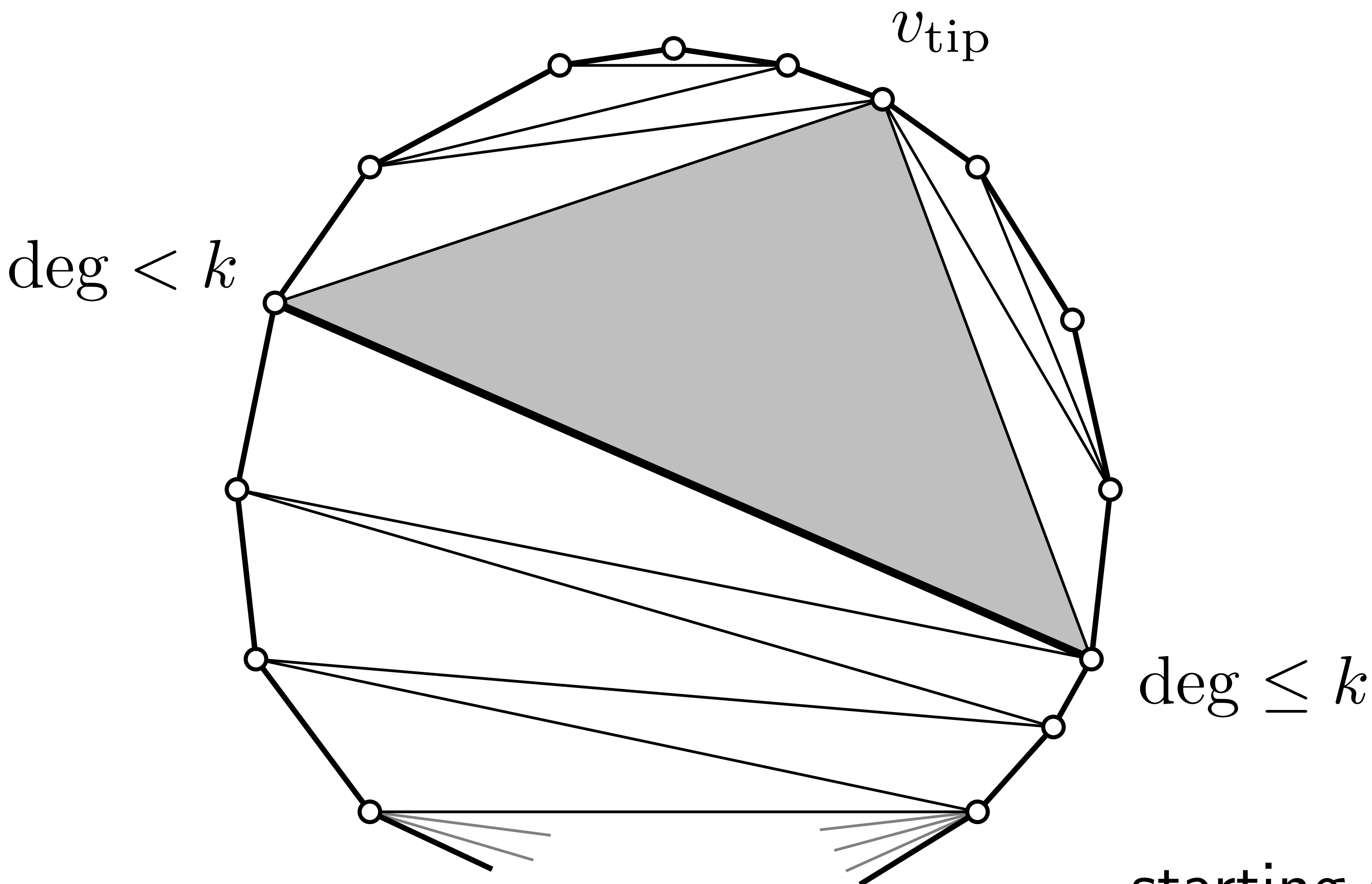
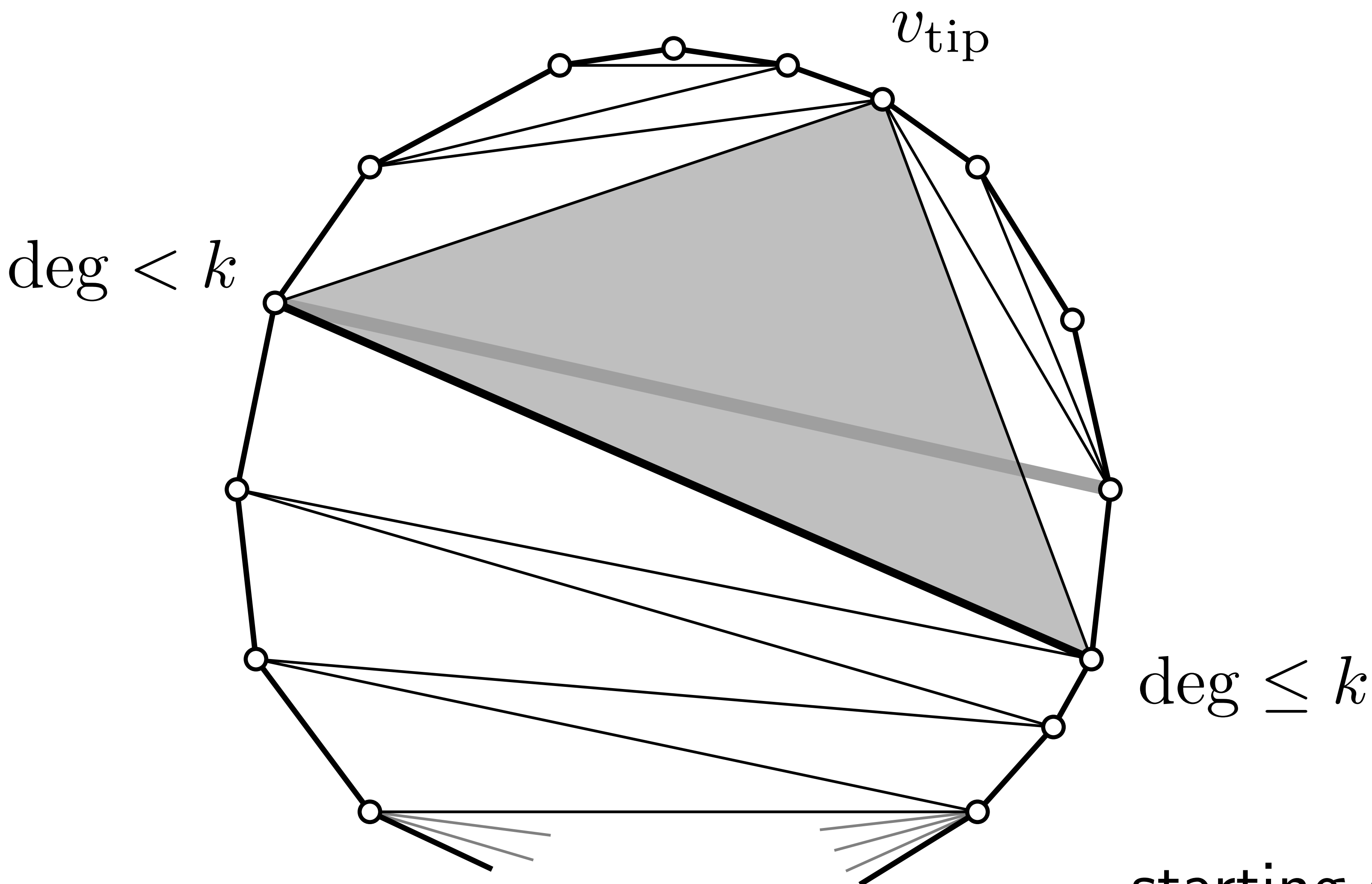


starting situation

Eliminate a Merge Triangle



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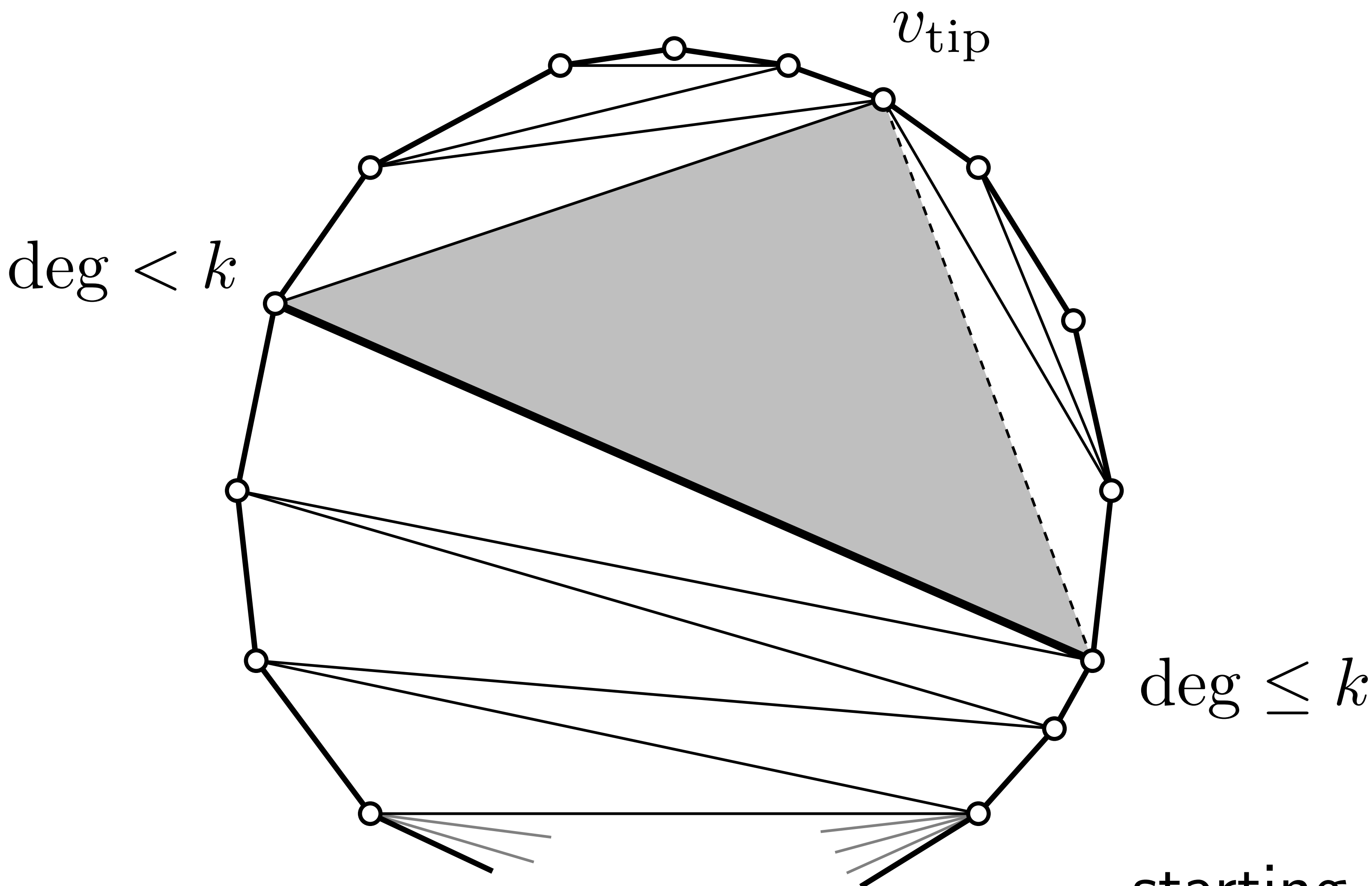

starting situation

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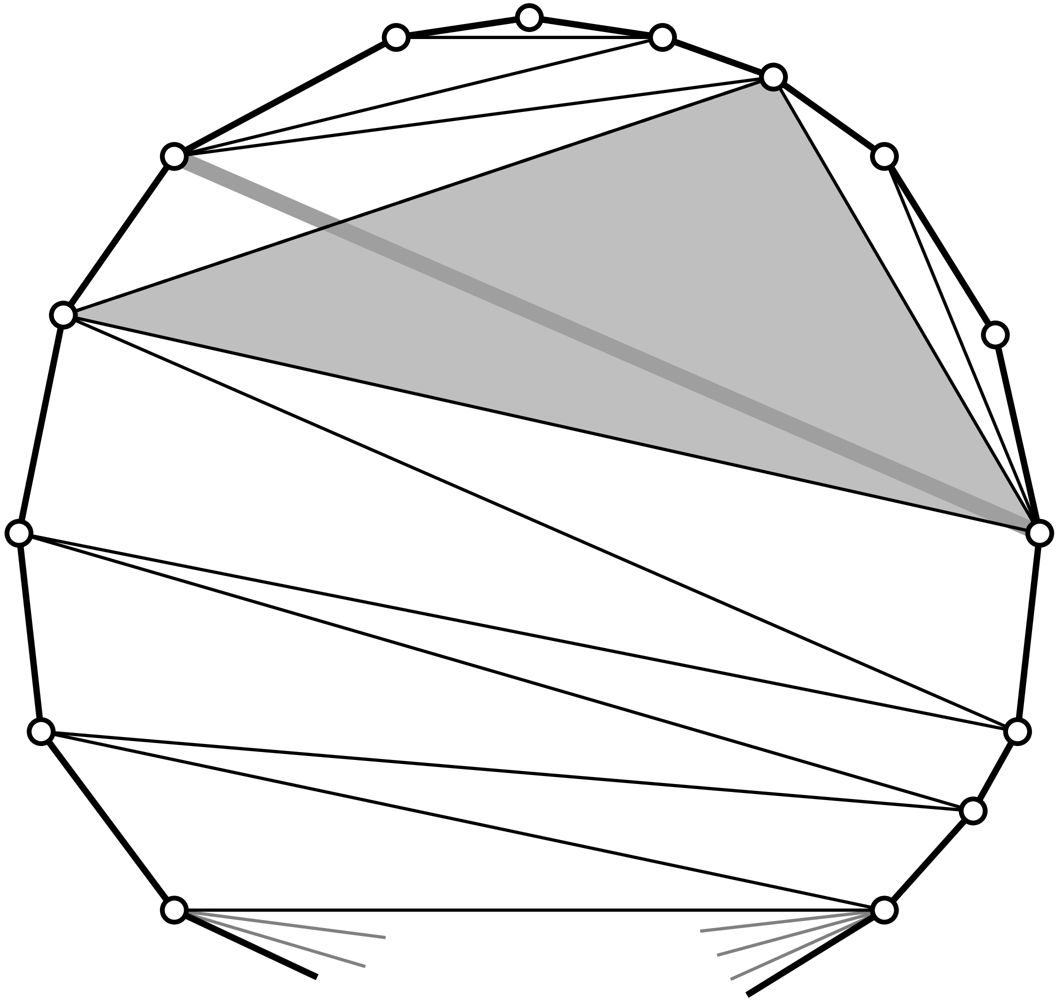

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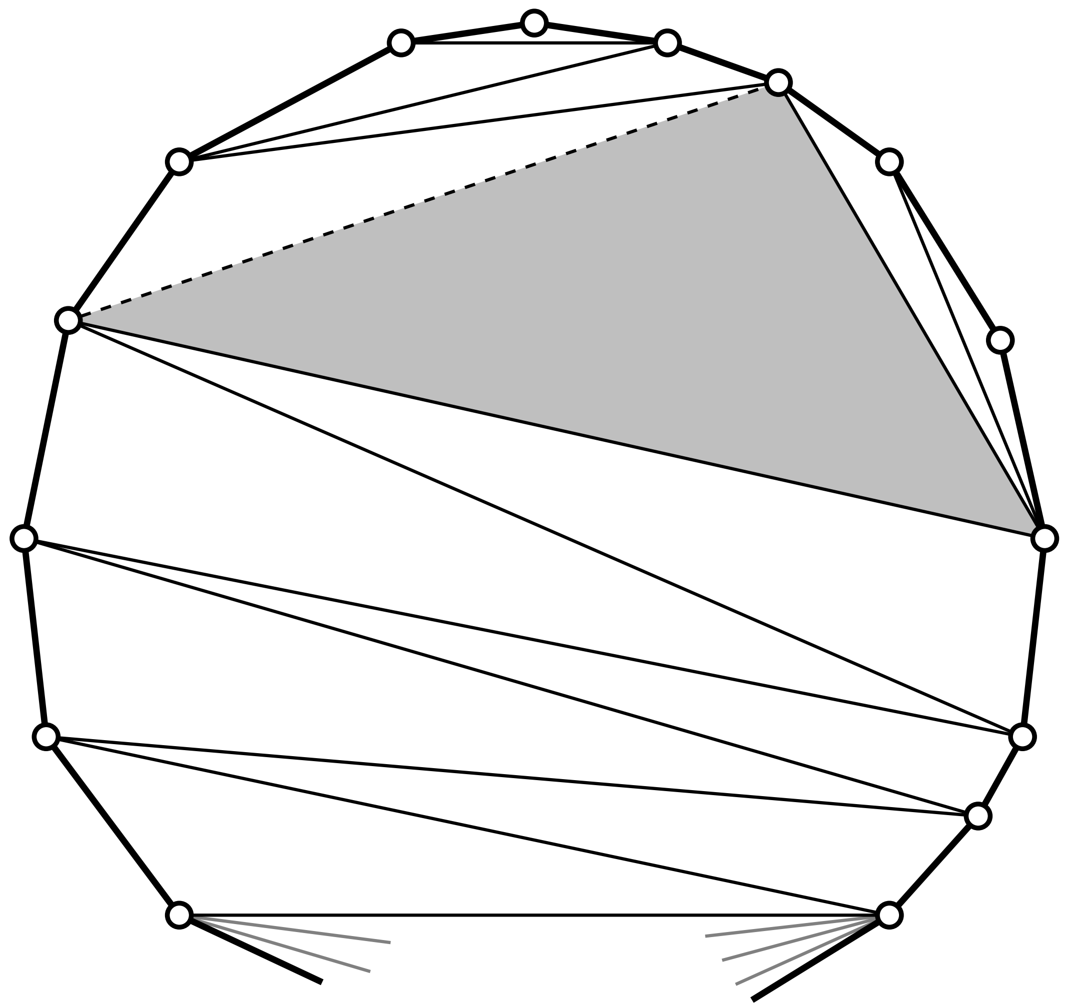

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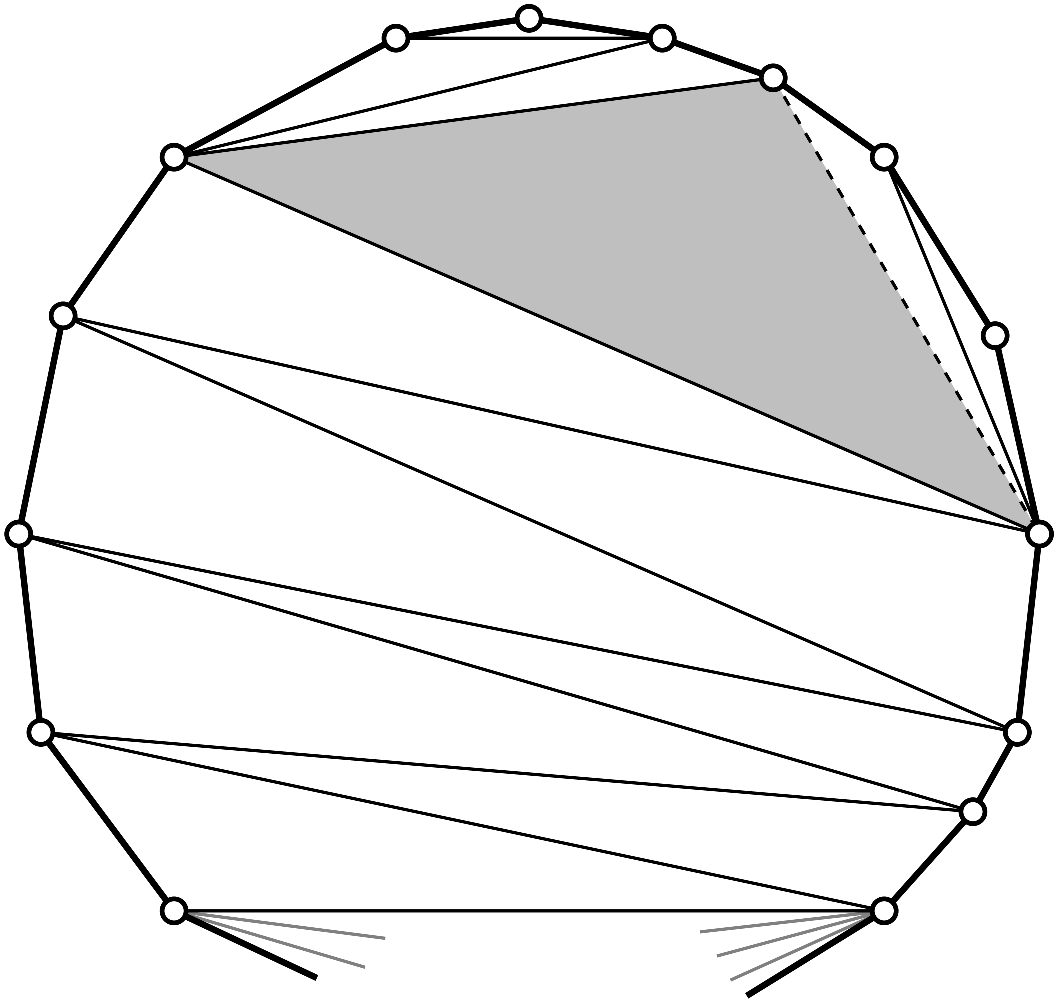

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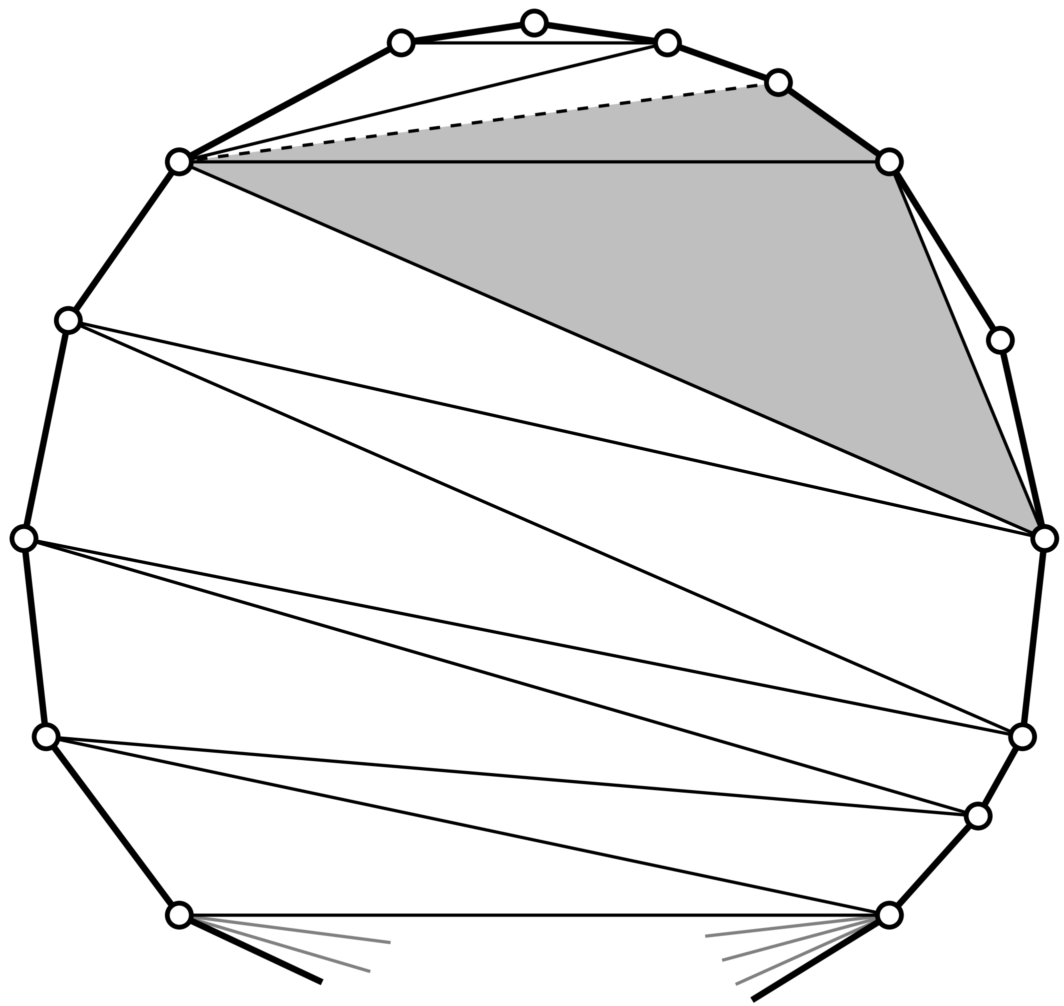

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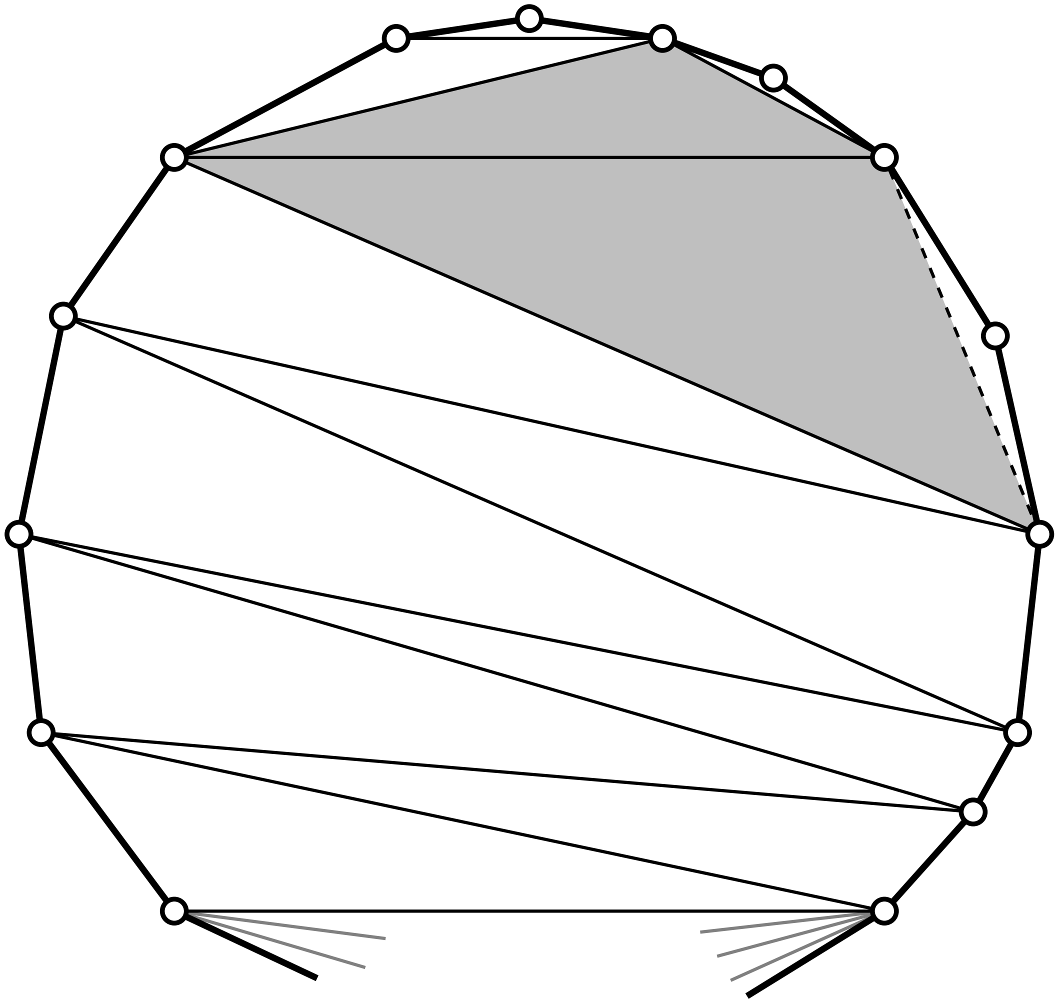

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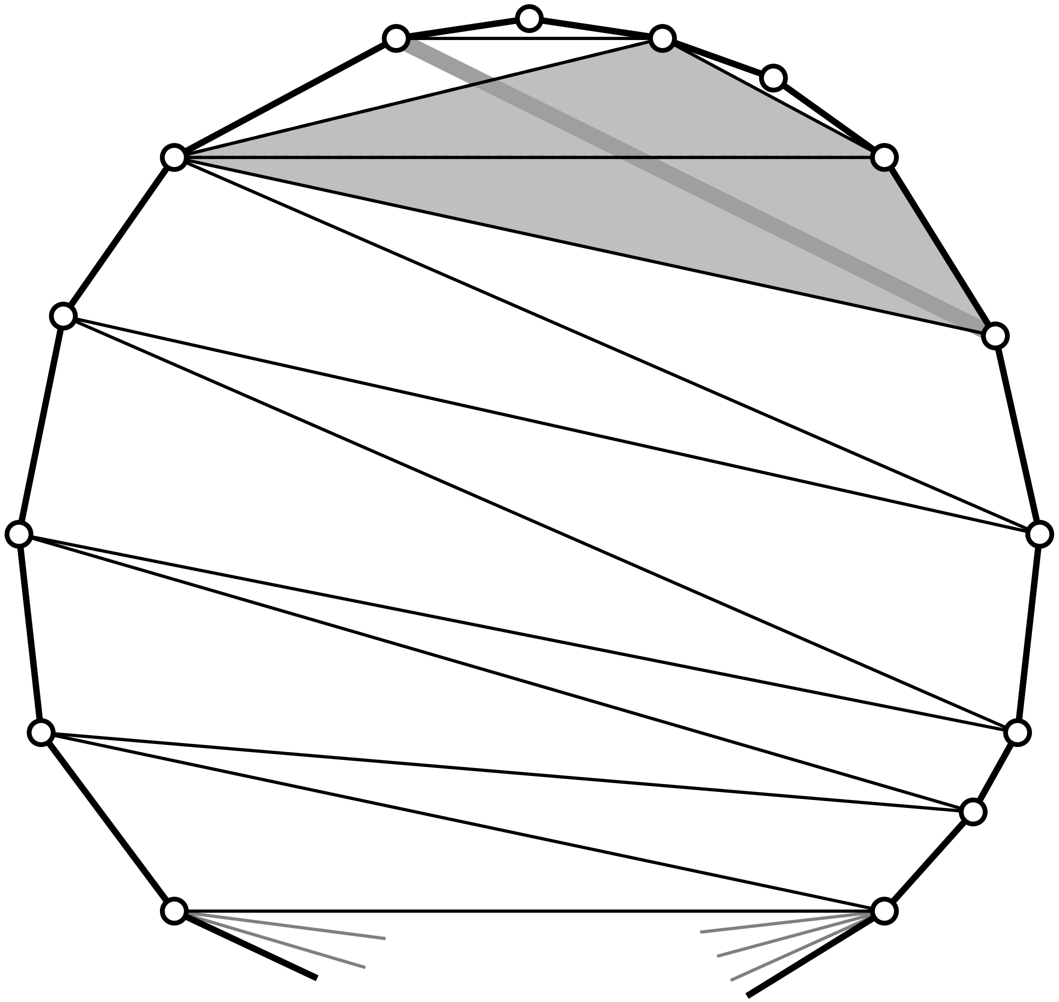

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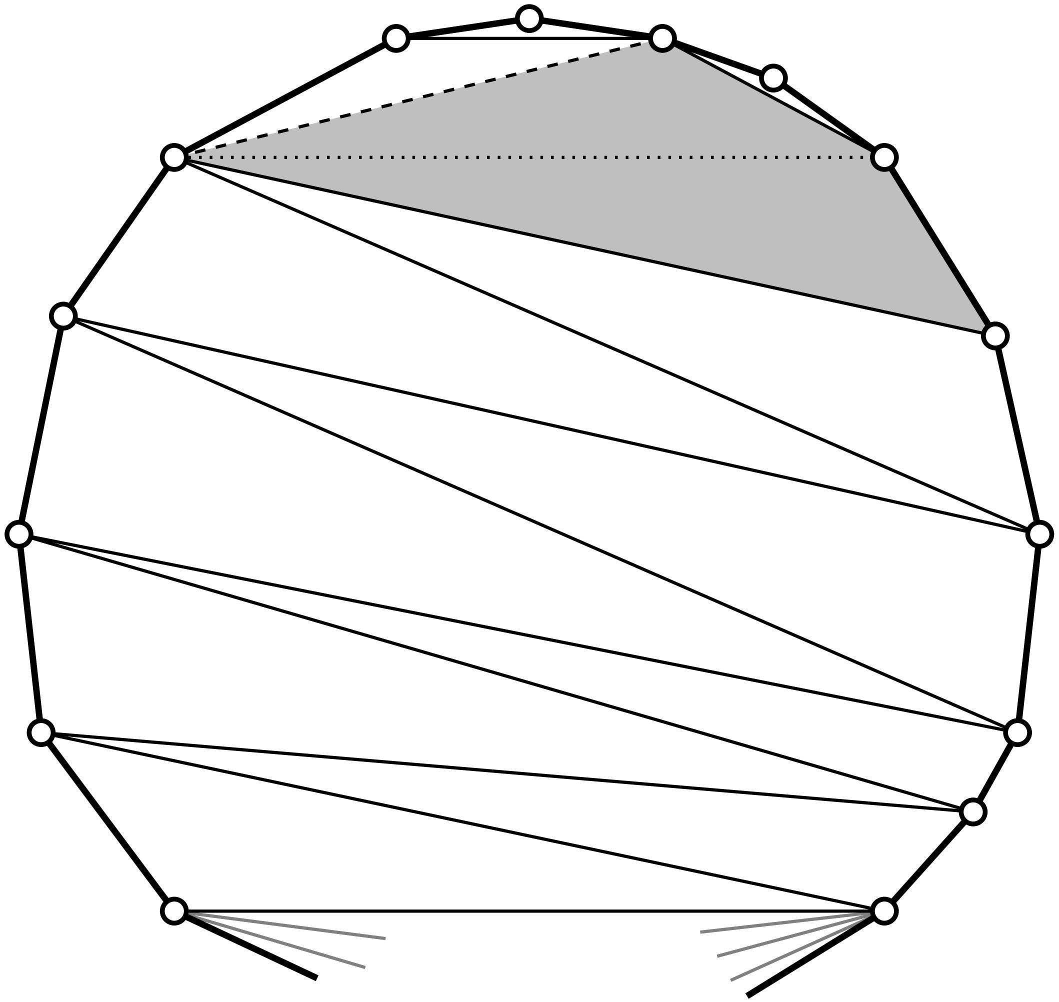

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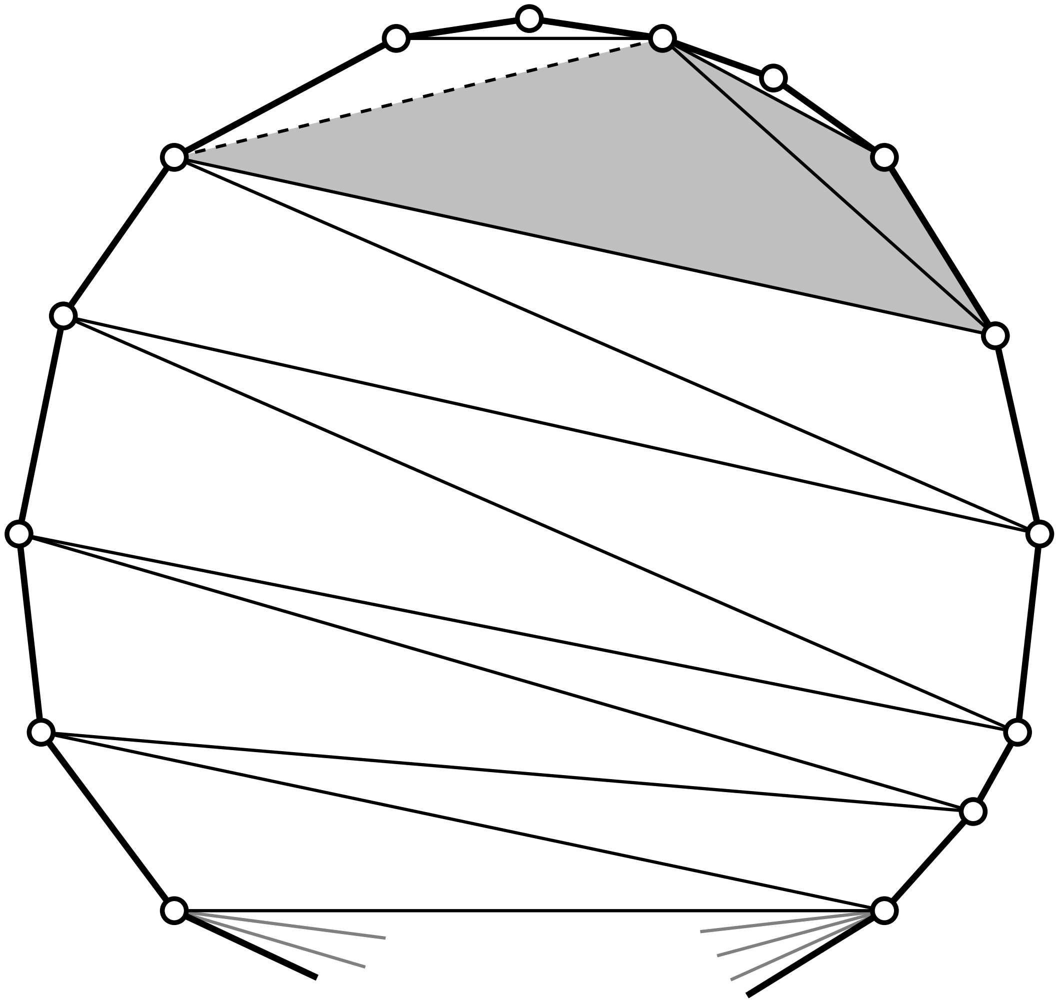

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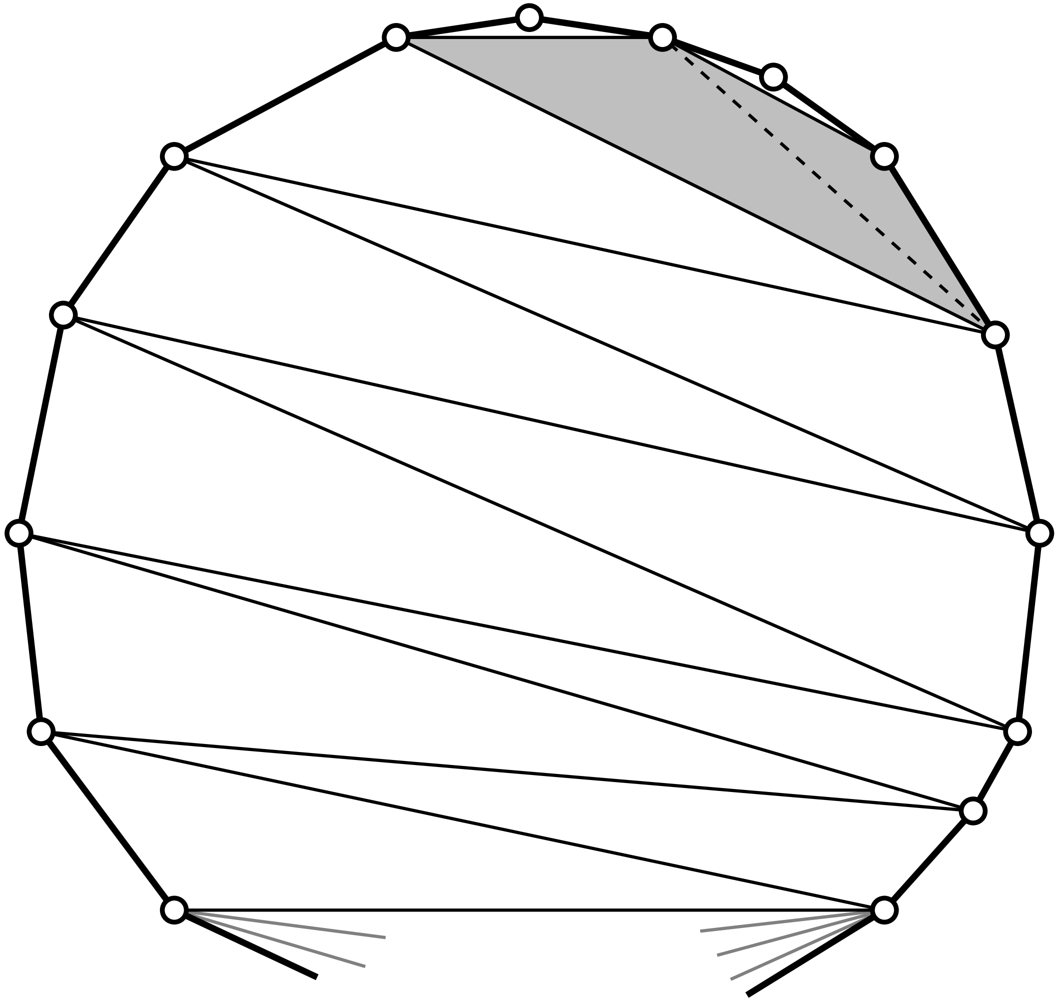

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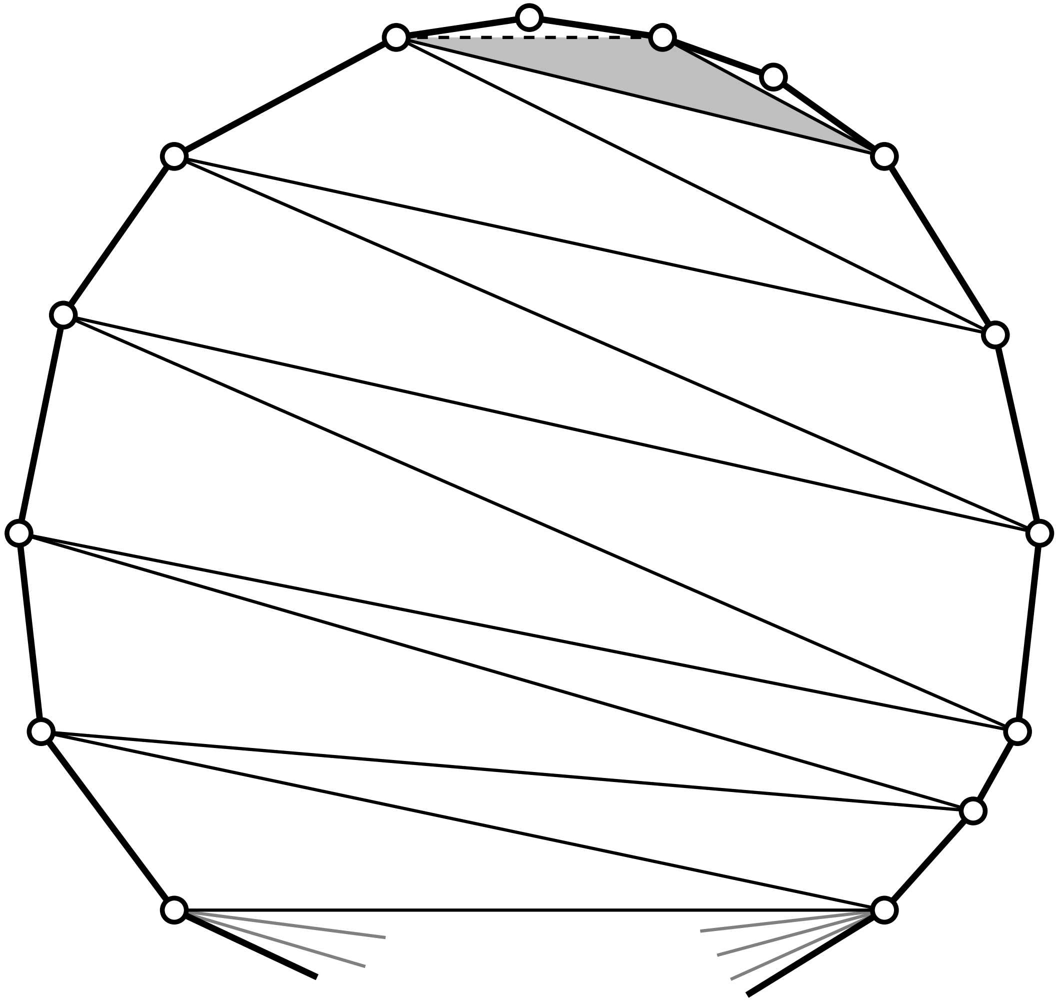

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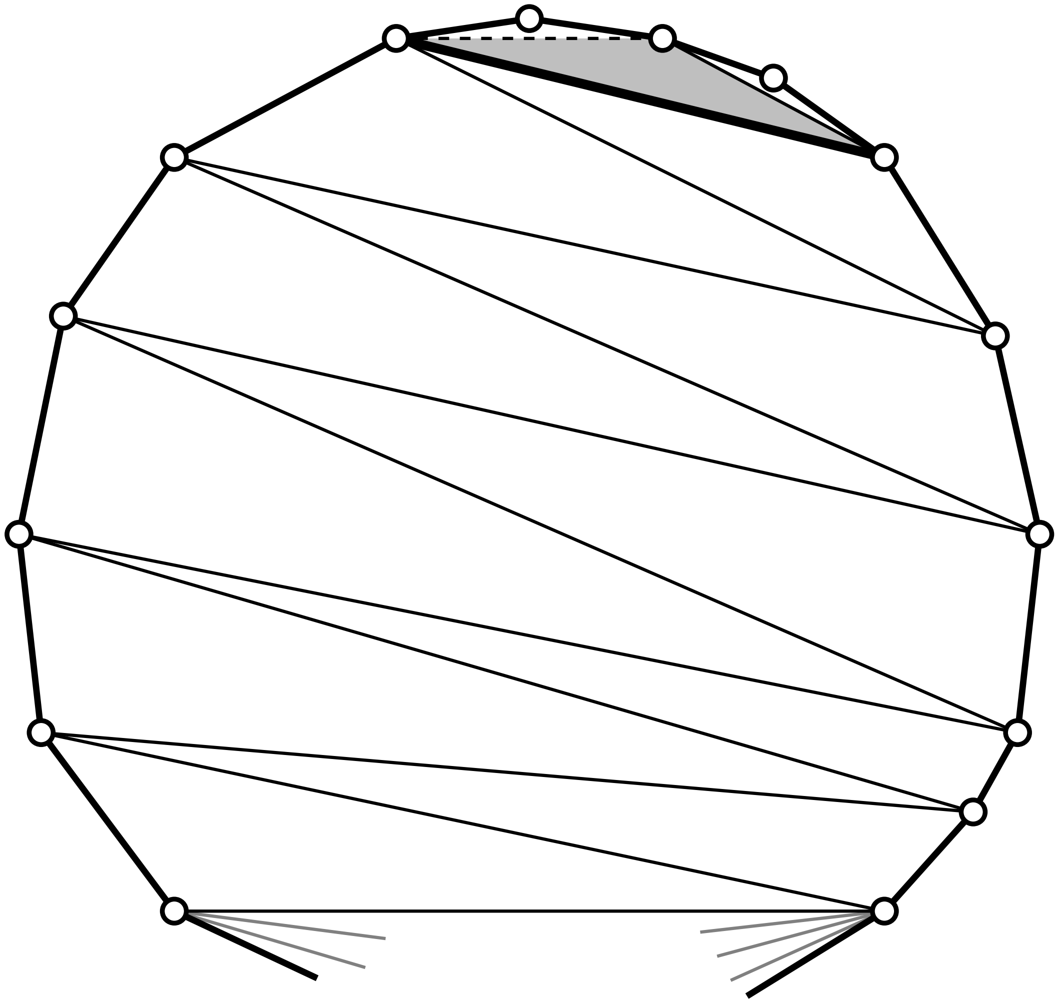

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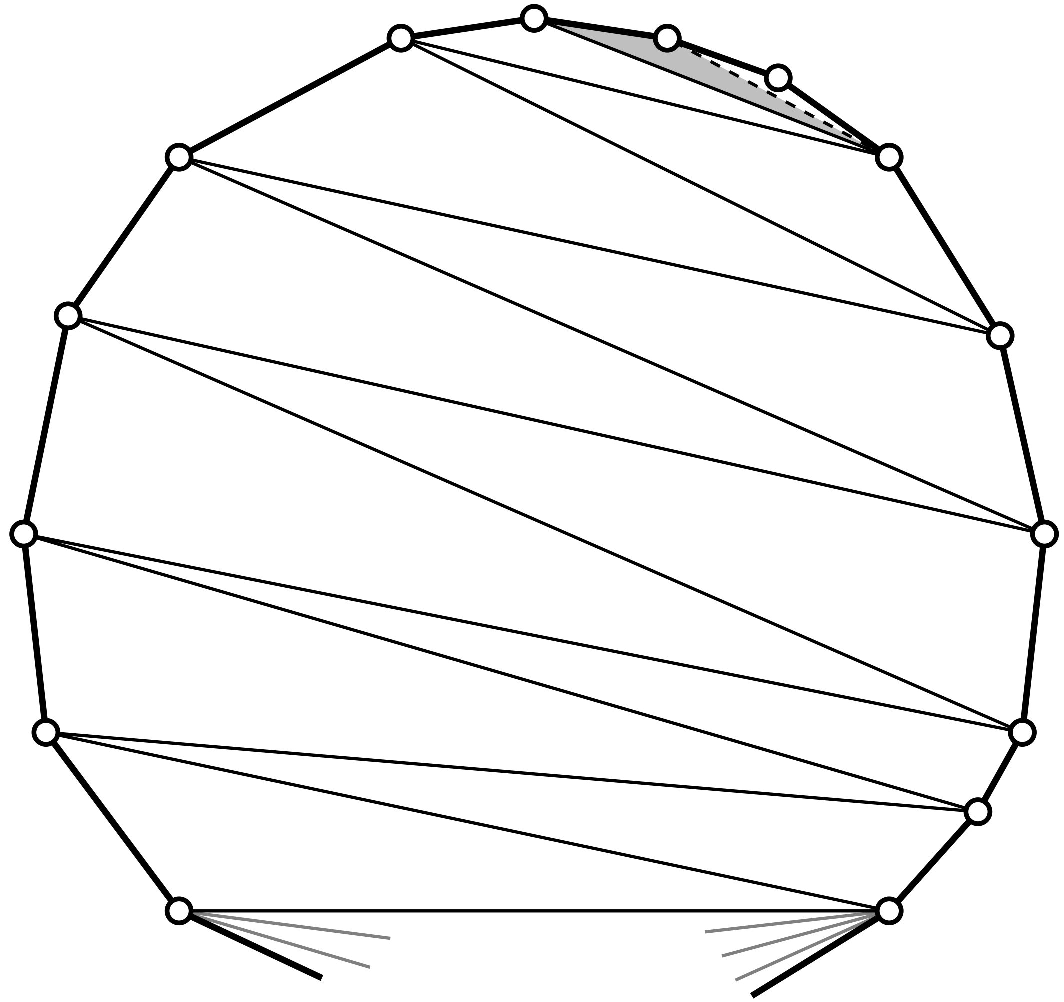

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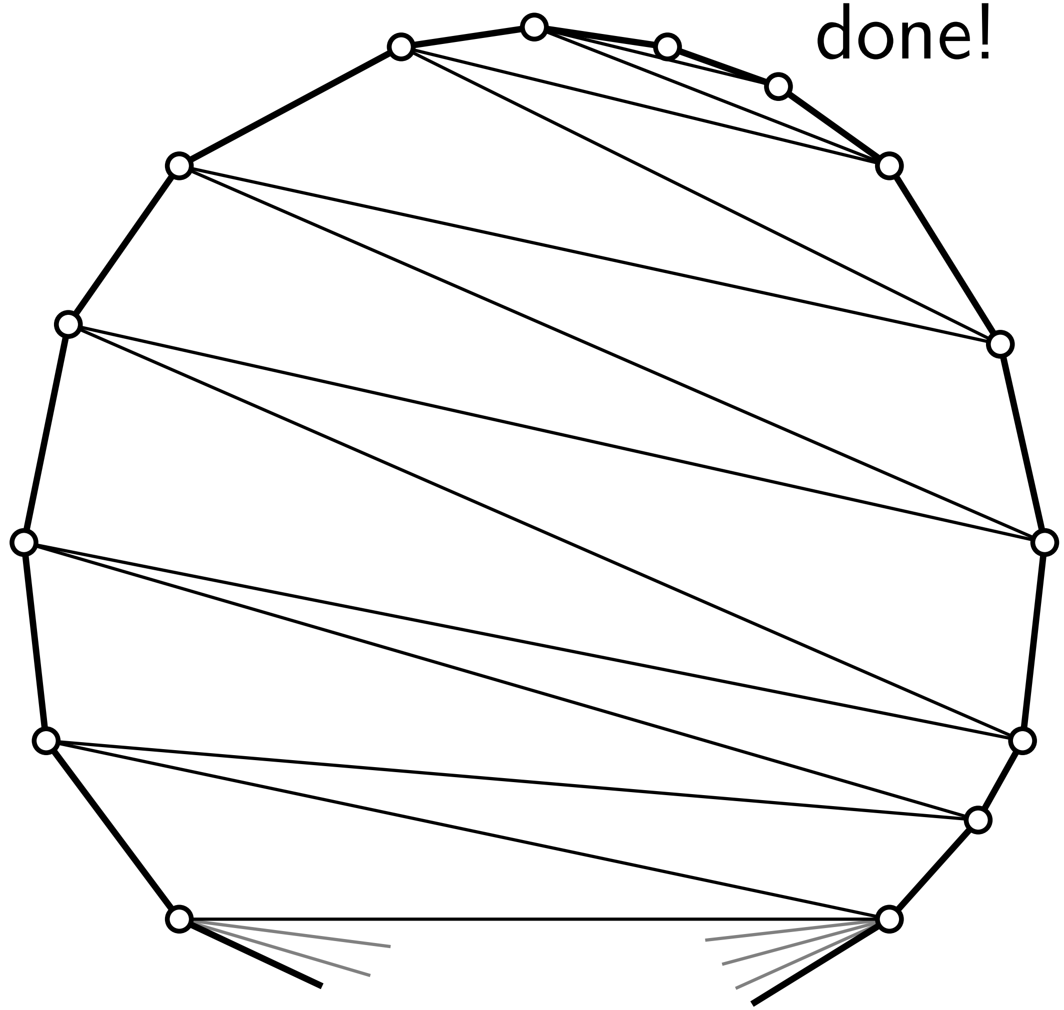
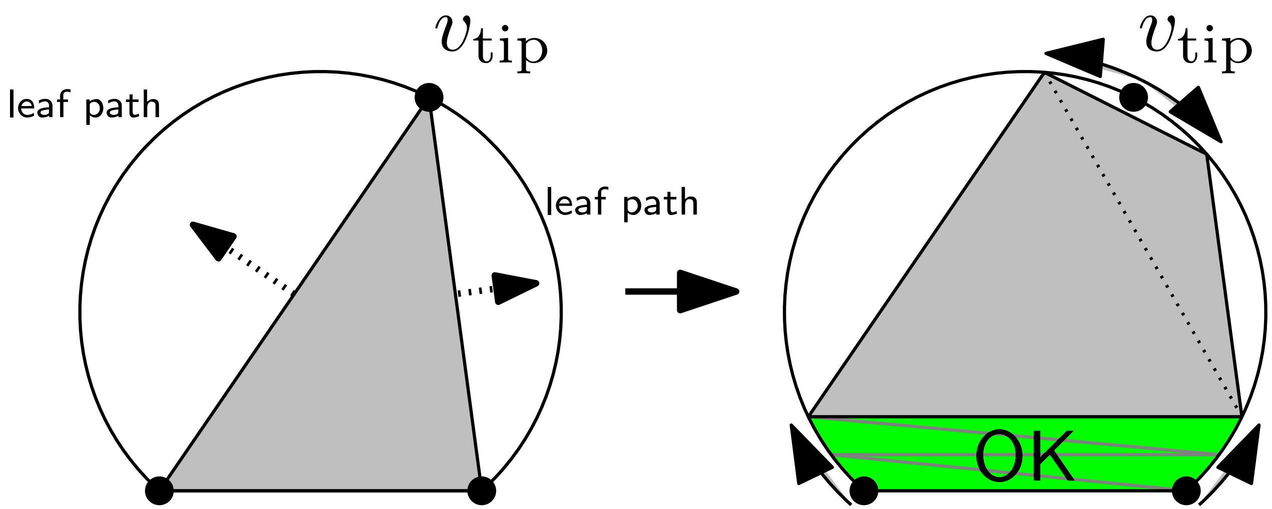


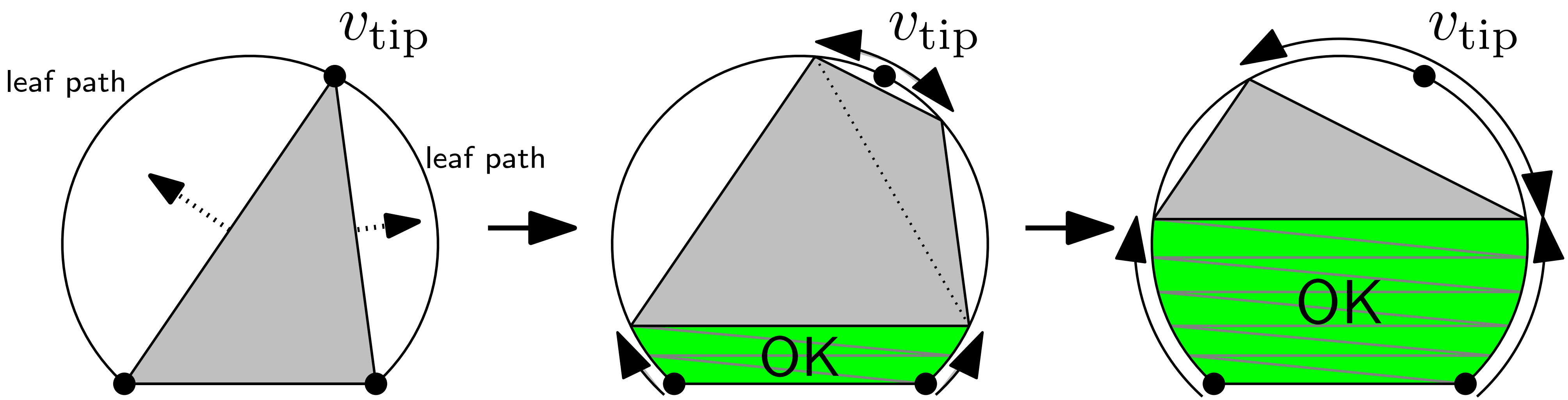
done!

$O(n)$ flips.

Schematic view



Schematic view



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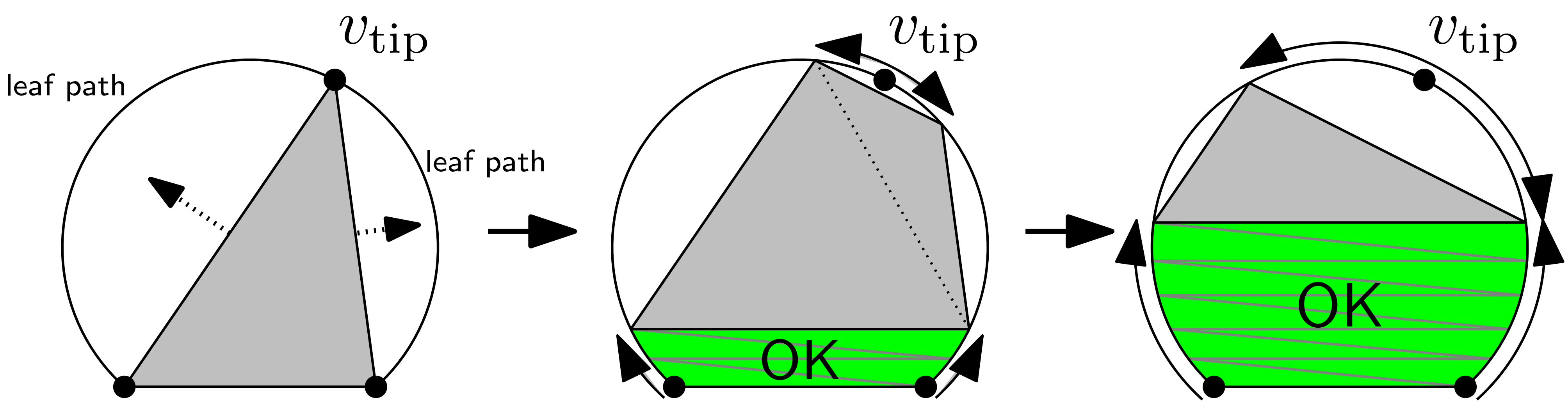


ANALYSIS:

n operations:

extending a zigzag by 1

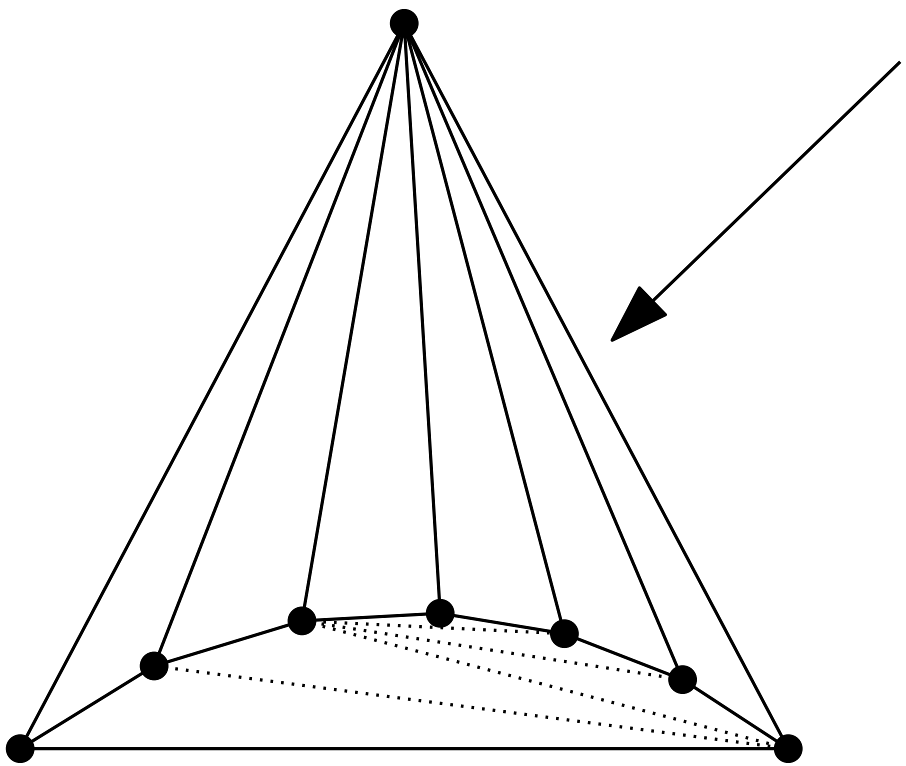
merging to leaf paths

Each operation takes $O(n)$ flips. $\rightarrow O(n^2)$ in total.

$+$ final rotation: $O(n^2)$ flips. $\square$

General Point Sets

Bounded degree is not always possible:



These edges are part of every triangulation.

maximum degree $= n - 1$

General Point Sets

Theorem. For any $k \geq 4$, there are arbitrarily large point sets with two triangulations T, T' with maximum degree k that cannot be transformed into each other by a sequence of flips without exceeding vertex degree k.

General Point Sets

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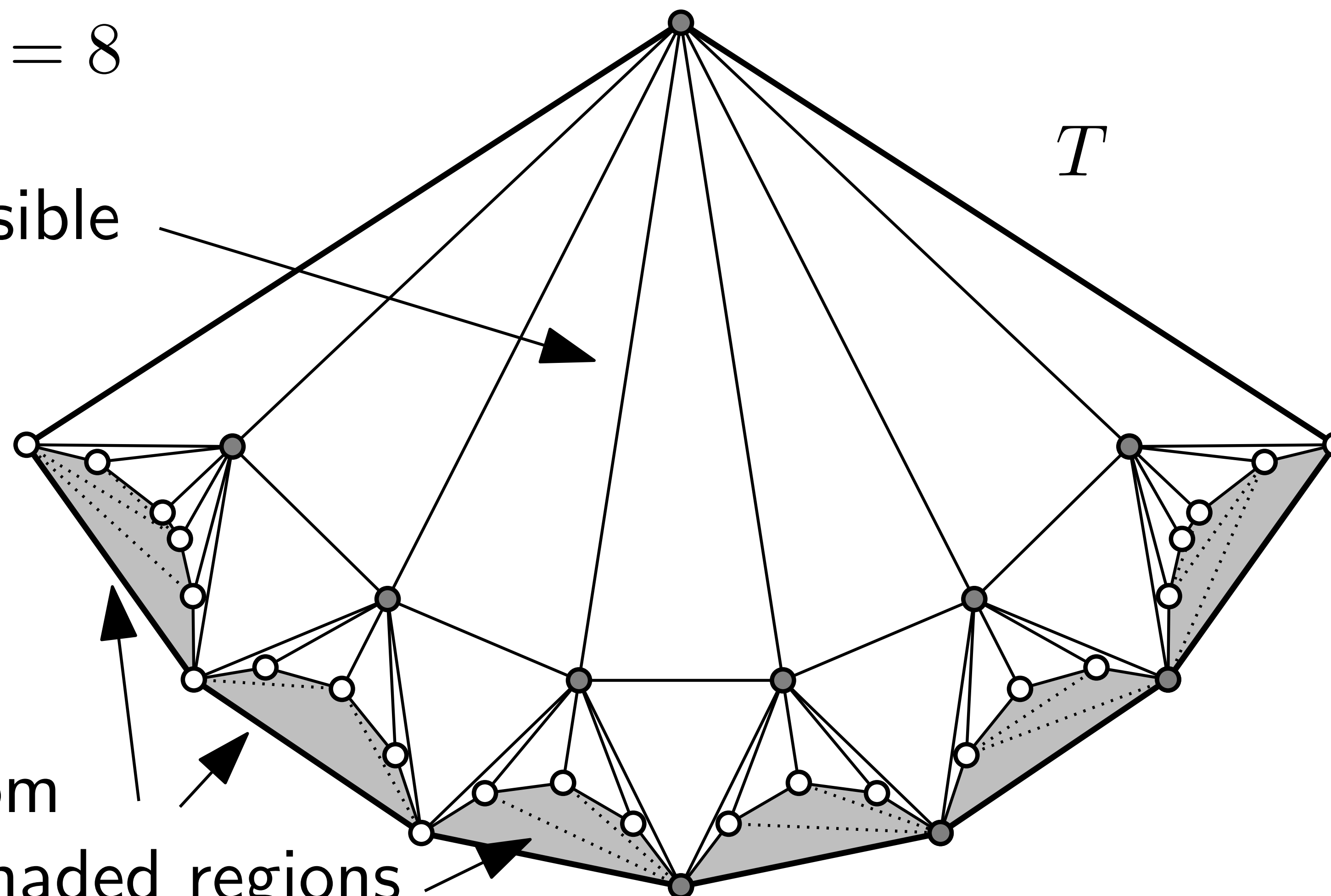
Example: $k = 8$

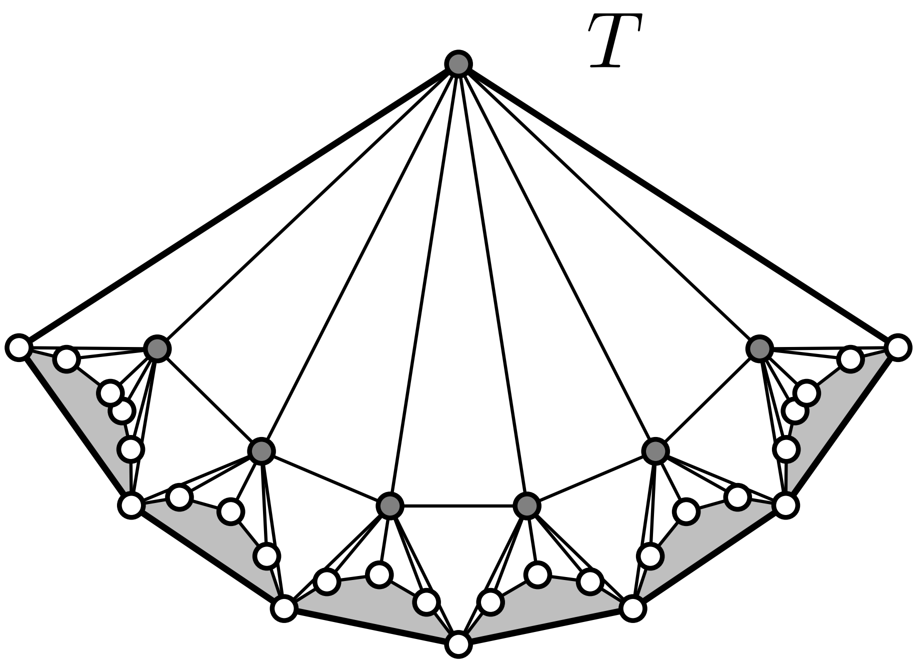
no flips possible

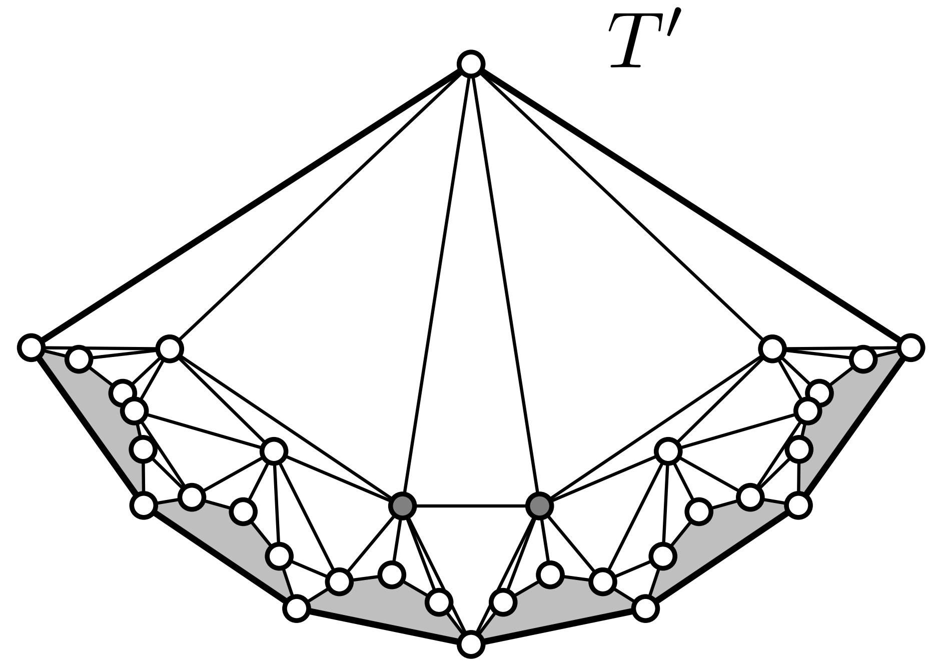


some freedom
inside the shaded regions

(for example: appropriate zigzag triangulations)

General Point Sets





General Point Sets

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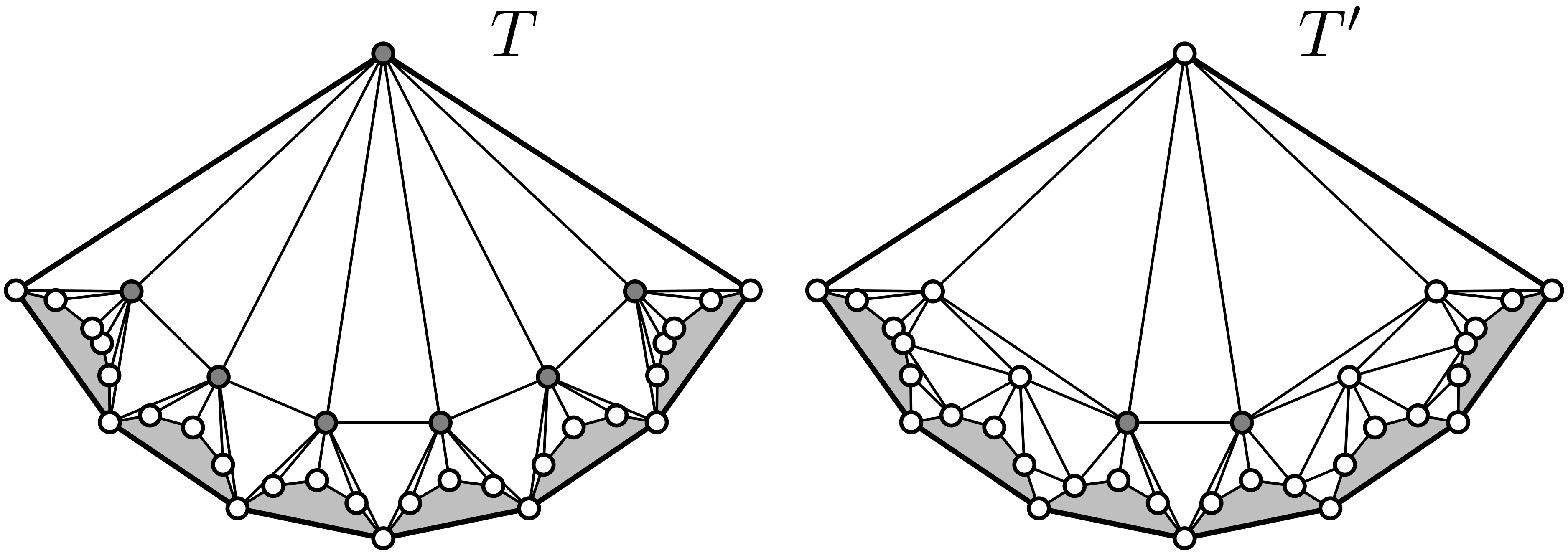

OPEN QUESTION:

1. Can two triangulations with degree $\leq k$ be transformed into each other without exceeding degree $k+1$? (Or $2k$? Or $f(k)$?)

Open Questions

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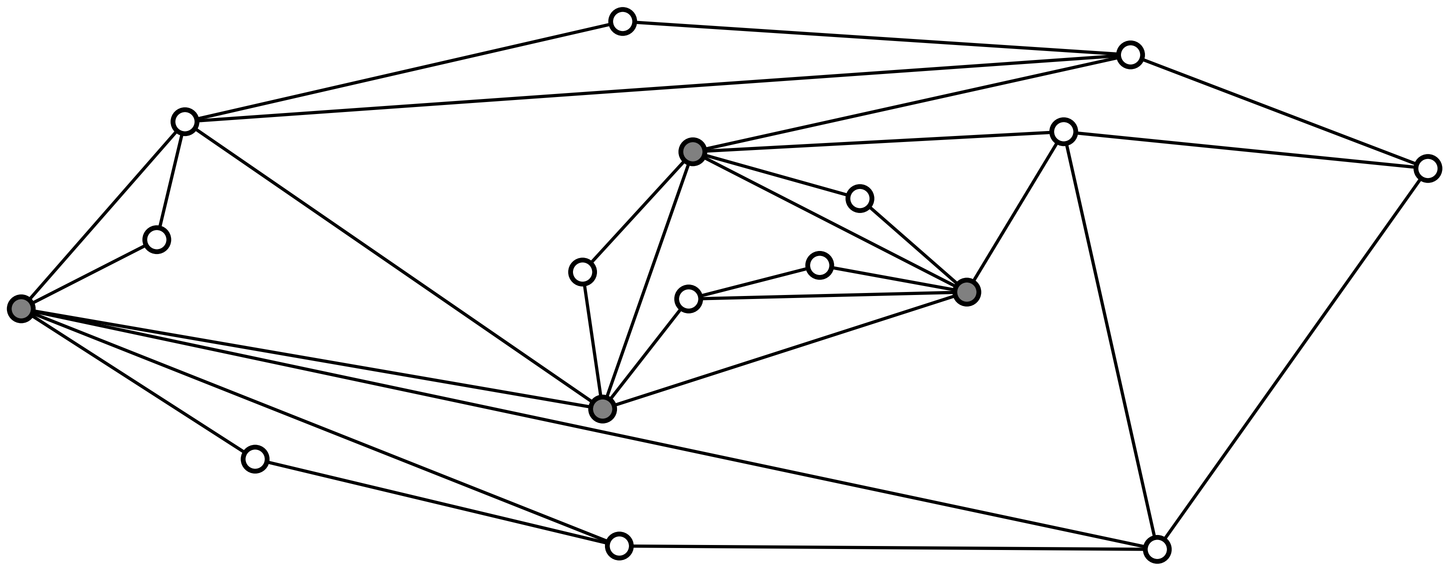
1. Can two triangulations with degree $\leq k$ be transformed into each other without exceeding degree $k+1$? (Or $2k$? Or $f(k)$?)

2. How about *pseudotriangulations*? ($k \geq 10$!)



3. Flip diameter of bounded-degree triangulations of *convex* point sets is $O(n)$? $O(n \log n)$?