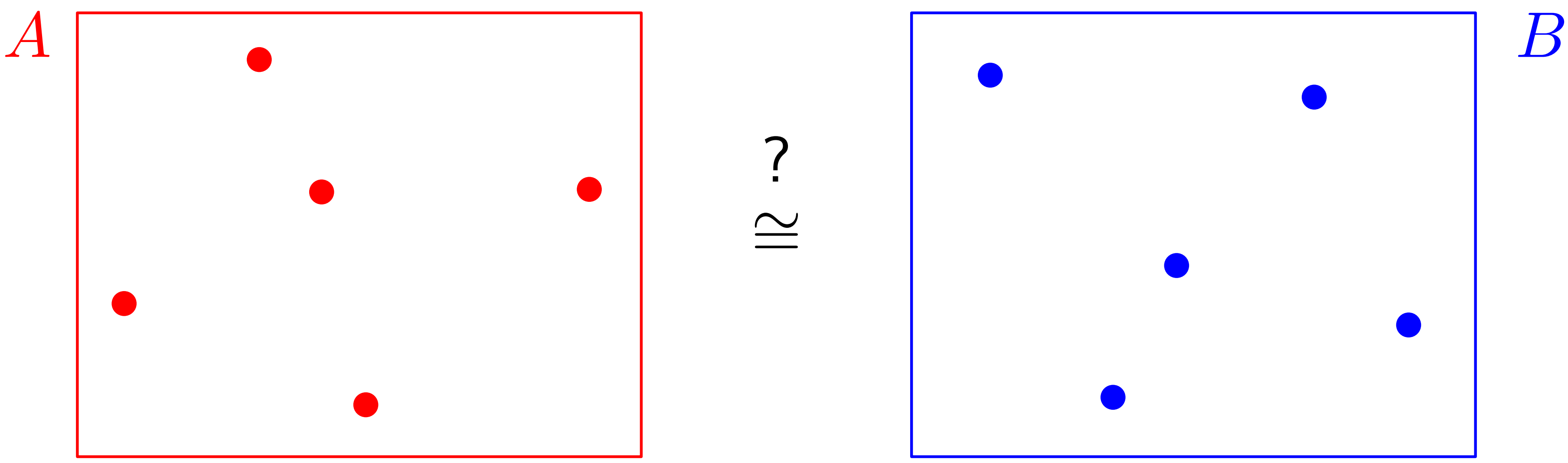
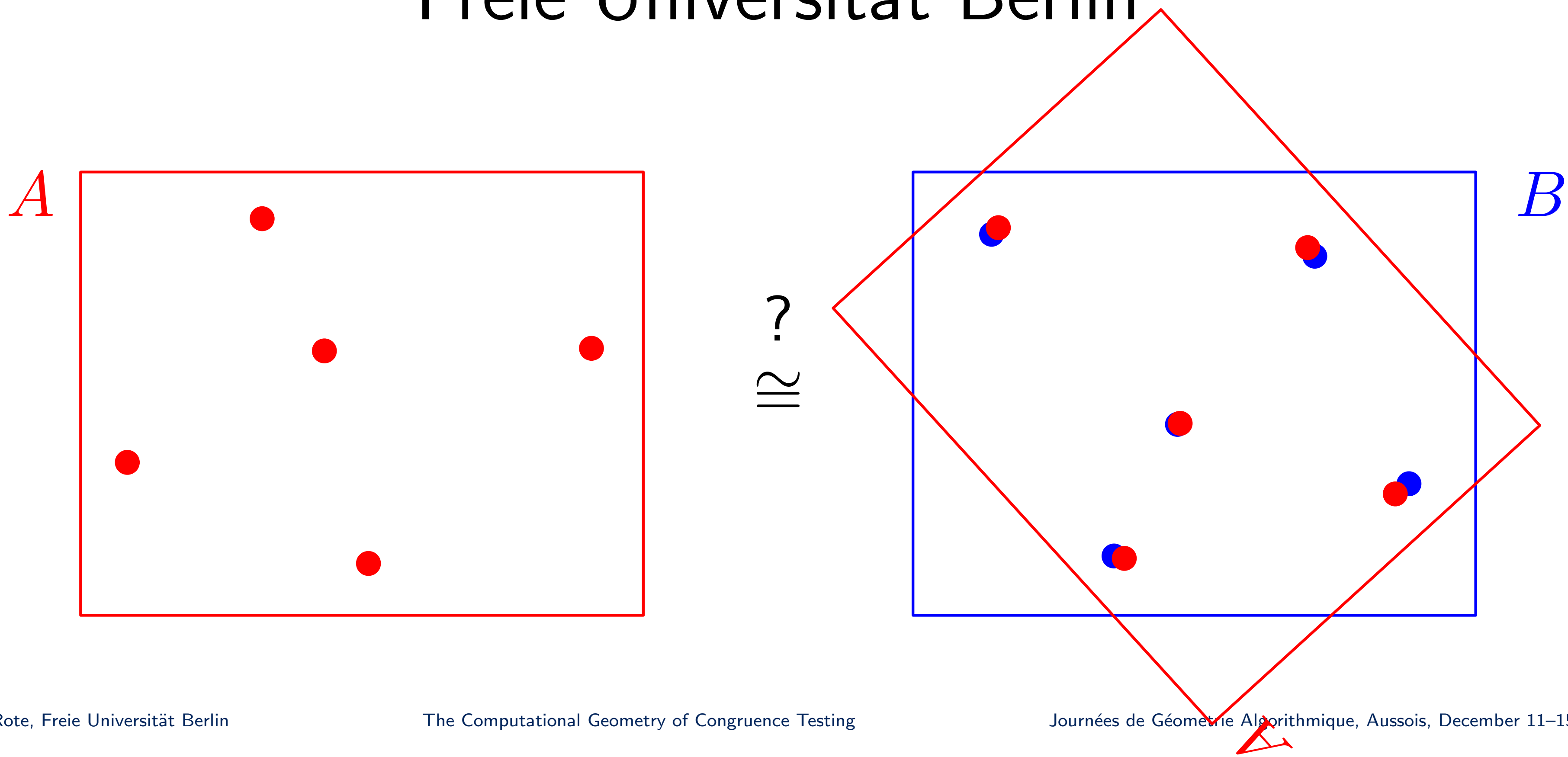
The Computational Geometry of Congruence Testing, Part II

Günter Rote
Freie Universität Berlin



The Computational Geometry of Congruence Testing, Part II

Günter Rote
Freie Universität Berlin



Overview

- 1 dimension
- 2 dimensions
- 3 dimensions
- 4 dimensions

} $O(n \log n)$ time

← today (joint work with Heuna Kim)

- d dimensions $O(n^{\lceil d/3 \rceil} \log n)$ time [Brass and Knauer 2002]

$O(n^{\lfloor (d+2)/2 \rfloor /2} \log n)$ Monte Carlo [Akutsu 1998/Matoušek]

$\downarrow O(n^{\lfloor (d+1)/2 \rfloor /2} \log n)$ time

Overview

- 1 dimension
- 2 dimensions
- 3 dimensions
- 4 dimensions

} $O(n \log n)$ time

← today (joint work with Heuna Kim)

- d dimensions $O(n^{\lceil d/3 \rceil} \log n)$ time [Brass and Knauer 2002]

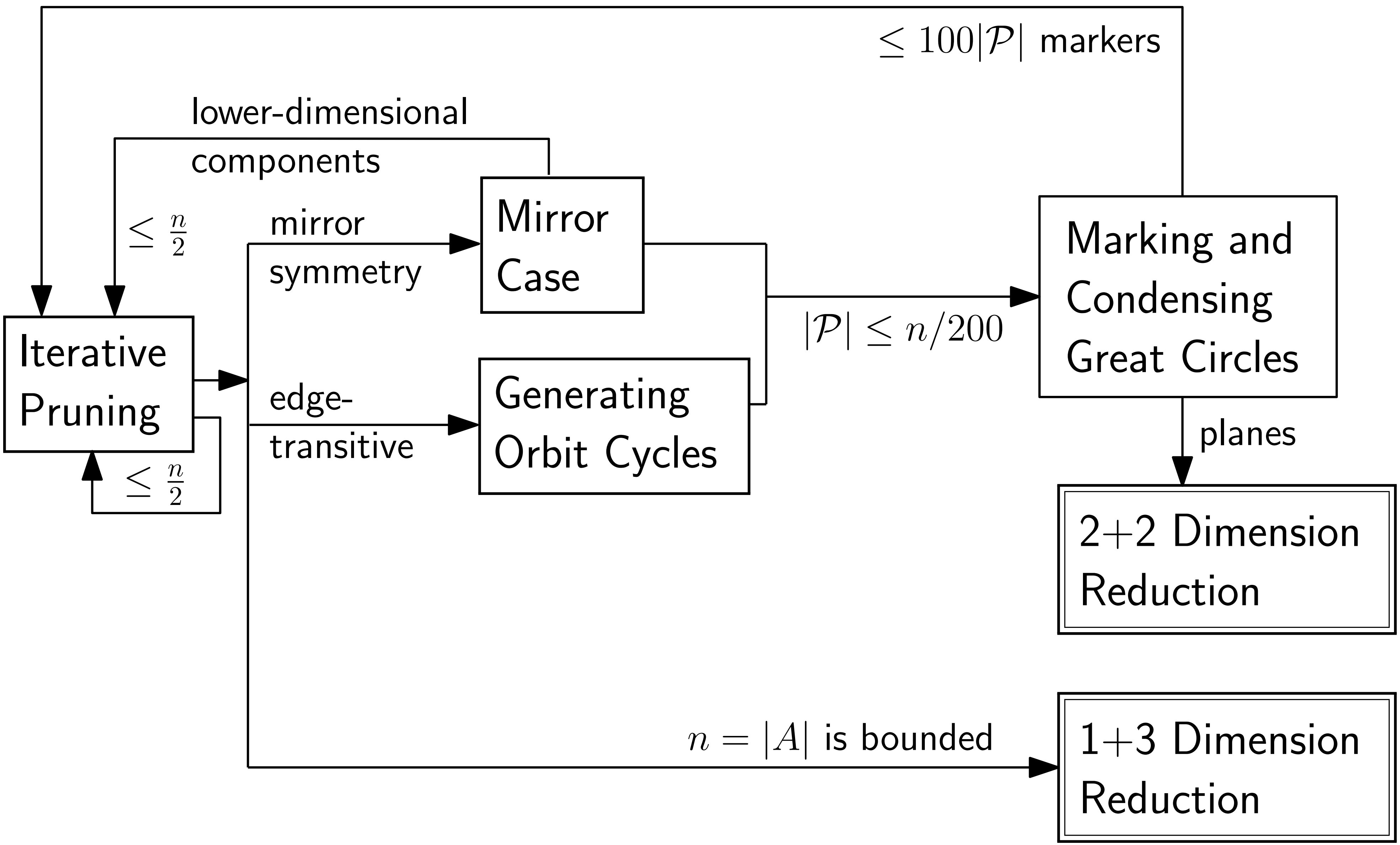
$O(n^{\lfloor (d+2)/2 \rfloor /2} \log n)$ Monte Carlo [Akutsu 1998/Matoušek]

$\downarrow O(n^{\lfloor (d+1)/2 \rfloor /2} \log n)$ time

- Rotations in 4-space
- Plücker coordinates for 2-planes in 4-space
- The Hopf fibration of $\mathbb{S}^3$
- Closest pair graph
- 2+2 dimension reduction
- Coxeter classification of reflection groups

4 Dimensions: Algorithm Overview

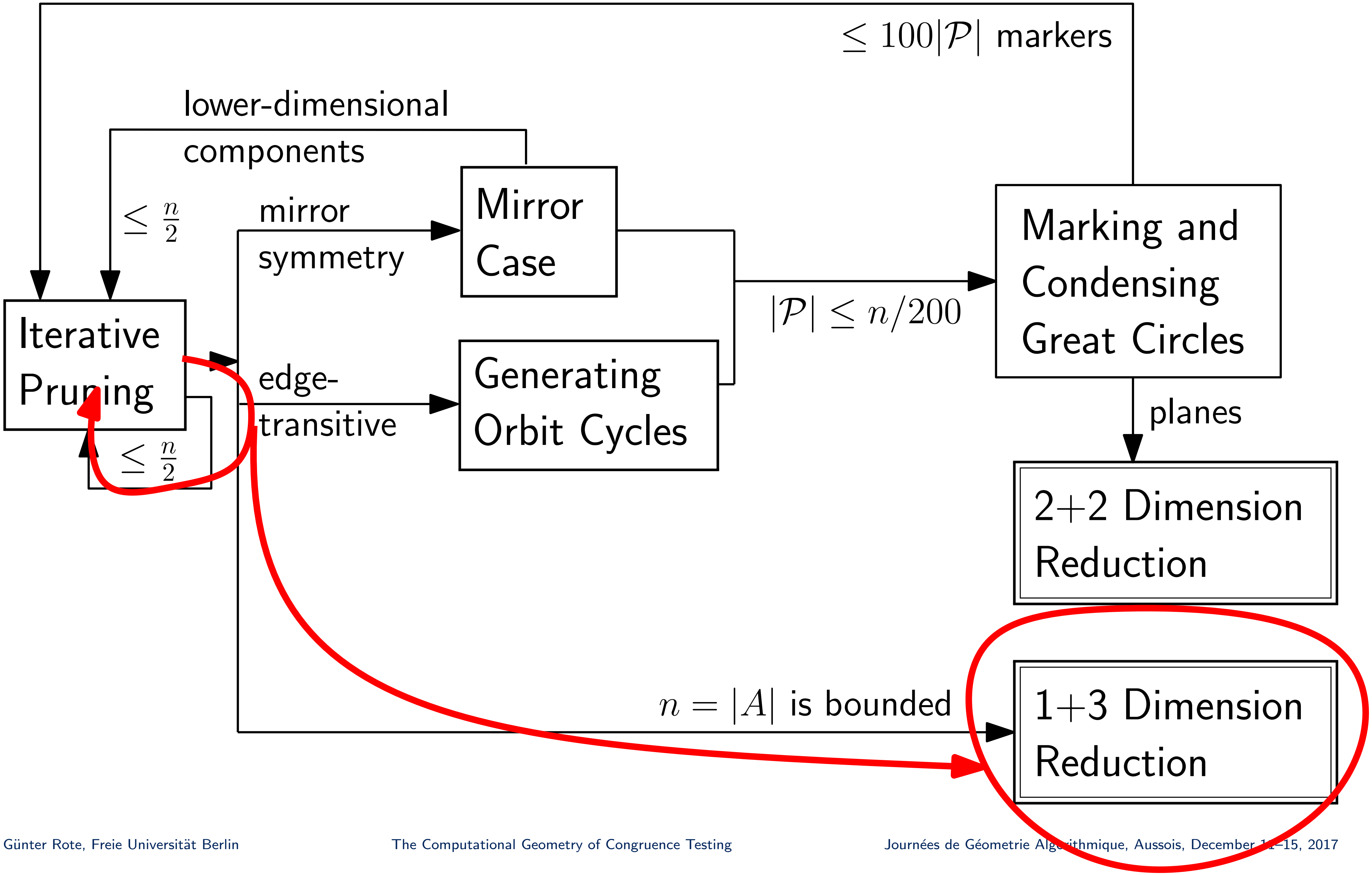
joint work with Heuna Kim



4 Dimensions: Algorithm Overview

Freie Universität Berlin

joint work with Heuna Kim



Initialization: Closest-Pair Graph

1) PRUNE by distance from the origin.

- $\Longrightarrow$ we can assume that A lies on the 3-sphere $\mathbb{S}^3$.

2) Compute the closest pair graph

$$G(A) = (A, \{\, uv : \|u - v\| = \delta \,\})$$

where $\delta :=$ the distance of the closest pair, in $O(n \log n)$ time.

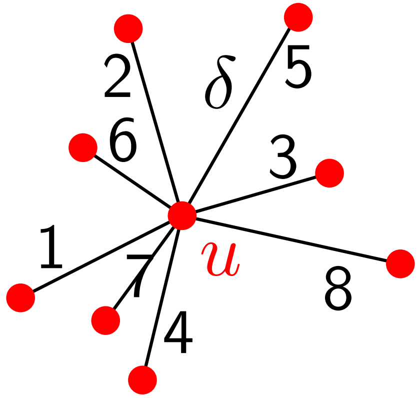
[Bentley and Shamos, STOC 1976]

- We can assume that δ is SMALL: $\delta \le \delta_0 := 0.0005$.
  (Otherwise, $|A| \le n_0$, by a packing argument.)

Everything looks the same!

By the PRUNING principle, we can assume that all points look locally the same:

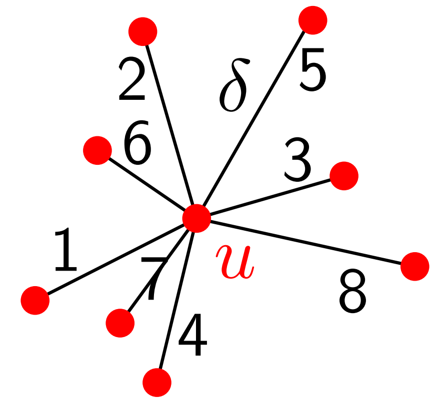
- All points have congruent neighborhoods in $G(A)$.
  (The neighbors of u lie on a 2-sphere in $\mathbb{S}^3$;
  There are at most $K_3 = 12$ neighbors.)



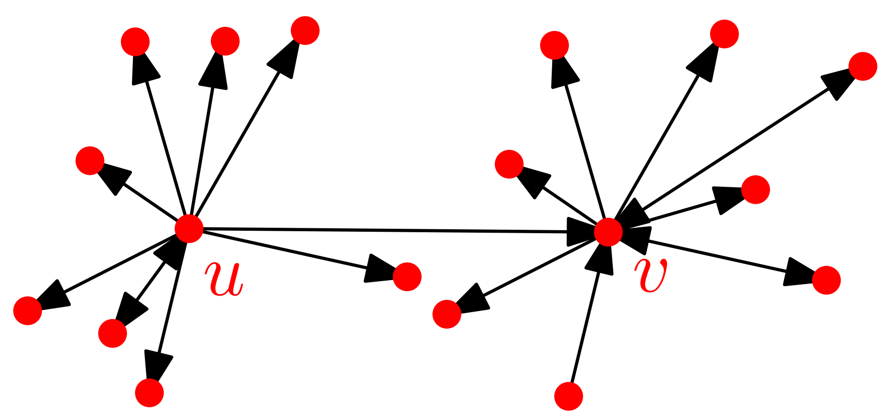
Everything looks the same!

By the PRUNING principle, we can assume that all points look locally the same:

- All points have congruent neighborhoods in $G(A)$.
  (The neighbors of u lie on a 2-sphere in $\mathbb{S}^3$;
  There are at most $K_3 = 12$ neighbors.)



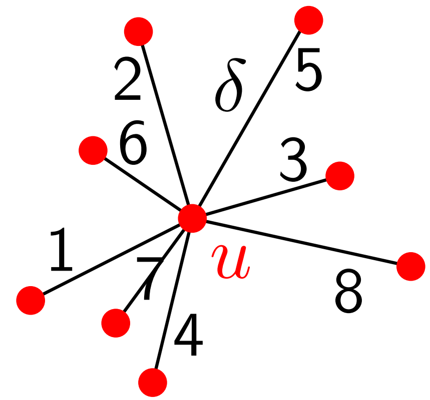
- Make a directed graph D from $G(A)$
  and PRUNE its arcs uv by
  the joint neighborhood of u and v.



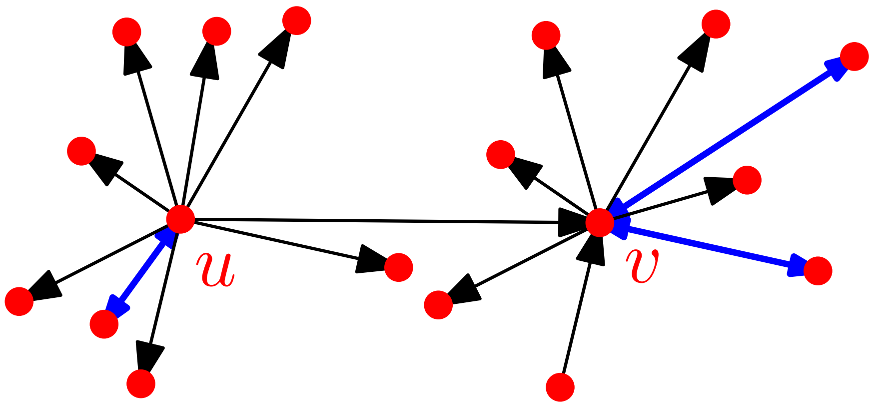
Everything looks the same!

By the PRUNING principle, we can assume that all points look locally the same:

- All points have congruent neighborhoods in $G(A)$.
  (The neighbors of u lie on a 2-sphere in $\mathbb{S}^3$;
  There are at most $K_3 = 12$ neighbors.)



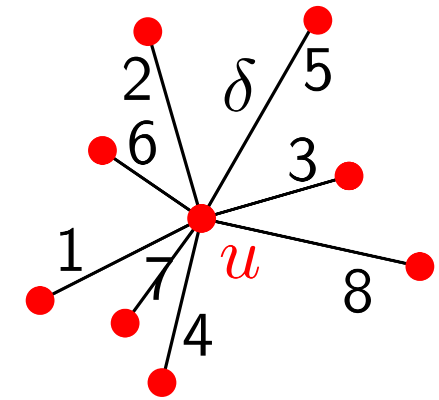
- Make a directed graph D from $G(A)$
  and PRUNE its arcs uv by
  the joint neighborhood of u and v.



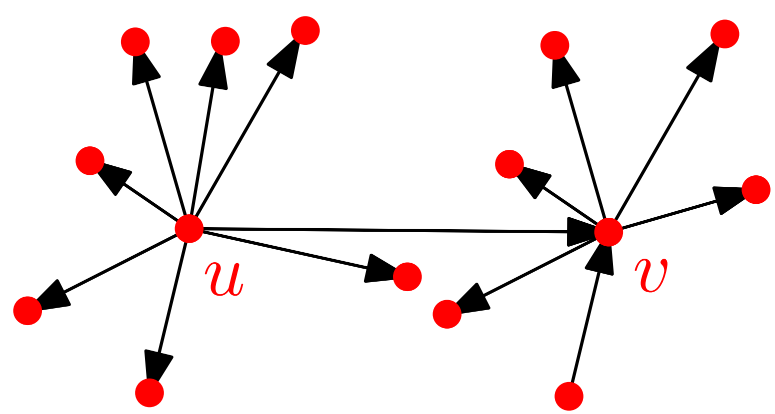
Everything looks the same!

By the PRUNING principle, we can assume that all points look locally the same:

- All points have congruent neighborhoods in $G(A)$. (The neighbors of u lie on a 2-sphere in $\mathbb{S}^3$; There are at most $K_3 = 12$ neighbors.)



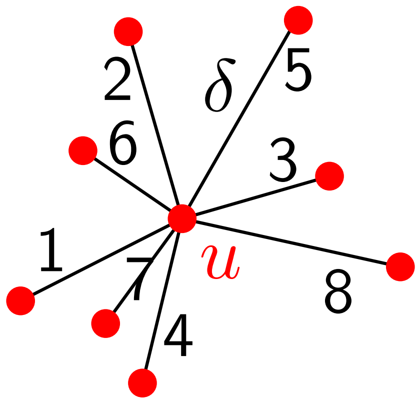
- Make a directed graph D from $G(A)$ and PRUNE its arcs uv by the joint neighborhood of u and v.



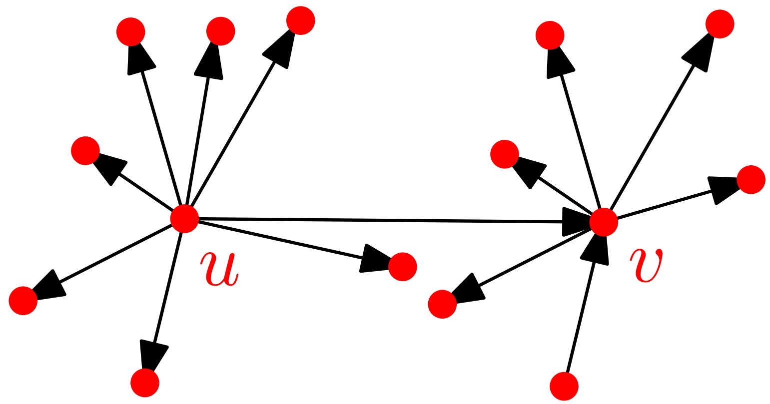
Everything looks the same!

By the PRUNING principle, we can assume that all points look locally the same:

- All points have congruent neighborhoods in $G(A)$.
  (The neighbors of u lie on a 2-sphere in $\mathbb{S}^3$;
  There are at most $K_3 = 12$ neighbors.)

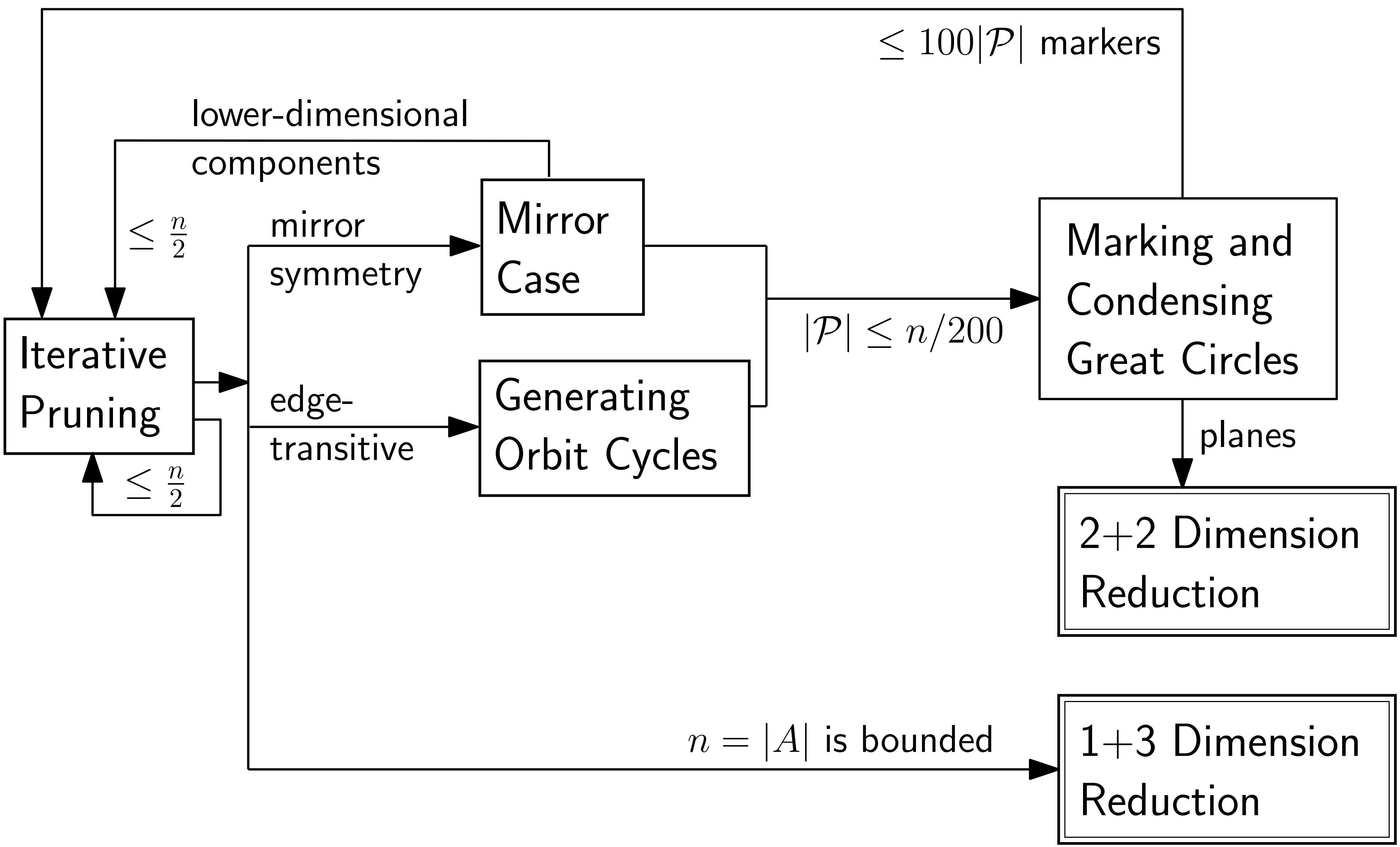


- Make a directed graph D from $G(A)$
  and PRUNE its arcs uv by
  the joint neighborhood of u and v.

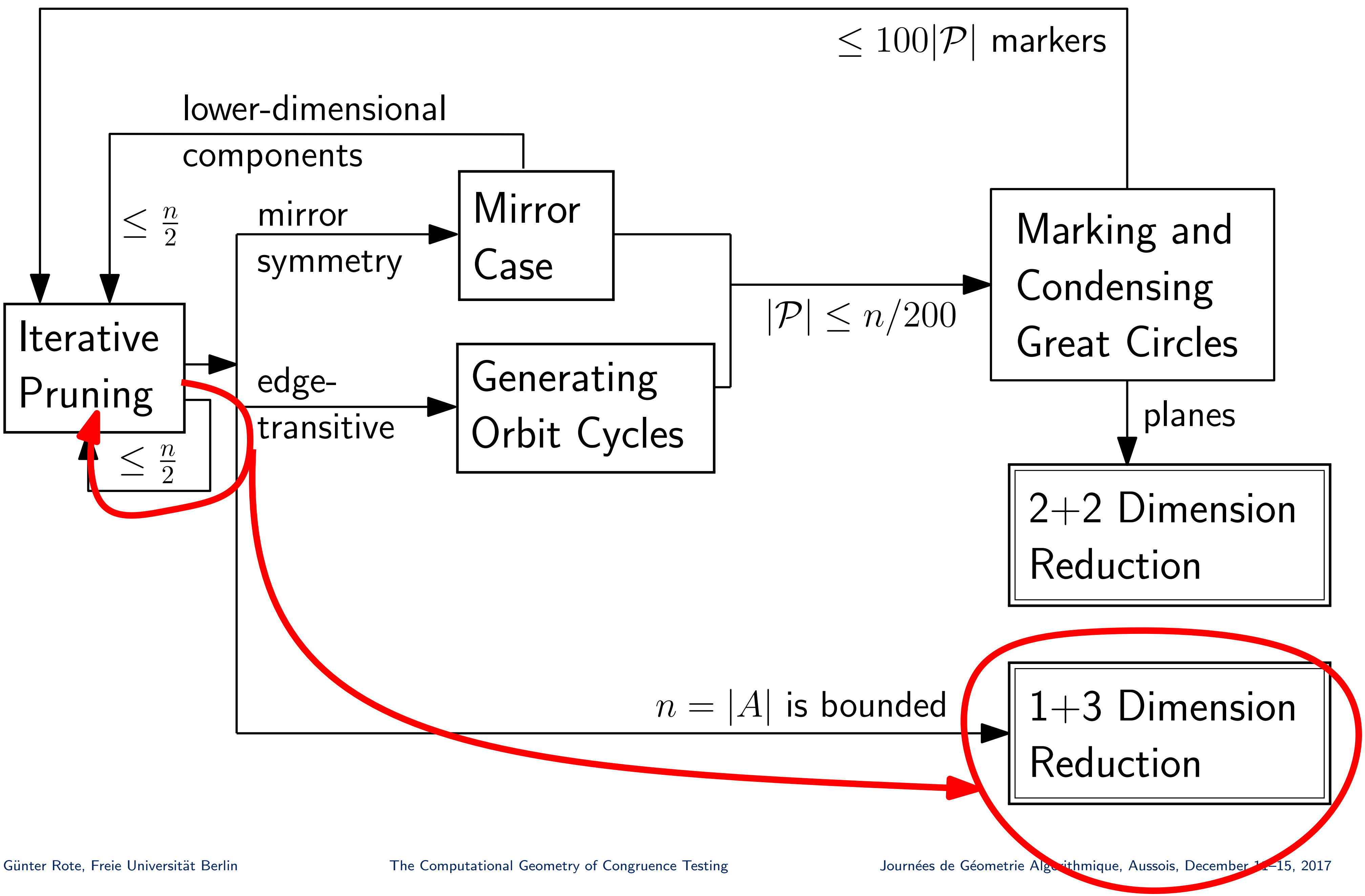


- . . . until all arcs uv
  look the same.

Algorithm Overview

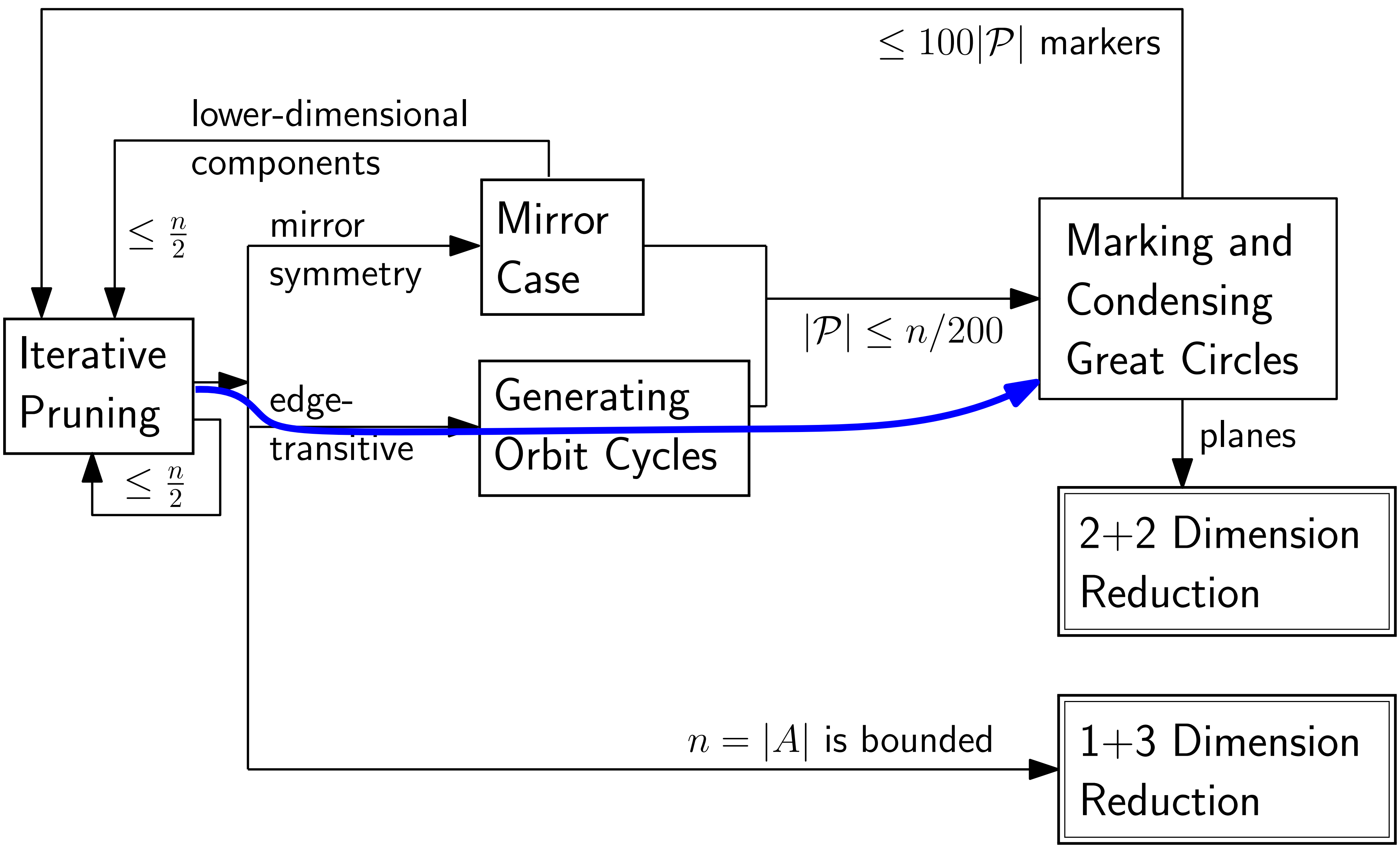


Algorithm Overview



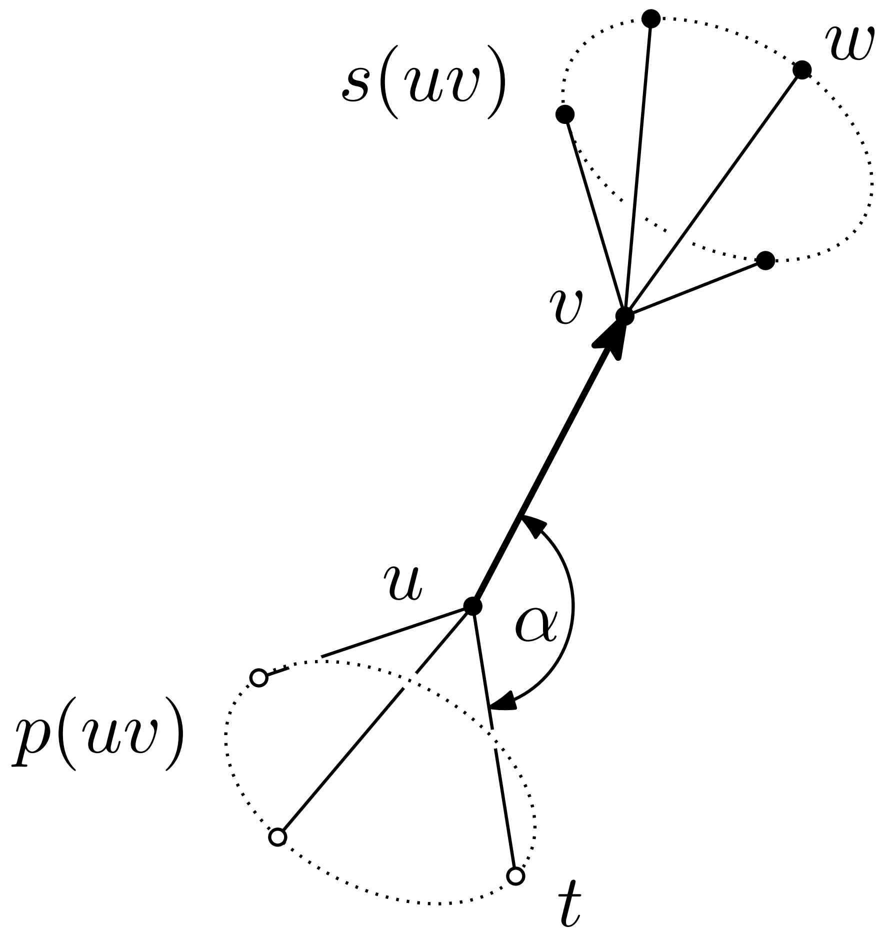
Algorithm Overview

Freie Universität Berlin



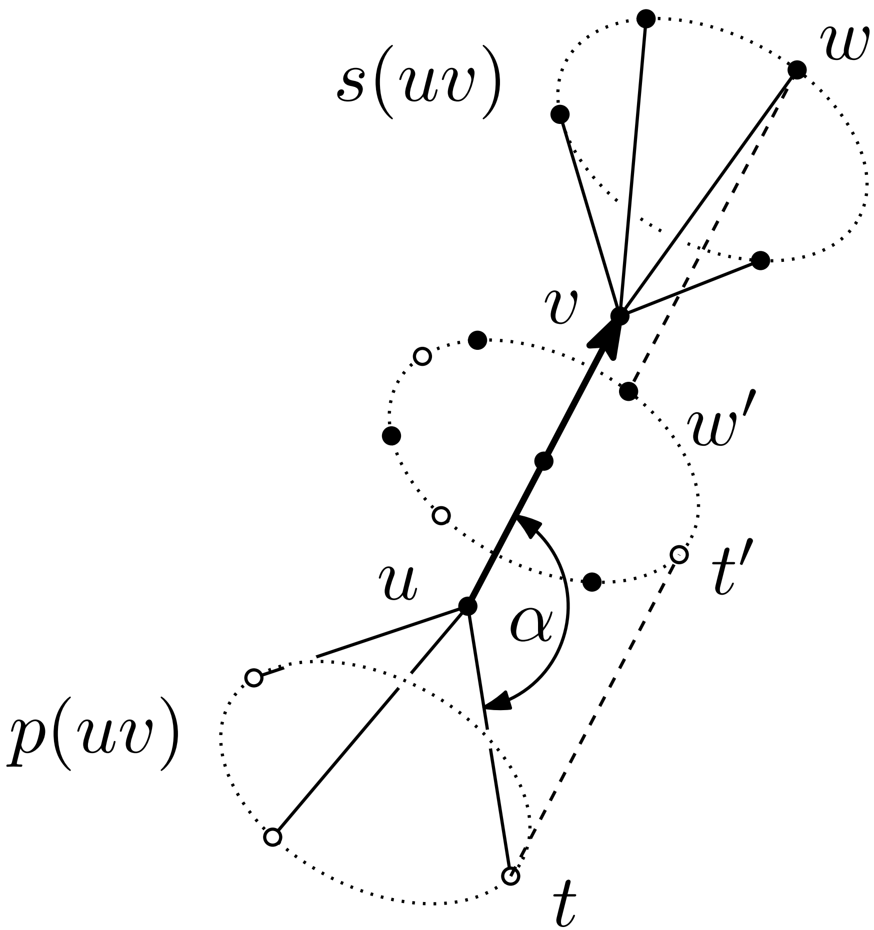
Predecessor-Successor Figure

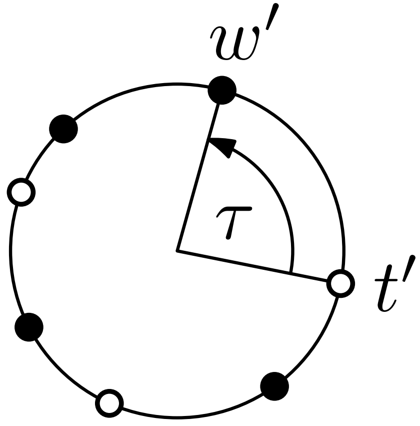
Pick some α. $s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$



Predecessor-Successor Figure

Pick some α. $s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$

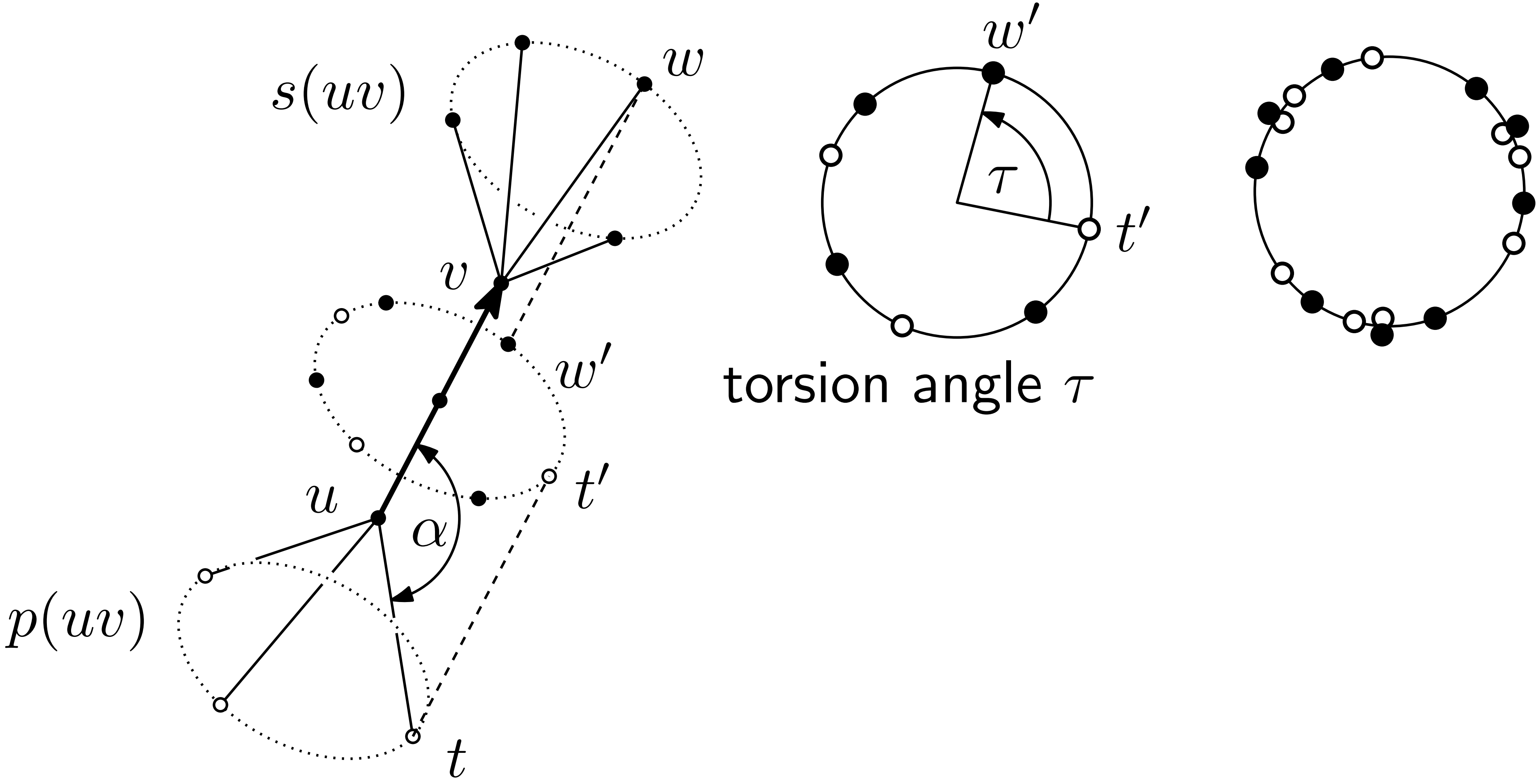




torsion angle τ

Predecessor-Successor Figure

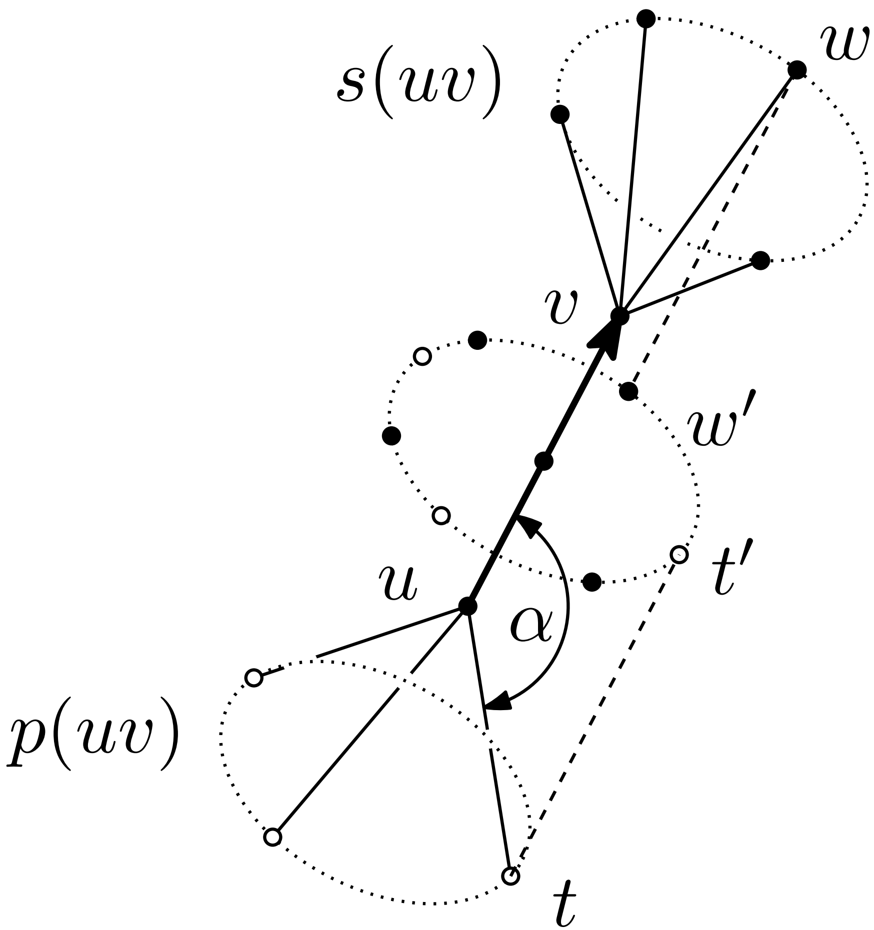
Pick some α. $\quad s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$

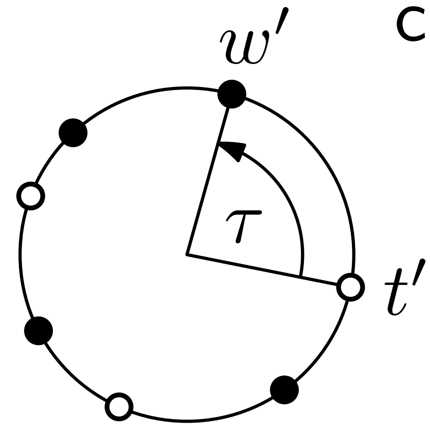


torsion angle τ

Predecessor-Successor Figure

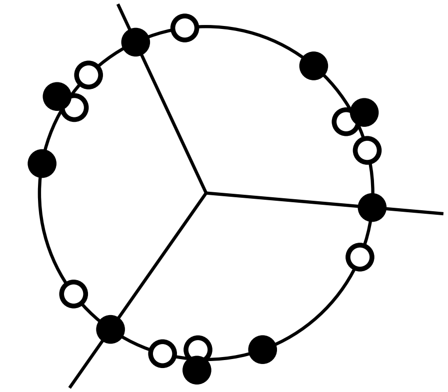
Pick some α. $\quad s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$





torsion angle τ

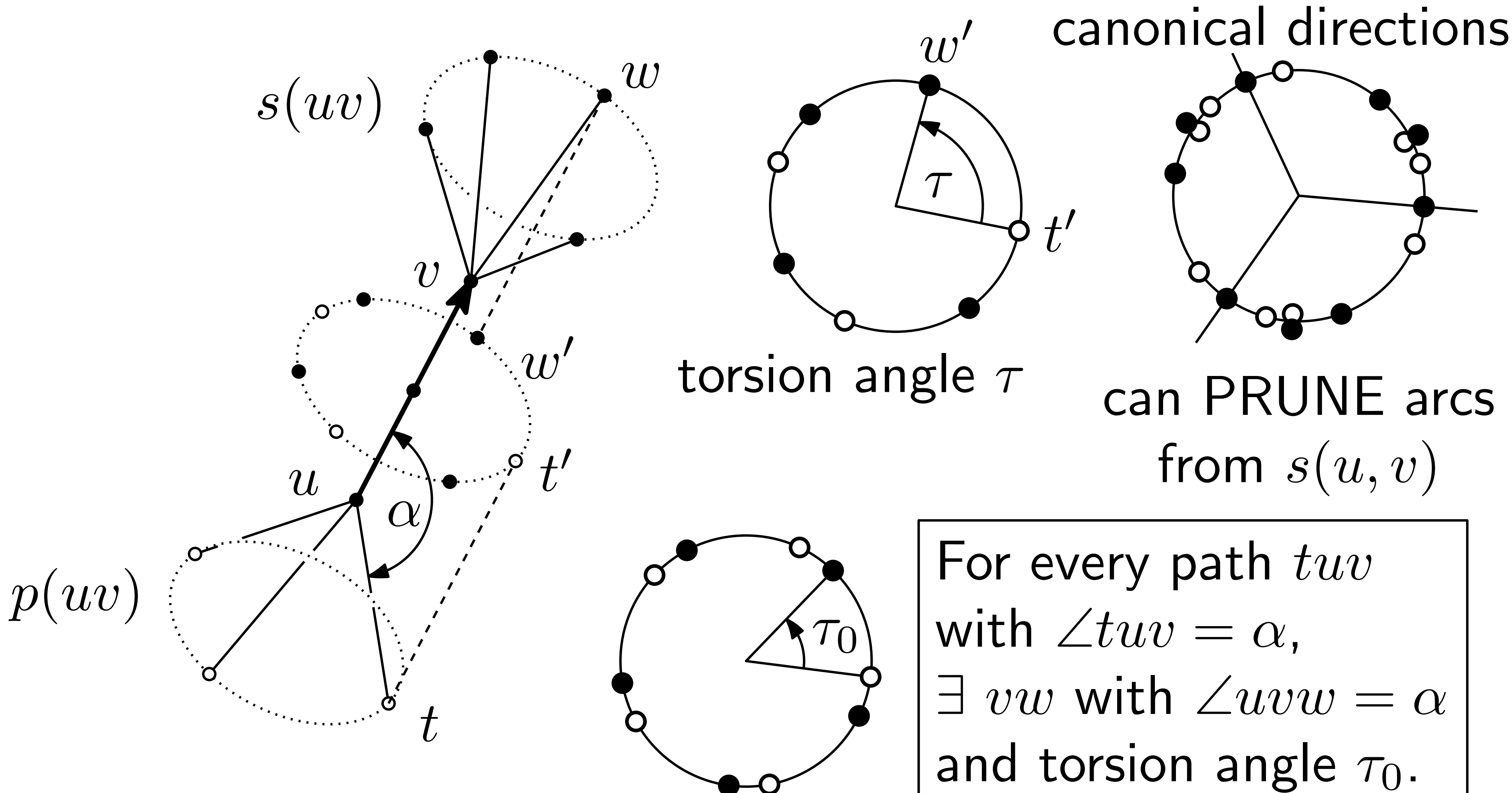
canonical directions



can PRUNE arcs from $s(u, v)$

Predecessor-Successor Figure

Pick some α. $s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$

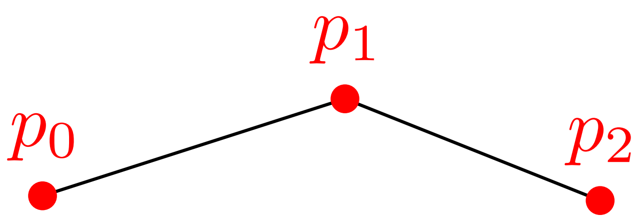


can PRUNE arcs from $s(u, v)$

For every path tuv with $\angle tuv = \alpha$, $\exists\ vw$ with $\angle uvw = \alpha$ and torsion angle τ_0.

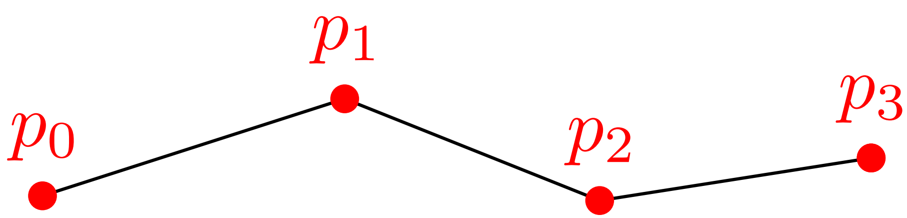
Orbit cycles

! For every path $p_i p_{i+1} p_{i+2}$ with $\angle p_i p_{i+1} p_{i+2} = \alpha$,
$\exists\ p_{i+3}$ with $\angle p_{i+1} p_{i+2} p_{i+3} = \alpha$ and torsion τ_0.



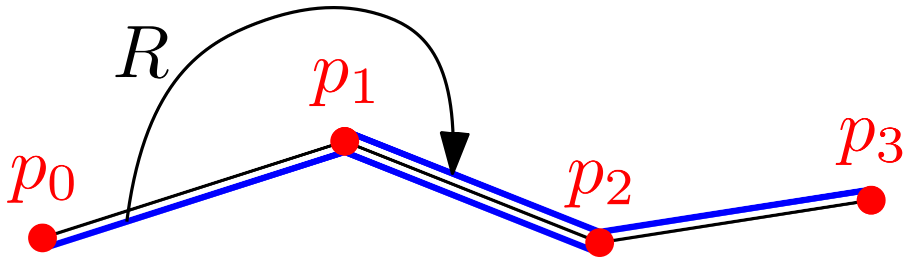
Orbit cycles

! For every path $p_i p_{i+1} p_{i+2}$ with $\angle p_i p_{i+1} p_{i+2} = \alpha$,
$\exists\ p_{i+3}$ with $\angle p_{i+1} p_{i+2} p_{i+3} = \alpha$ and torsion τ_0.



Orbit cycles

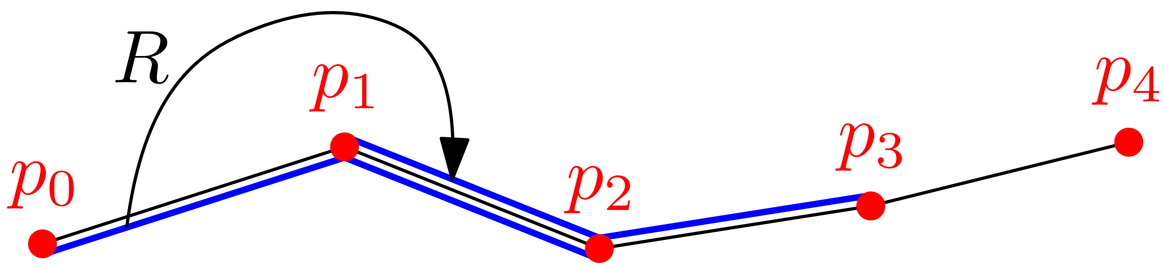
! For every path $p_i p_{i+1} p_{i+2}$ with $\angle p_i p_{i+1} p_{i+2} = \alpha$,
$\exists\ p_{i+3}$ with $\angle p_{i+1} p_{i+2} p_{i+3} = \alpha$ and torsion τ_0.



$$R(p_0, p_1, p_2) = (p_1, p_2, p_3)$$

Orbit cycles

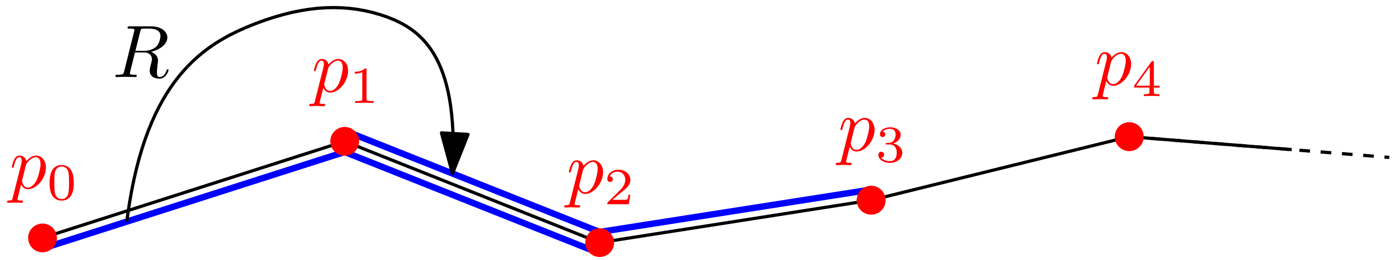
! For every path $p_i p_{i+1} p_{i+2}$ with $\angle p_i p_{i+1} p_{i+2} = \alpha$,
$\exists\ p_{i+3}$ with $\angle p_{i+1} p_{i+2} p_{i+3} = \alpha$ and torsion τ_0.



$$R(p_0, p_1, p_2) = (p_1, p_2, p_3)$$
$$R(p_0, p_1, p_2, p_3) = (p_1, p_2, p_3, p_4)$$

Orbit cycles

! For every path $p_i p_{i+1} p_{i+2}$ with $\angle p_i p_{i+1} p_{i+2} = \alpha$,
$\exists\ p_{i+3}$ with $\angle p_{i+1} p_{i+2} p_{i+3} = \alpha$ and torsion τ_0.



$$R(p_0, p_1, p_2) = (p_1, p_2, p_3)$$
$$R(p_0, p_1, p_2, p_3) = (p_1, p_2, p_3, p_4)$$
$$R(p_1, p_2, p_3, p_4) = (p_2, p_3, p_4, p_5)$$
$\ldots$

$Rp_i = p_{i+1}$: The orbit of p_0 under R, a helix

Rotations in 4 dimensions

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix} = \begin{pmatrix} R_\varphi & 0 \\ 0 & R_\psi \end{pmatrix}$$

in some appropriate coordinate system.

$\varphi \neq \pm\psi$: $\to$ unique decomposition $\mathbb{R}^4 = P \oplus Q$ into two completely orthogonal 2-dimensional *axis planes* P and Q

$\varphi = \pm\psi$: isoclinic rotations

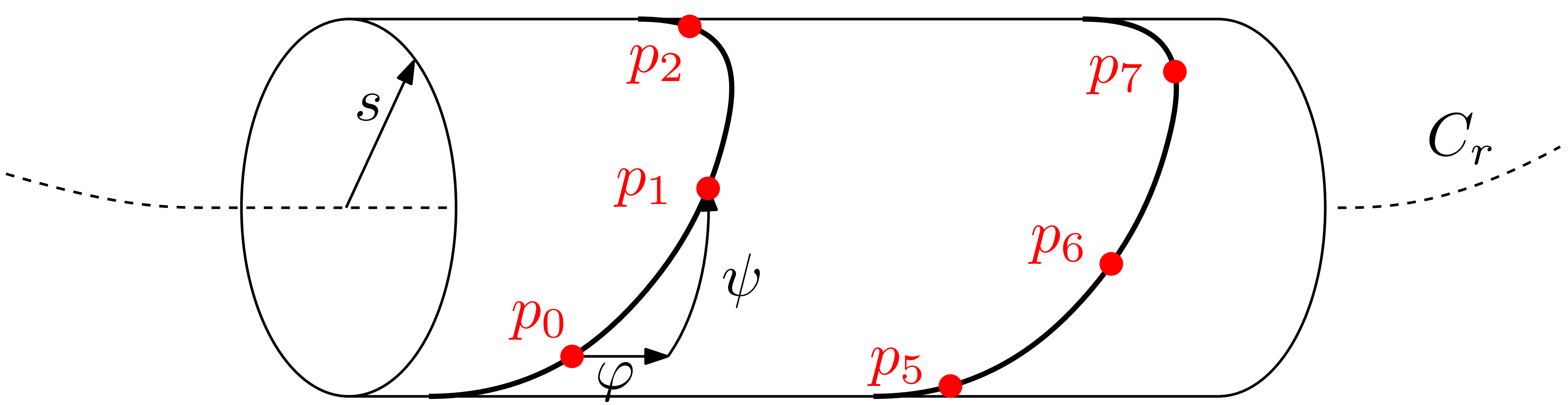
The orbit of a point $p_0 = (x_1, y_1, x_2, y_2)$ lies on a *helix* on a *flat torus* $C_r \times C_s$, with $r = \sqrt{x_1^2 + y_1^2}$, $s = \sqrt{x_2^2 + y_2^2}$

↑ circle with radius r

Rotations in 4 dimensions

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix} = \begin{pmatrix} R_\varphi & 0 \\ 0 & R_\psi \end{pmatrix}$$

The orbit of a point $p_0 = (x_1, y_1, x_2, y_2)$ lies on a *helix* on a *flat torus* $C_r \times C_s$, with $r = \sqrt{x_1^2 + y_1^2}$, $s = \sqrt{x_2^2 + y_2^2}$

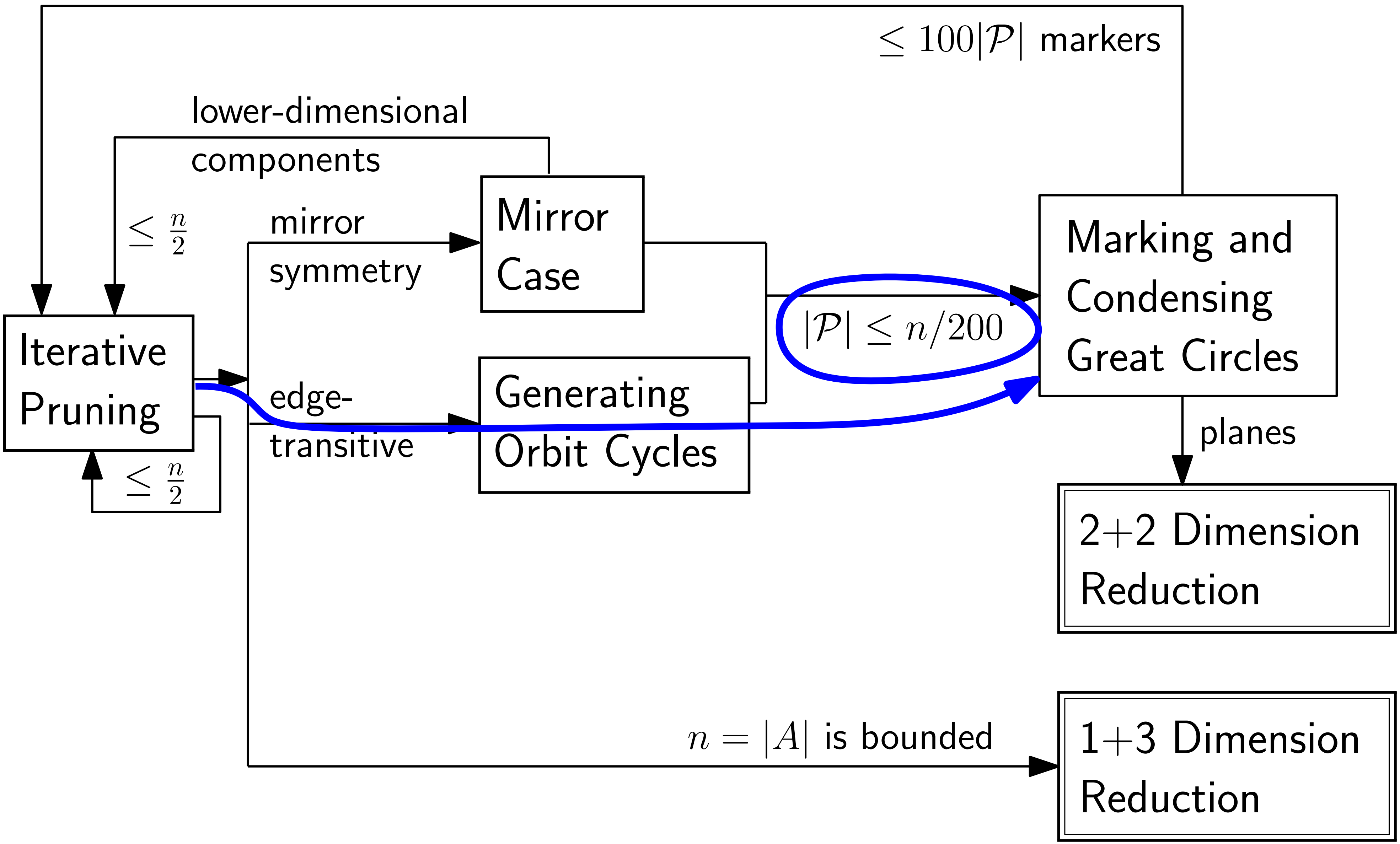


Planes in 4 dimensions

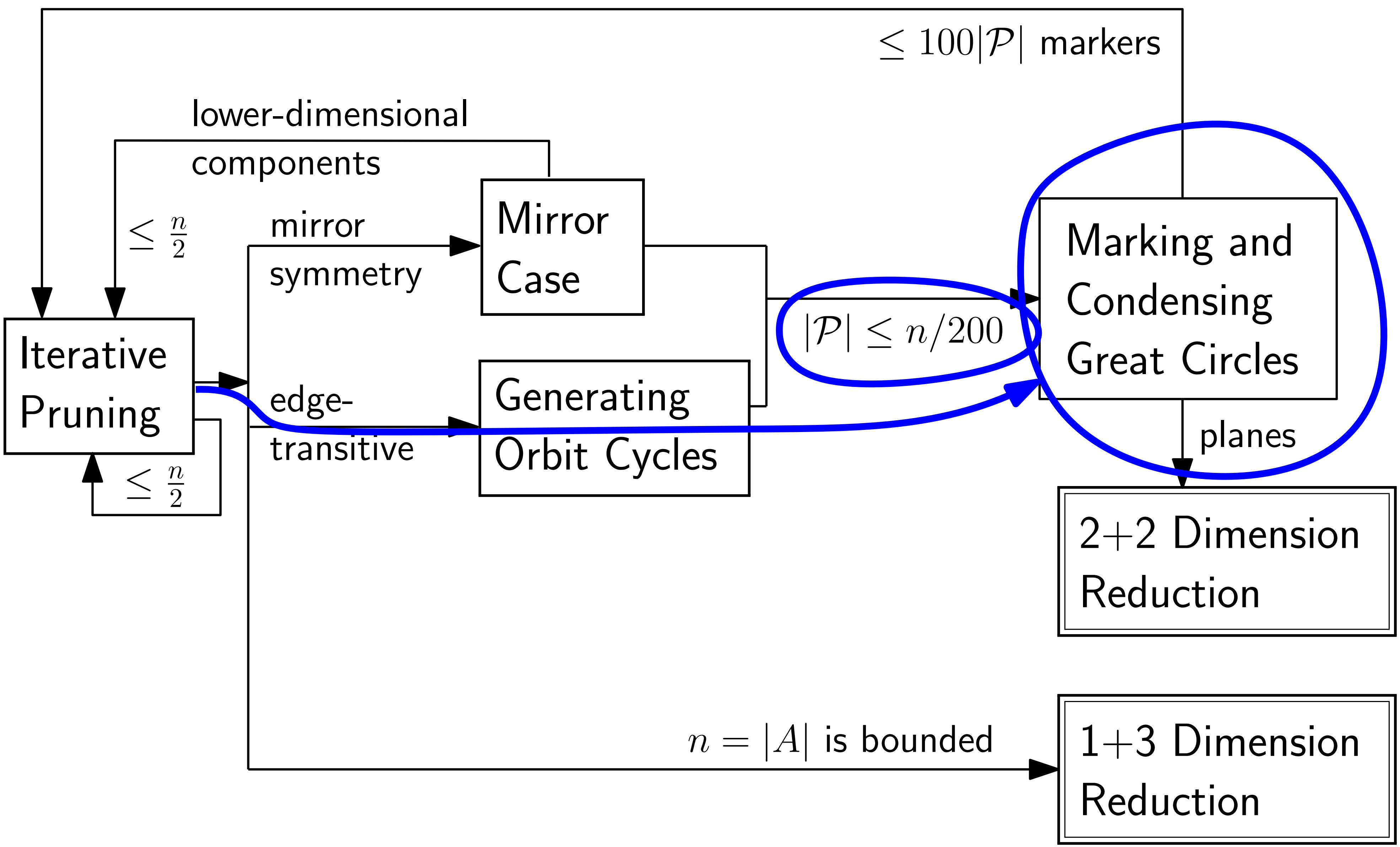
- Every point lies on ≤ 60 orbit cycles.
- Every orbit cycle contains ≥ 12000 points, because δ is small.
- Every orbit cycle generates 1 plane (corresponding to the smaller of φ and ψ.)

$\Longrightarrow$ a collection of $\leq n/200$ planes (or: great circles)

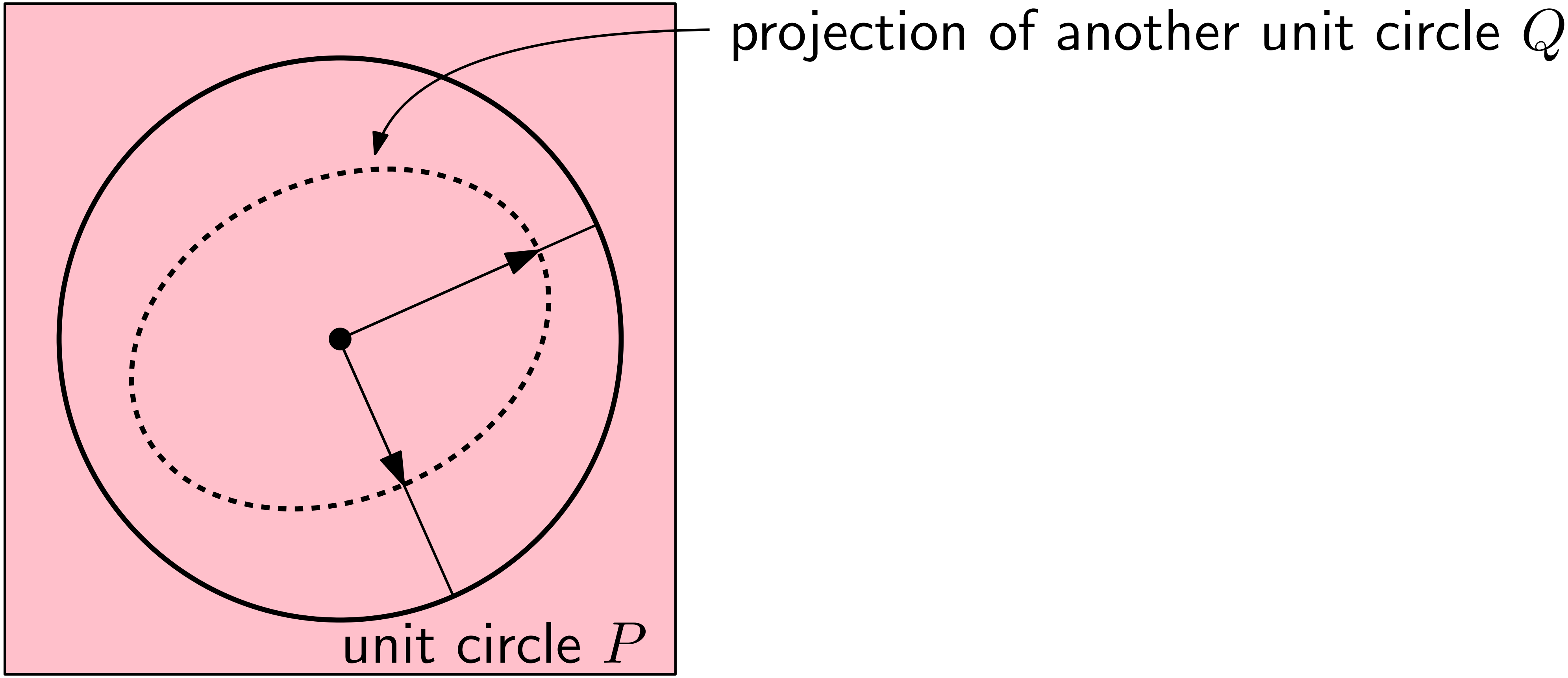
Algorithm Overview



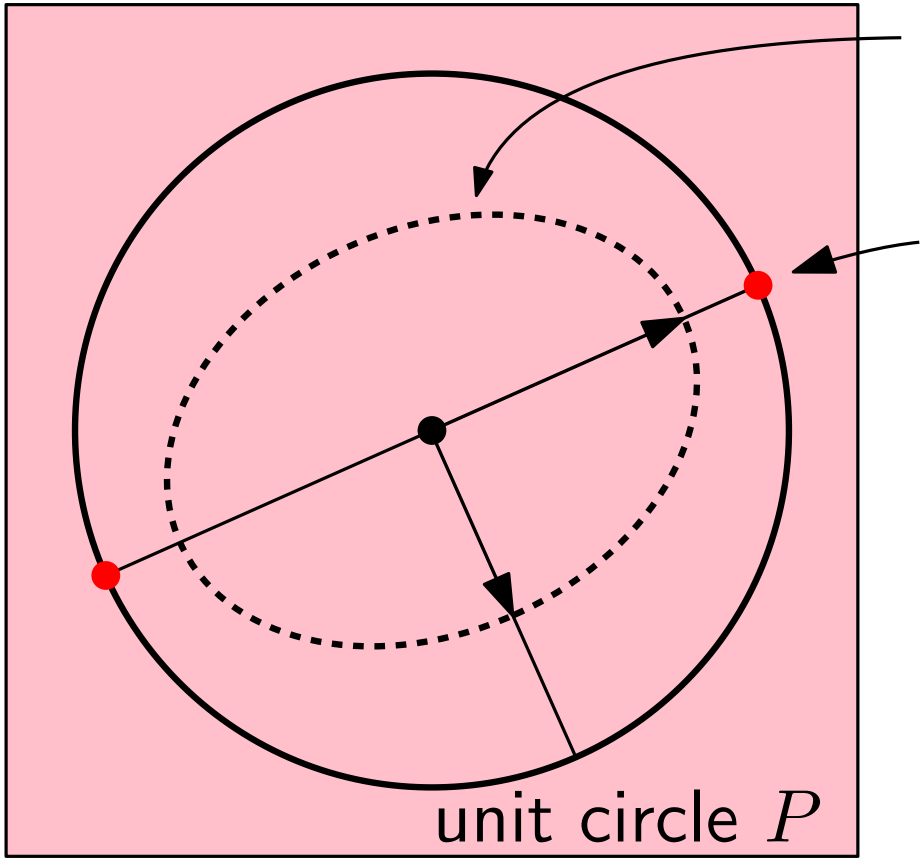
Algorithm Overview



Marking Points on Great Circles



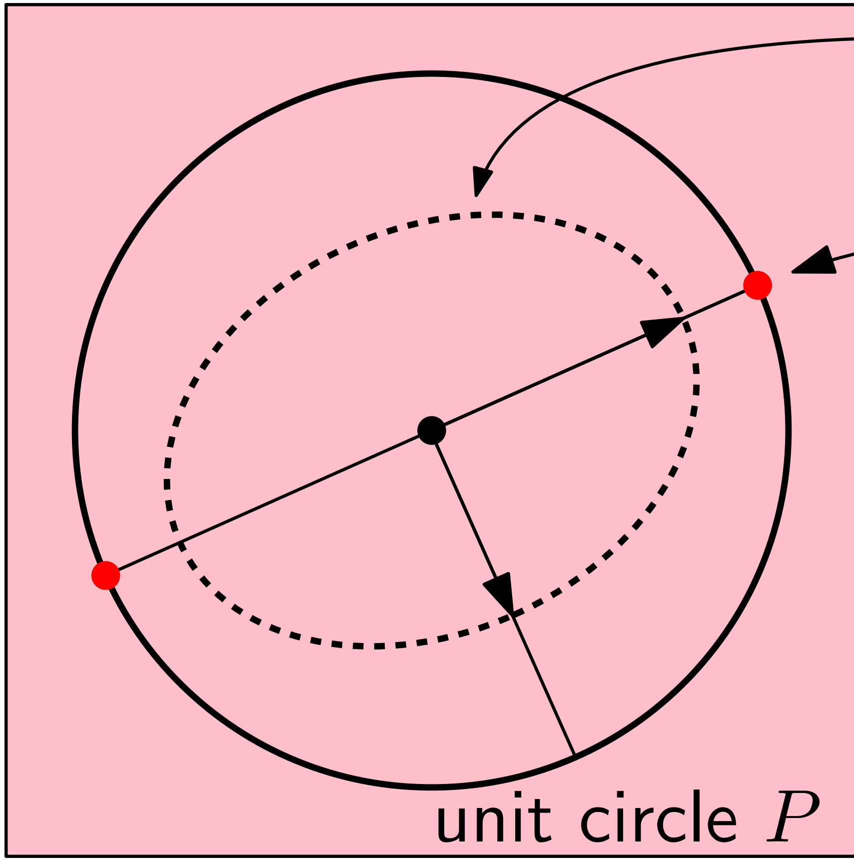
Marking Points on Great Circles



projection of another unit circle Q

IDEA: mark those two points in P

Marking Points on Great Circles



projection of a~~nother unit circle~~ Q
a neighbor of P

IDEA: mark those two points in P

IDEA 2: Construct the closest-pair graph in the space of great circles, in $O(n \log n)$ time.

Plücker coordinates

planes in 4-space $\Leftrightarrow$ great circles on $\mathbb{S}^3$ $\Leftrightarrow$ a.k.a. lines in $\mathbb{R}P^3$

plane through (x_1, y_1, x_2, y_2) and (x'_1, y'_1, x'_2, y'_2) :

$$(v_1, \dots, v_6) = \left(\begin{vmatrix} x_1 & y_1 \\ x'_1 & y'_1 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ x'_1 & x'_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_2 \\ x'_1 & y'_2 \end{vmatrix}, \begin{vmatrix} y_1 & x_2 \\ y'_1 & x'_2 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, \begin{vmatrix} x_2 & y_2 \\ x'_2 & y'_2 \end{vmatrix} \right)$$

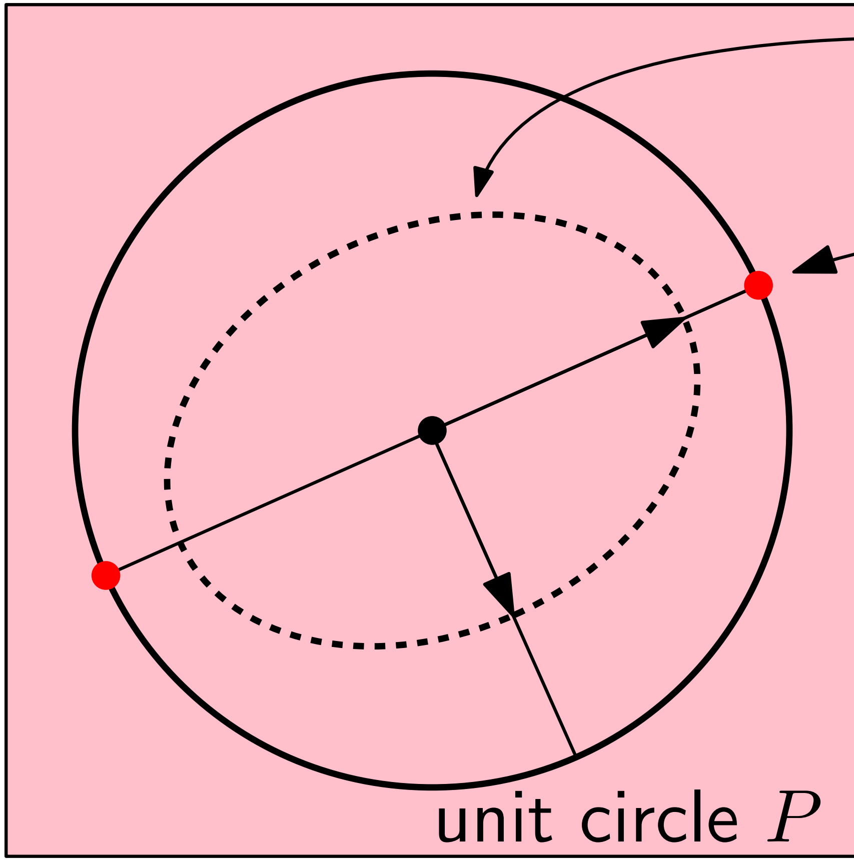
$(v_1, \dots, v_6) \in \mathbb{R}P^5$. [Plücker relations $v_1 v_6 - v_2 v_5 + v_3 v_4 = 0$]

Normalize:
$\rightarrow$ A great circle is represented by two antipodal points on $\mathbb{S}^5$.

This representation is geometrically meaningful:
Distances on $\mathbb{S}^5$ are preserved under rotations of $\mathbb{R}^4$ / $\mathbb{S}^3$.

(Packings of 2-planes in 4-space were considered by [Conway, Hardin and Sloane 1996], with different distances.)

Marking Points on Great Circles

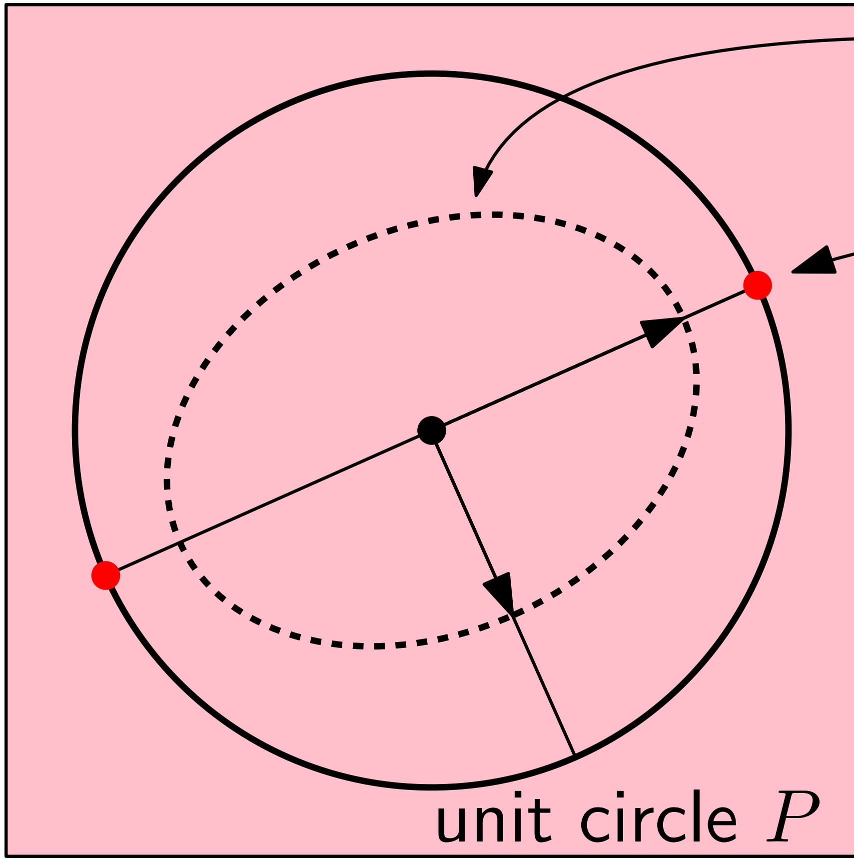


projection of ~~another unit circle~~ Q
a neighbor of P

IDEA: mark those two points in P

IDEA 2: Construct the closest-pair graph in the space of great circles, in $O(n \log n)$ time.
Every plane has at most $K_5 \leq 44$ neighbors.

Marking Points on Great Circles



projection of ~~another unit circle~~ Q
a neighbor of P

IDEA: mark those two points in P

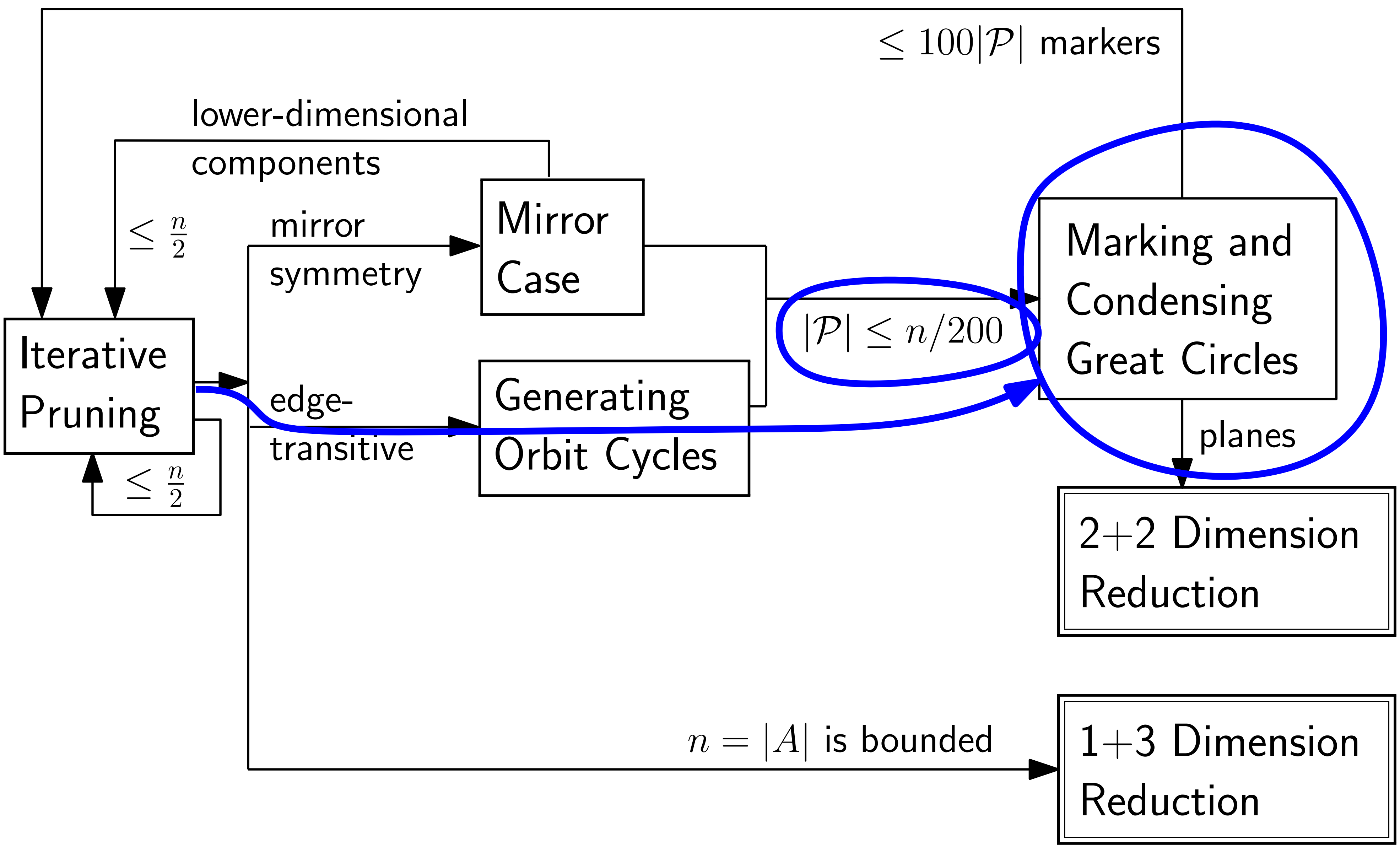
IDEA 2: Construct the closest-pair graph in the space of great circles, in $O(n \log n)$ time.
Every plane has at most $K_5 \leq 44$ neighbors.

$m \leq \frac{n}{200}$ great circles in $\mathbb{R}^4$ $\longrightarrow$ m point pairs on $\mathbb{S}^5$

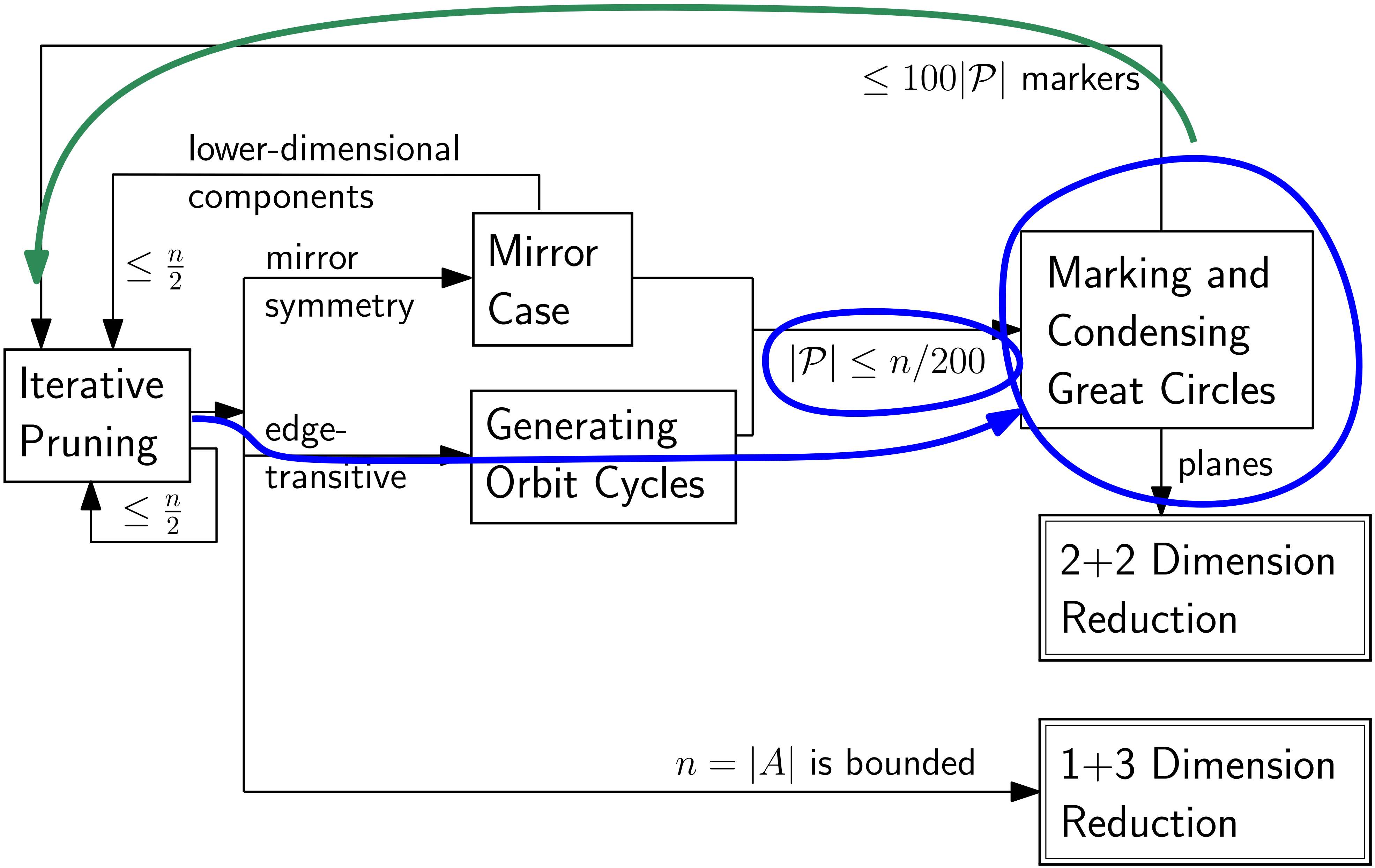
At most 88 (≤ 100) points are marked on every great circle.

These points replace A. $\rightarrow$ successful CONDENSATION

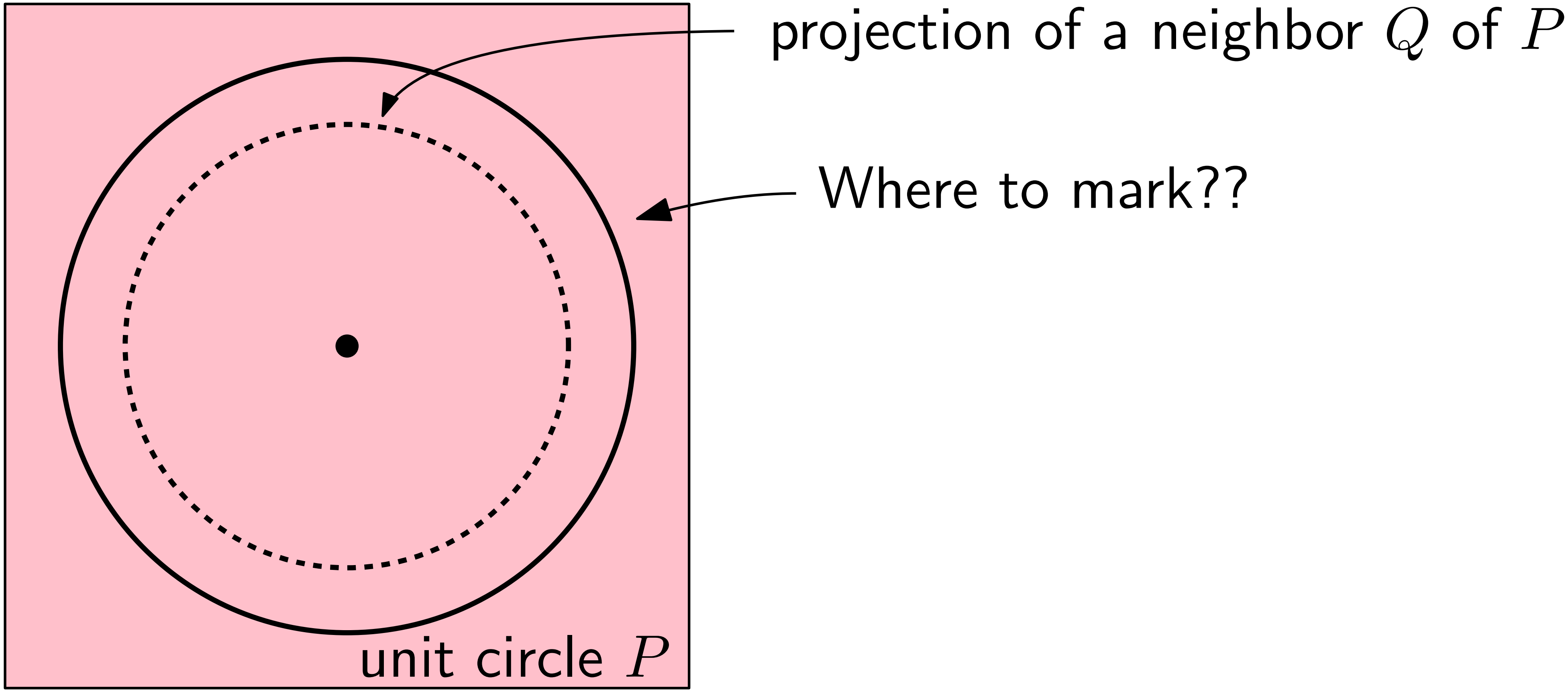
Algorithm Overview



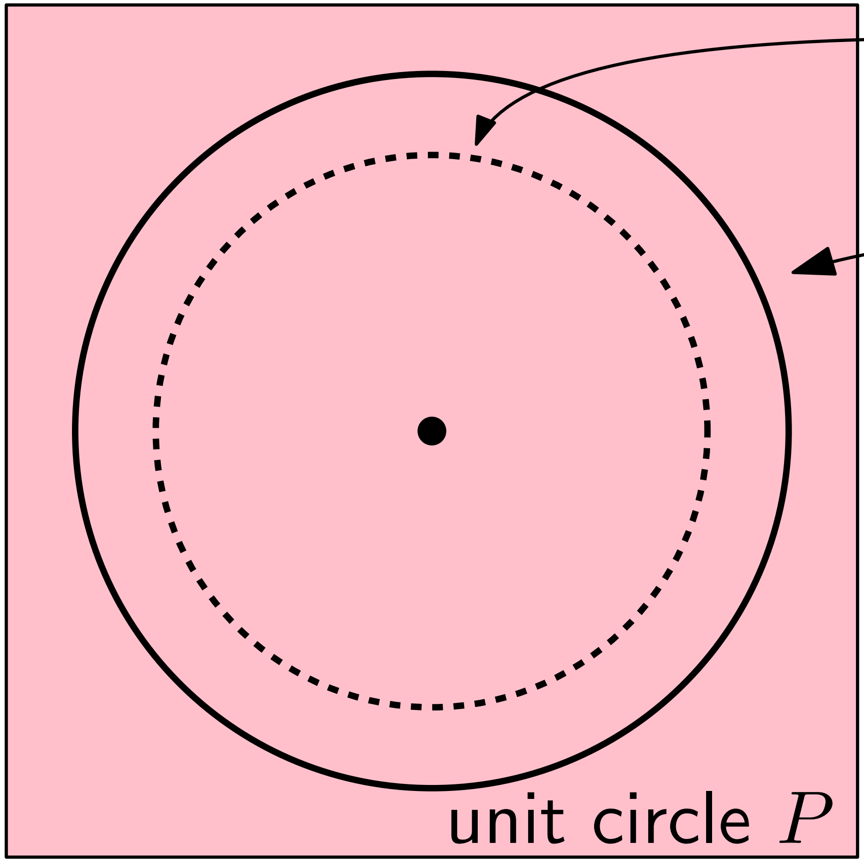
Algorithm Overview



Isoclinic planes



Isoclinic planes

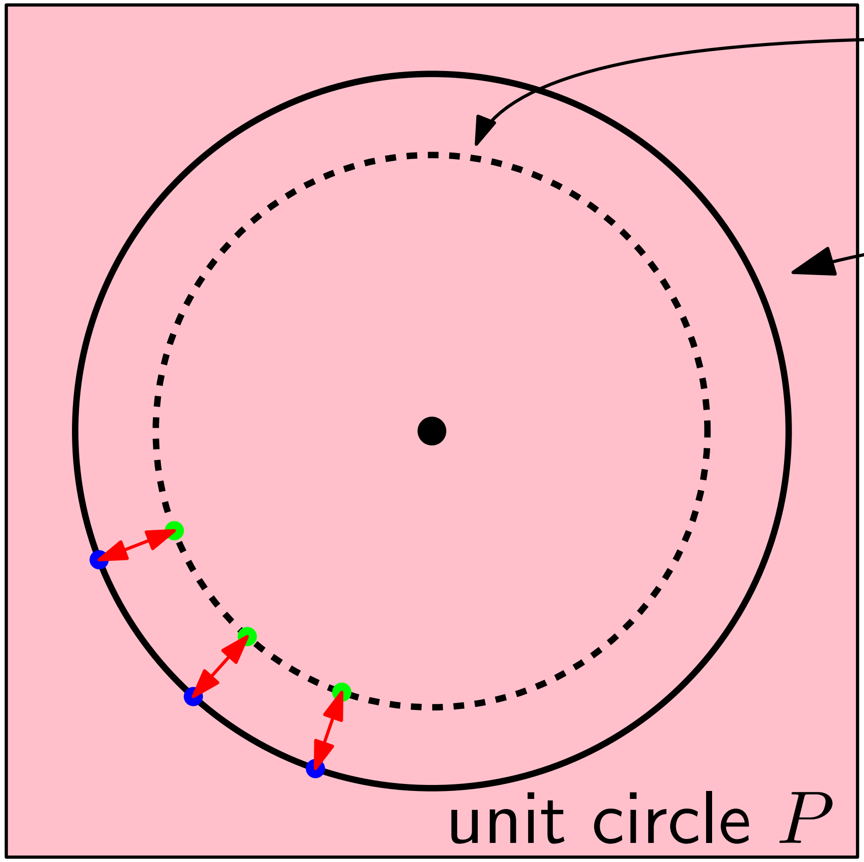


projection of a neighbor Q of P

Where to mark??

Problem if *all* closest pairs are isoclinic.

Isoclinic planes



projection of a neighbor Q of P

Where to mark??

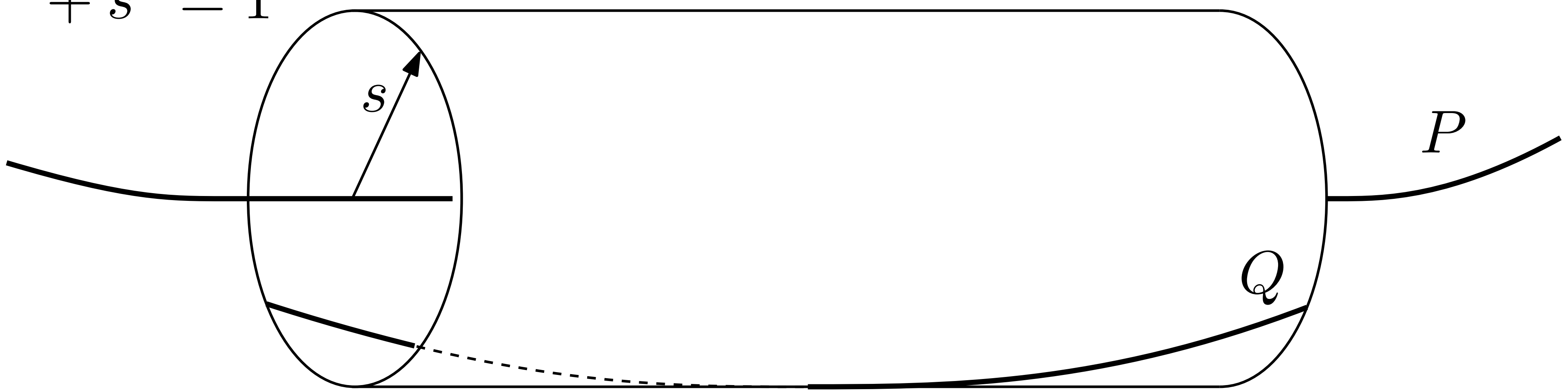
Problem if *all* closest pairs are isoclinic.

Constant distances from one circle to the other.
"Clifford-parallel" $\equiv$ isoclinic

Clifford-parallel circles

$$P\colon \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, \quad Q\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha + t) \\ s\sin(\alpha + t) \end{pmatrix}$$

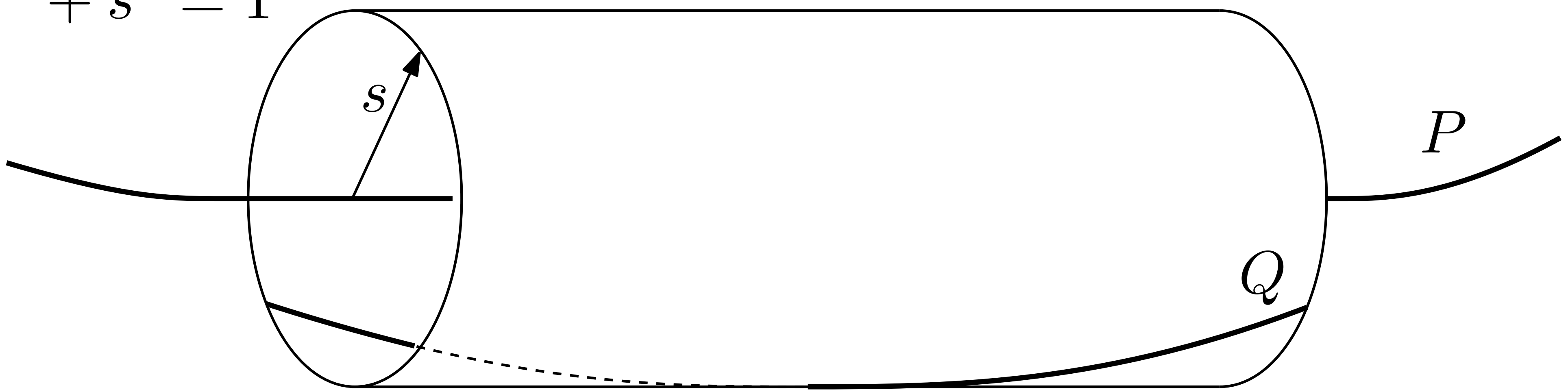
$$r^2 + s^2 = 1$$



Clifford-parallel circles

$$P\colon \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, \quad Q\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha+t) \\ s\sin(\alpha+t) \end{pmatrix}$$

$r^2 + s^2 = 1$



$h(x_1, y_1, x_2, y_2) =$ the *right* Hopf map $h\colon \mathbb{S}^3 \to \mathbb{S}^2$

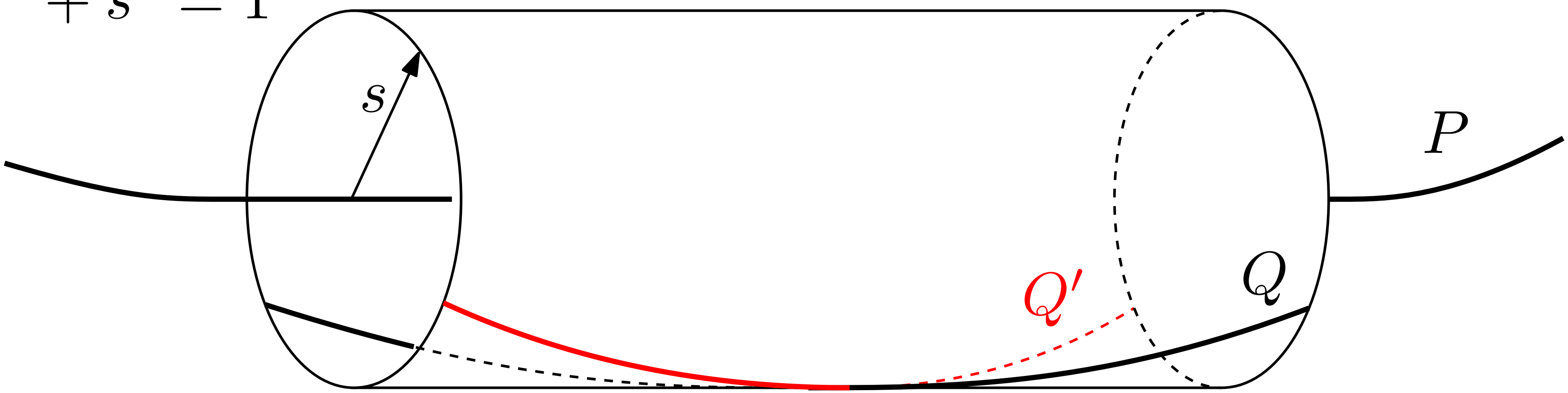
$$\Big(2(x_1y_2 - y_1x_2),\ 2(x_1x_2 + y_1y_2),\ 1 - 2(x_2^2 + y_2^2)\Big)$$

[Hopf 1931]

Clifford-parallel circles

$$P\colon \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, \quad Q\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha + t) \\ s\sin(\alpha + t) \end{pmatrix} \qquad Q'\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha - t) \\ s\sin(\alpha - t) \end{pmatrix}$$

$$r^2 + s^2 = 1$$



$$h(x_1, y_1, x_2, y_2) =$$

the *right* Hopf map $h\colon \mathbb{S}^3 \to \mathbb{S}^2$

$$\Big(2(x_1y_2 - y_1x_2),\ 2(x_1x_2 + y_1y_2),\ 1 - 2(x_2^2 + y_2^2)\Big)$$

[Hopf 1931]

The Hopf fibration

Right Hopf map $h\colon \mathbb{S}^3 \to \mathbb{S}^2$

The fibers $h^{-1}(p)$ for $p \in \mathbb{S}^2$ are great circles: a *Hopf bundle*

Every great circle belongs to a unique right Hopf bundle.

Isoclinic $\equiv$ belong to the same Hopf bundle
This is a transitive relation.

stereographic projection $\mathbb{S}^3 \to \mathbb{R}^3$

(Villarceau circles)

http://www.geom.uiuc.edu/~banchoff/script/b3d/hypertorus.html

The Hopf fibration

Right Hopf map $h\colon \mathbb{S}^3 \to \mathbb{S}^2$

The fibers $h^{-1}(p)$ for $p \in \mathbb{S}^2$ are great circles: a *Hopf bundle*

Every great circle belongs to a unique right Hopf bundle.

Isoclinic $\equiv$ belong to the same Hopf bundle
This is a transitive relation.

If all closest pairs are isoclinic
$\to$ all great circles in a connected component of the closest-pair graph belong to the same bundle.
$\to$ h maps them to points on $\mathbb{S}^2$.

We know how to deal with $\mathbb{S}^2$!

http://www.geom.uiuc.edu/~banchoff/script/b3d/hypertorus.html

Condensing on the 2-Sphere

Freie Universität Berlin

Equivariant condensation on the 2-sphere:

Input: $A \subseteq \mathbb{S}^2$.

Output: $A' \subseteq \mathbb{S}^2$, $|A'| \leq \min\{|A|, 12\}$.

- $A' =$ vertices of a regular icosahedron
- $A' =$ vertices of a regular octahedron
- $A' =$ vertices of a regular tetrahedron
- $A' =$ two antipodal points, or
- $A' =$ a single point.

Condensing on the 2-Sphere

Freie Universität Berlin

Equivariant condensation on the 2-sphere:

Input: $A \subseteq \mathbb{S}^2$.

Output: $A' \subseteq \mathbb{S}^2$, $|A'| \le \min\{|A|, 12\}$.

- $A' =$ vertices of a regular icosahedron
- $A' =$ vertices of a regular octahedron
- $A' =$ vertices of a regular tetrahedron
- $A' =$ two antipodal points, or
- $A' =$ a single point.

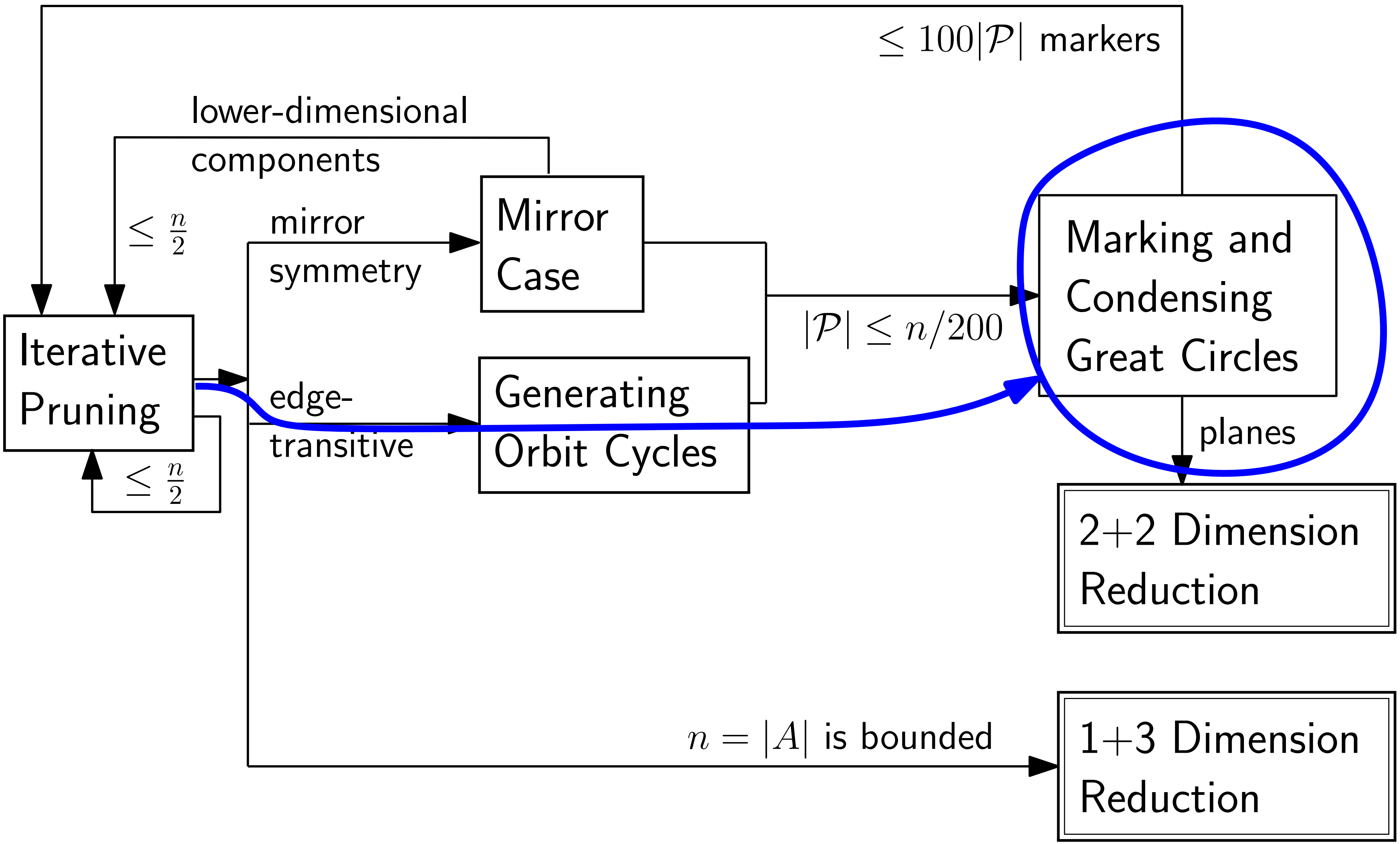
Condense each connected component of the closest-pair graph to ≤ 12 great circles.

Compute closest-pair graph (on $\mathbb{S}^5$) from scratch.

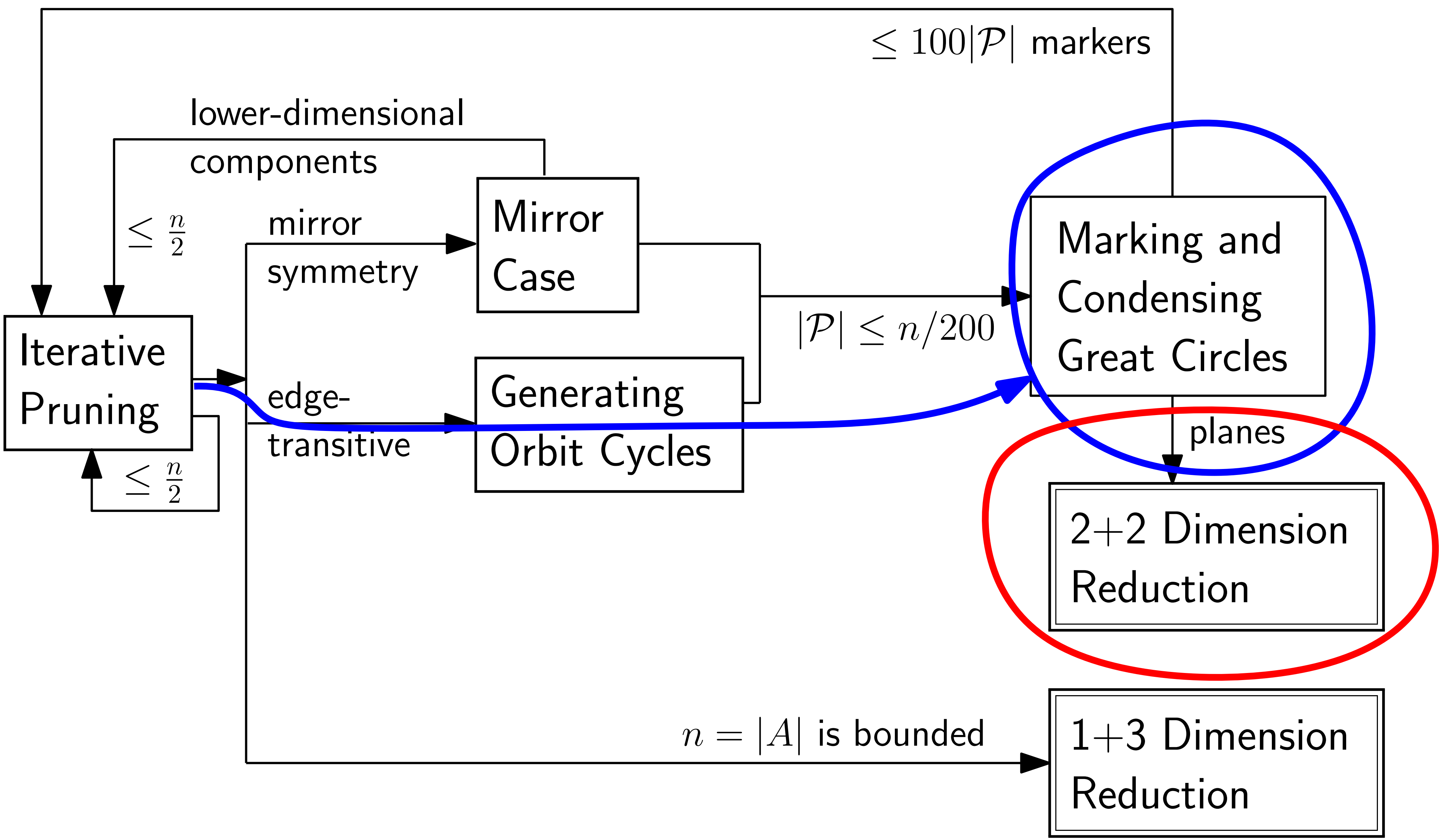
If no progress, distance between closest pairs is $\ge D_{\text{icosa}}$

$\rightarrow \le 829$ great circles $\rightarrow$ 2+2 DIMENSION REDUCTION

Algorithm Overview



Algorithm Overview

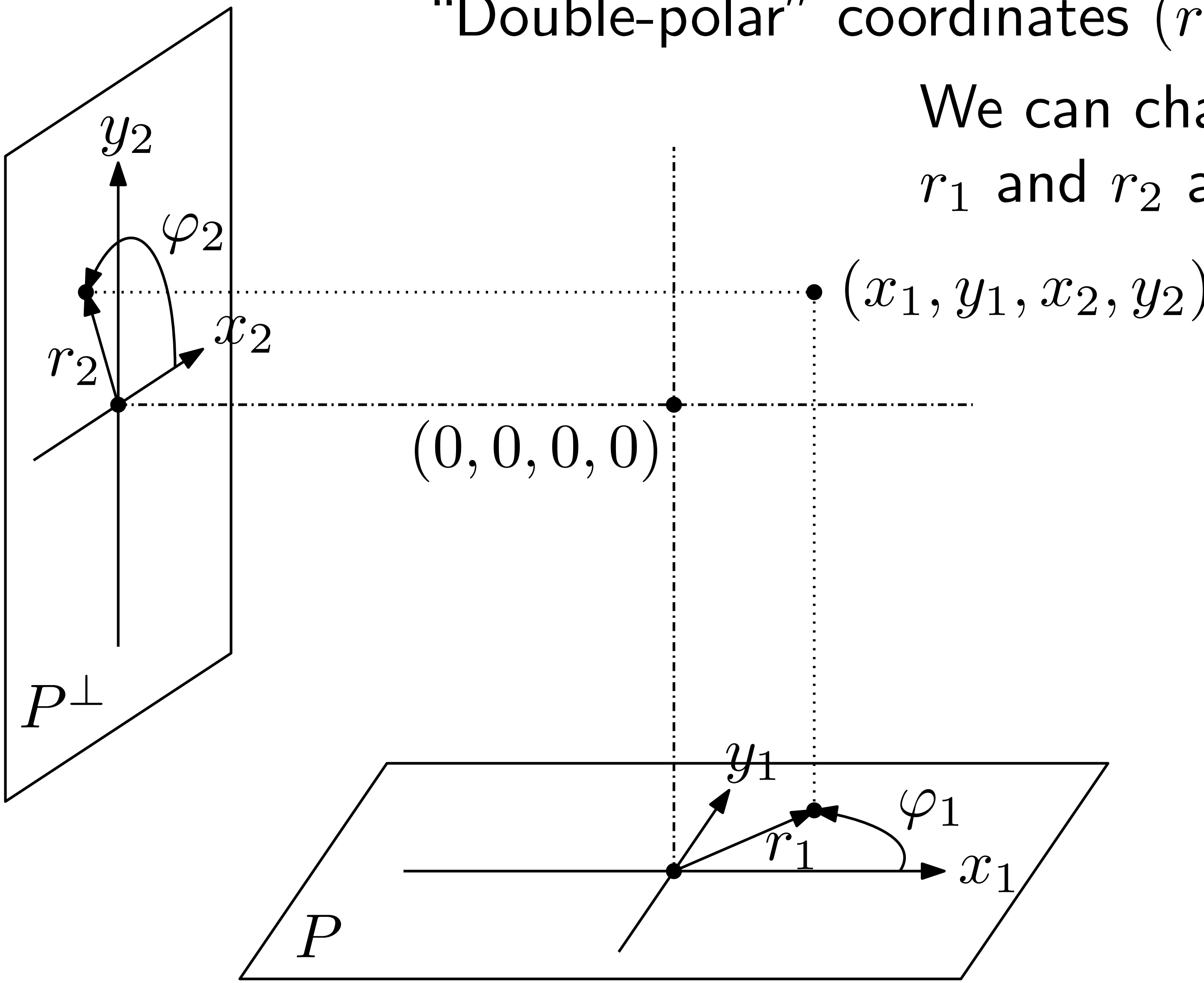


2+2 Dimension Reduction

We have a plane P and we know its image in B.

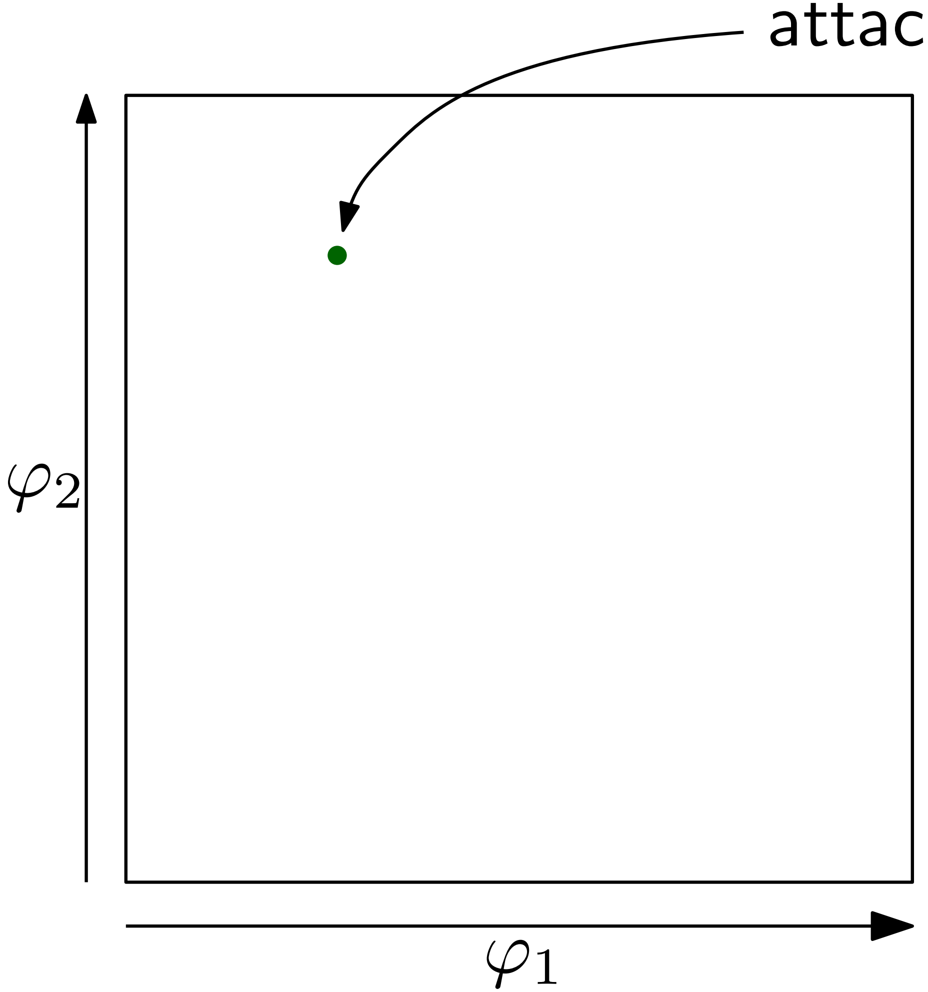
"Double-polar" coordinates $(r_1, \varphi_1, r_2, \varphi_2)$

We can change φ_1 and φ_2.
r_1 and r_2 are fixed.



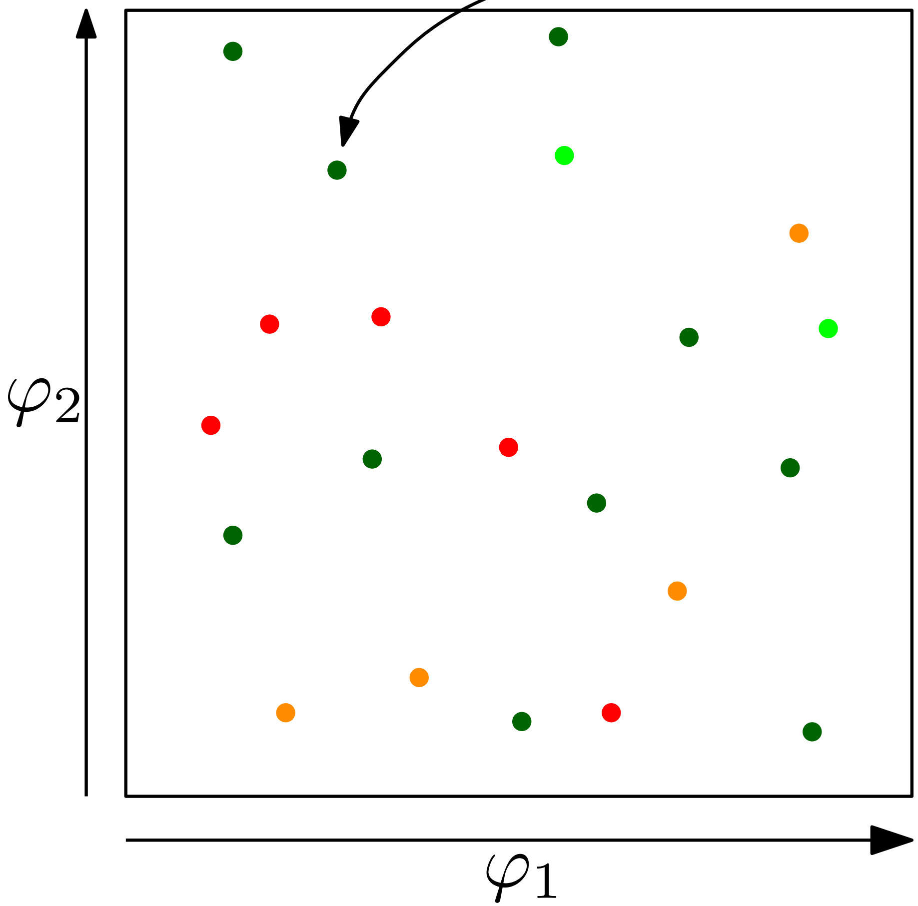
2+2 Dimension Reduction

attach (r_1, r_2) as a label (color)



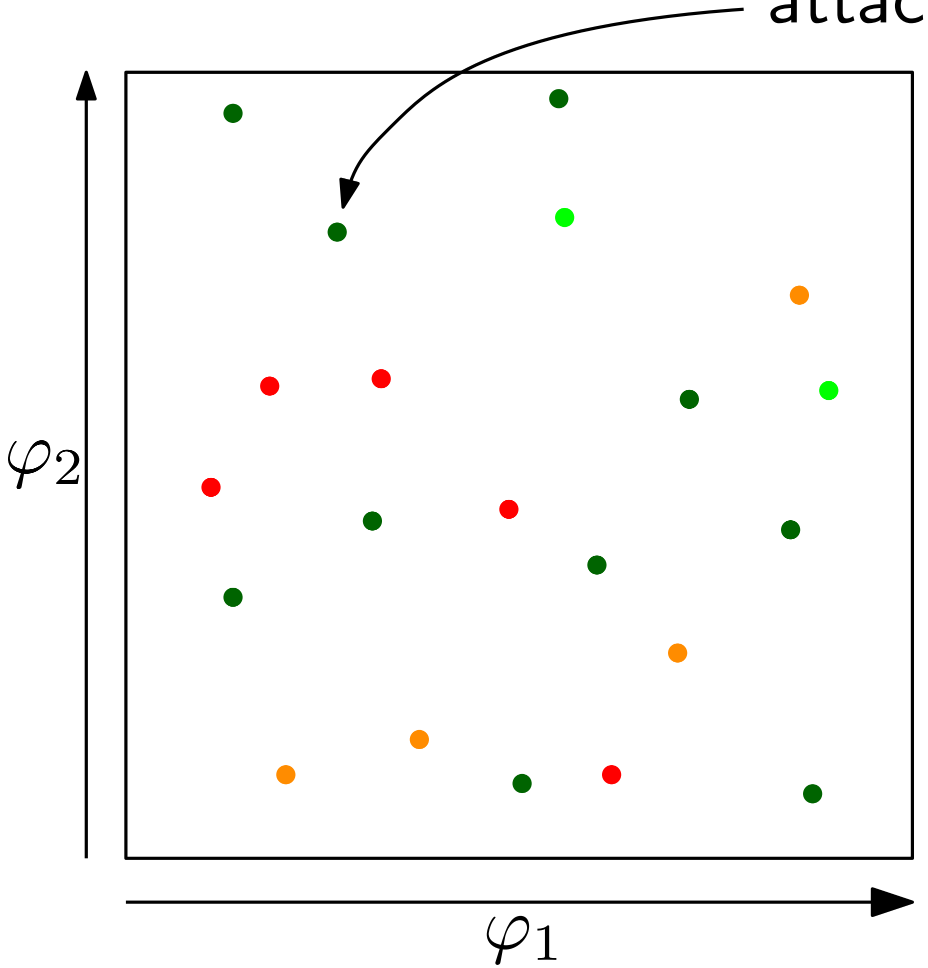
2+2 Dimension Reduction

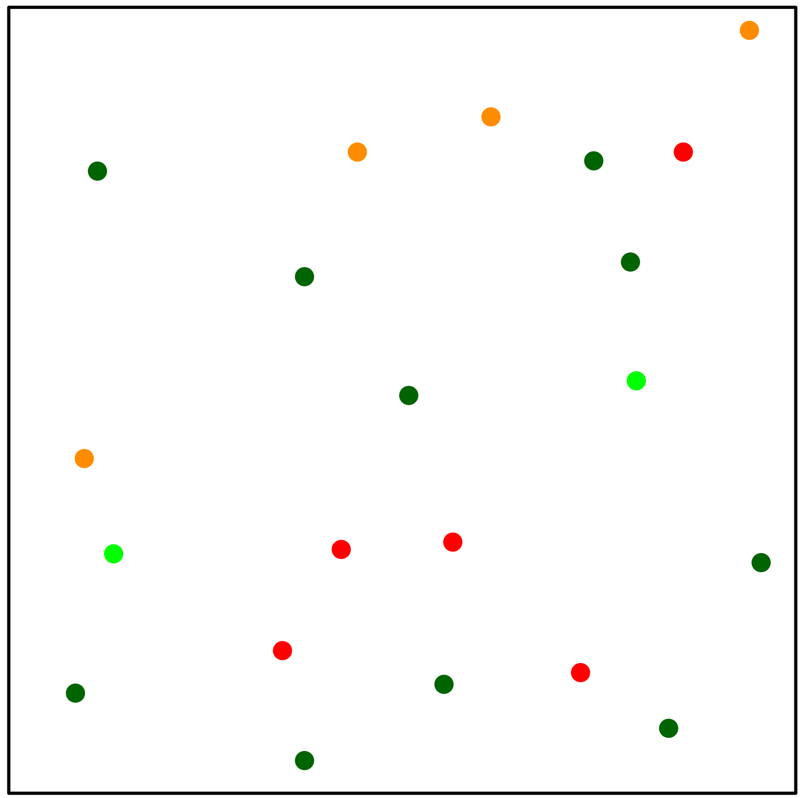
attach (r_1, r_2) as a label (color)



2+2 Dimension Reduction

attach (r_1, r_2) as a label (color)

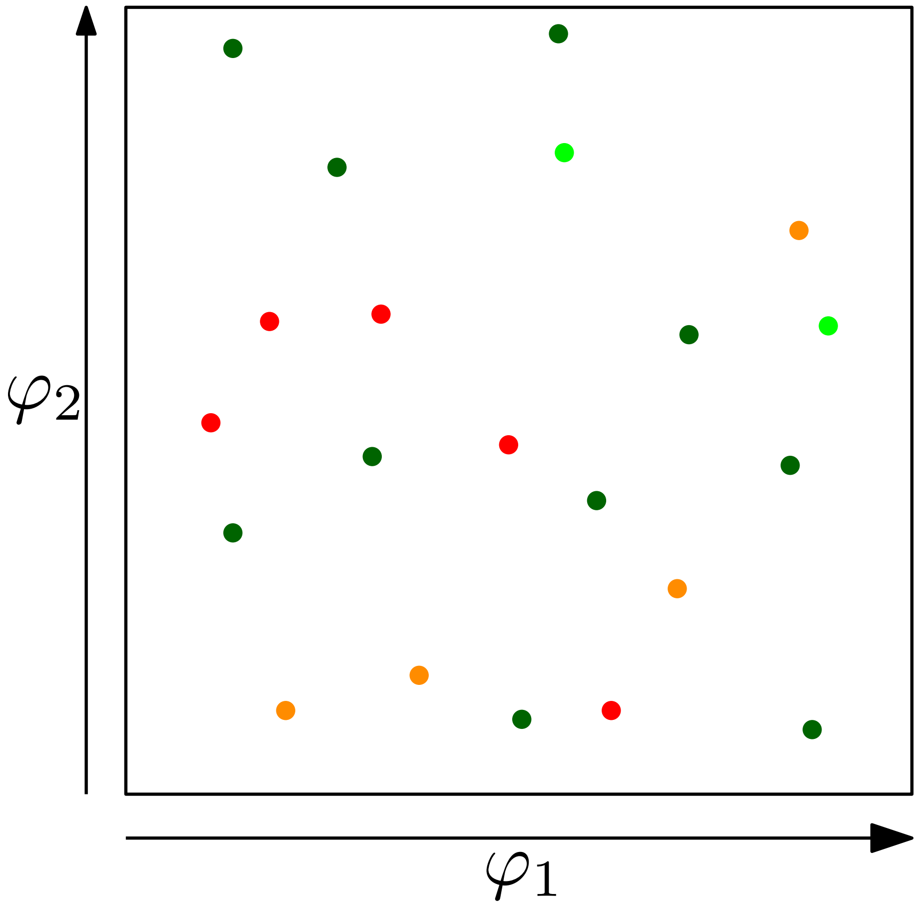




the picture for set B

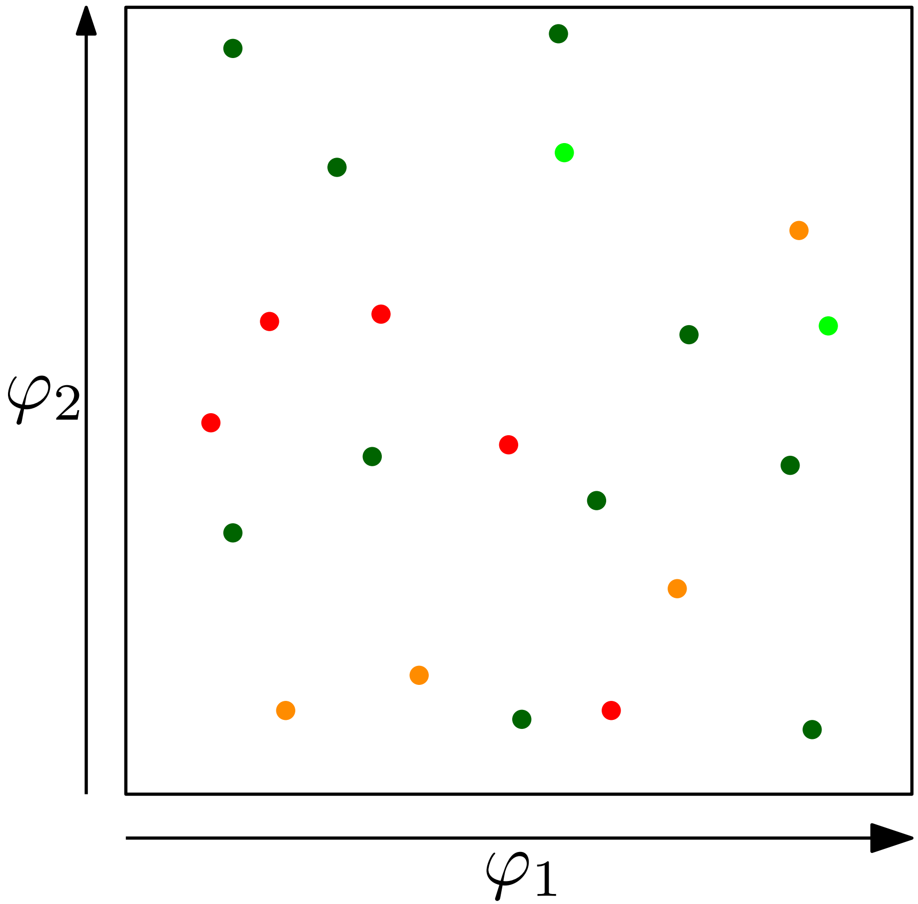
Are they the same up to translation on the φ_1, φ_2-torus?

2+2 Dimension Reduction



Prune without losing information:
(CANONICAL SET)

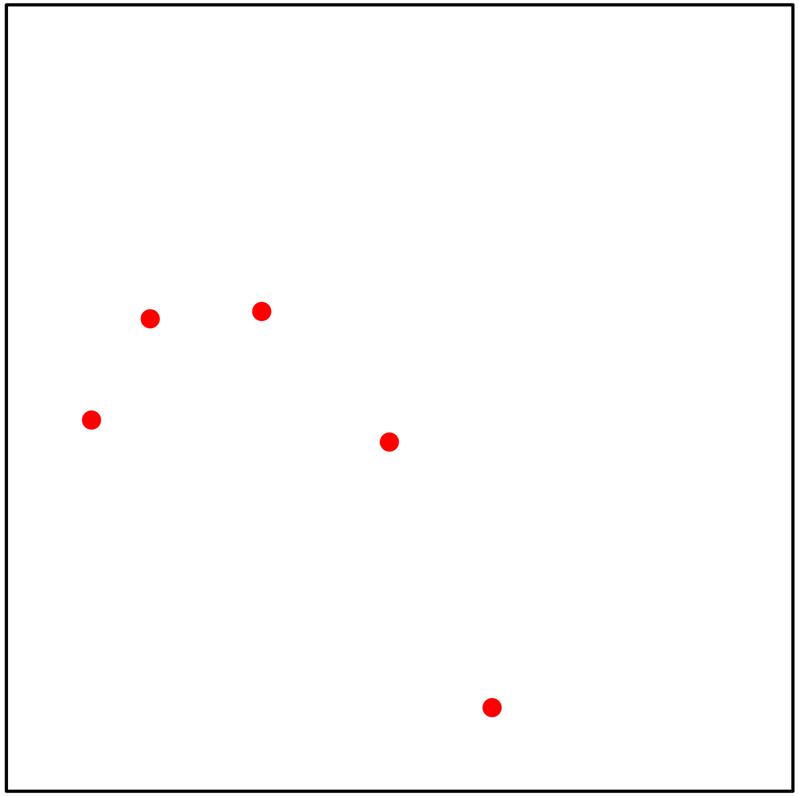
2+2 Dimension Reduction



Prune without losing information:
(CANONICAL SET)

Pick a color class

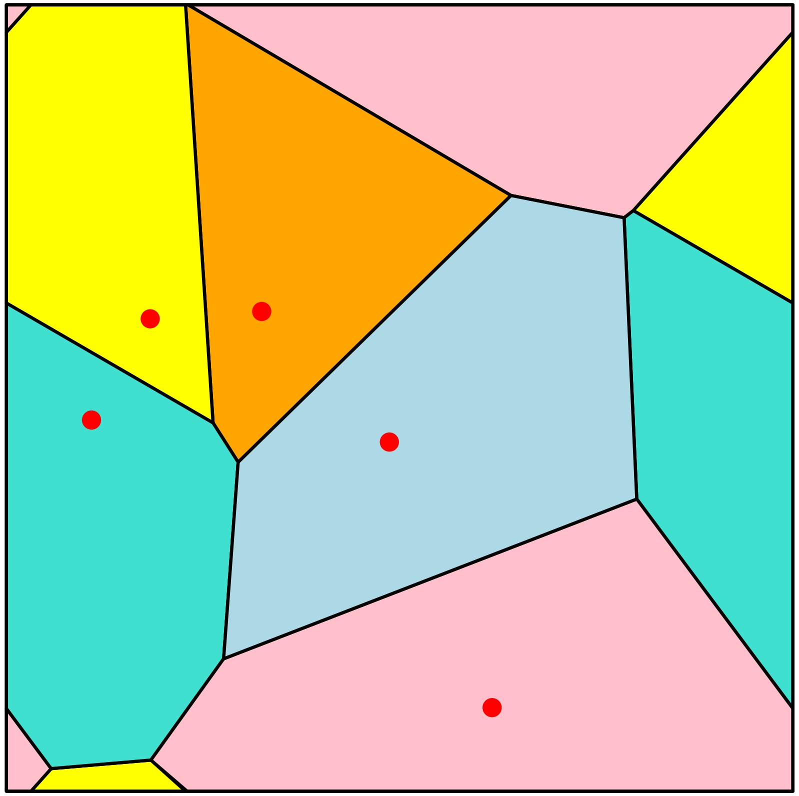
2+2 Dimension Reduction



Prune without losing information:
(CANONICAL SET)

Pick a color class

2+2 Dimension Reduction

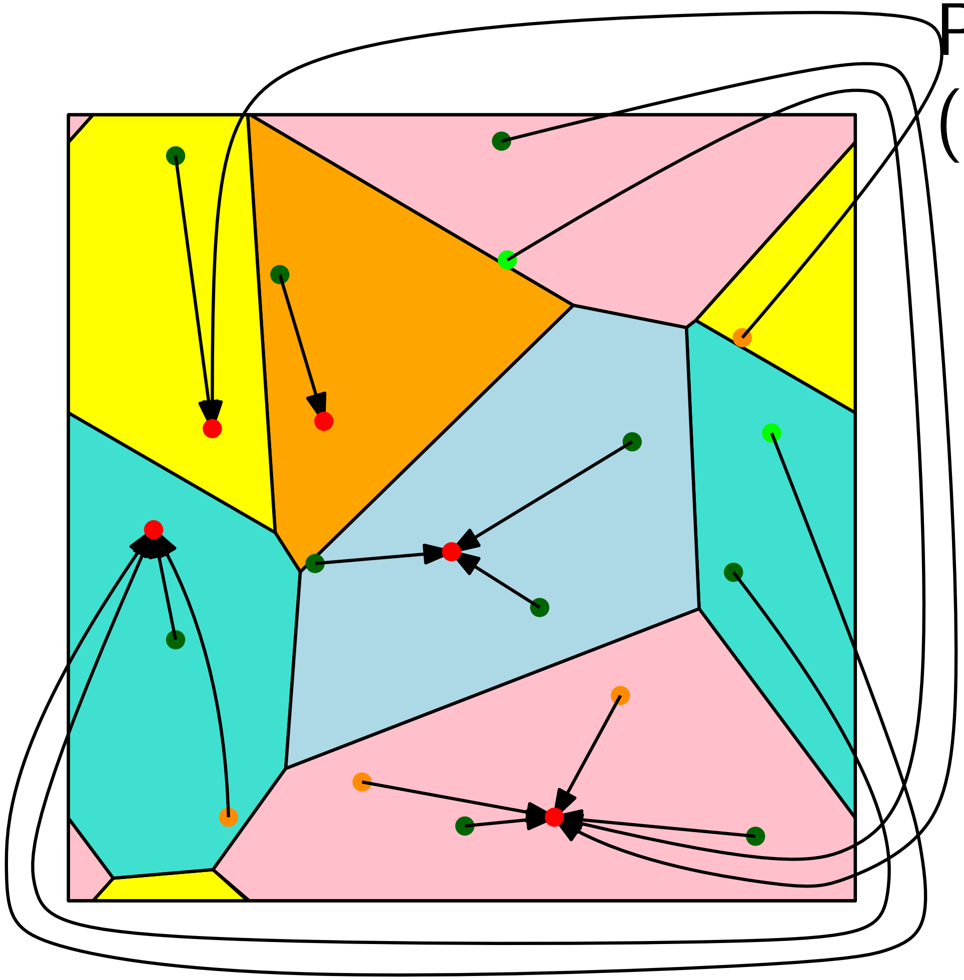


Prune without losing information: (CANONICAL SET)

Pick a color class

Compute the Voronoi diagram

2+2 Dimension Reduction

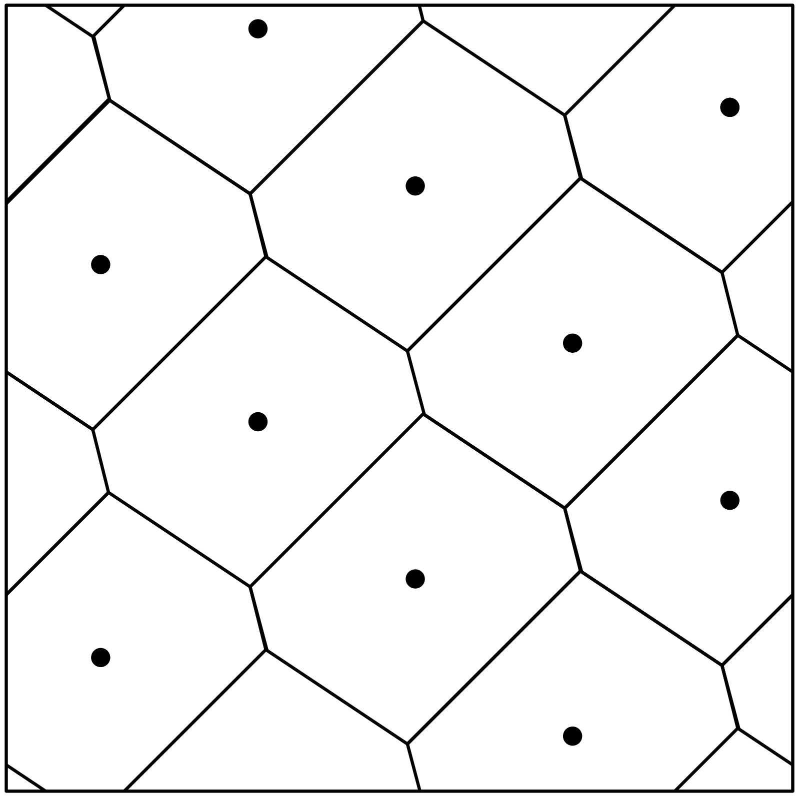


Prune without losing information:
(CANONICAL SET)

- Pick a color class
- Compute the Voronoi diagram
- Assign other points to cells.
- Refine the coloring, based on color and relative position of assigned points, shape of Voronoi cell.
- Repeat.

After recoloring, the reduced set has THE SAME translational symmetries as the old set.

2+2 Dimension Reduction



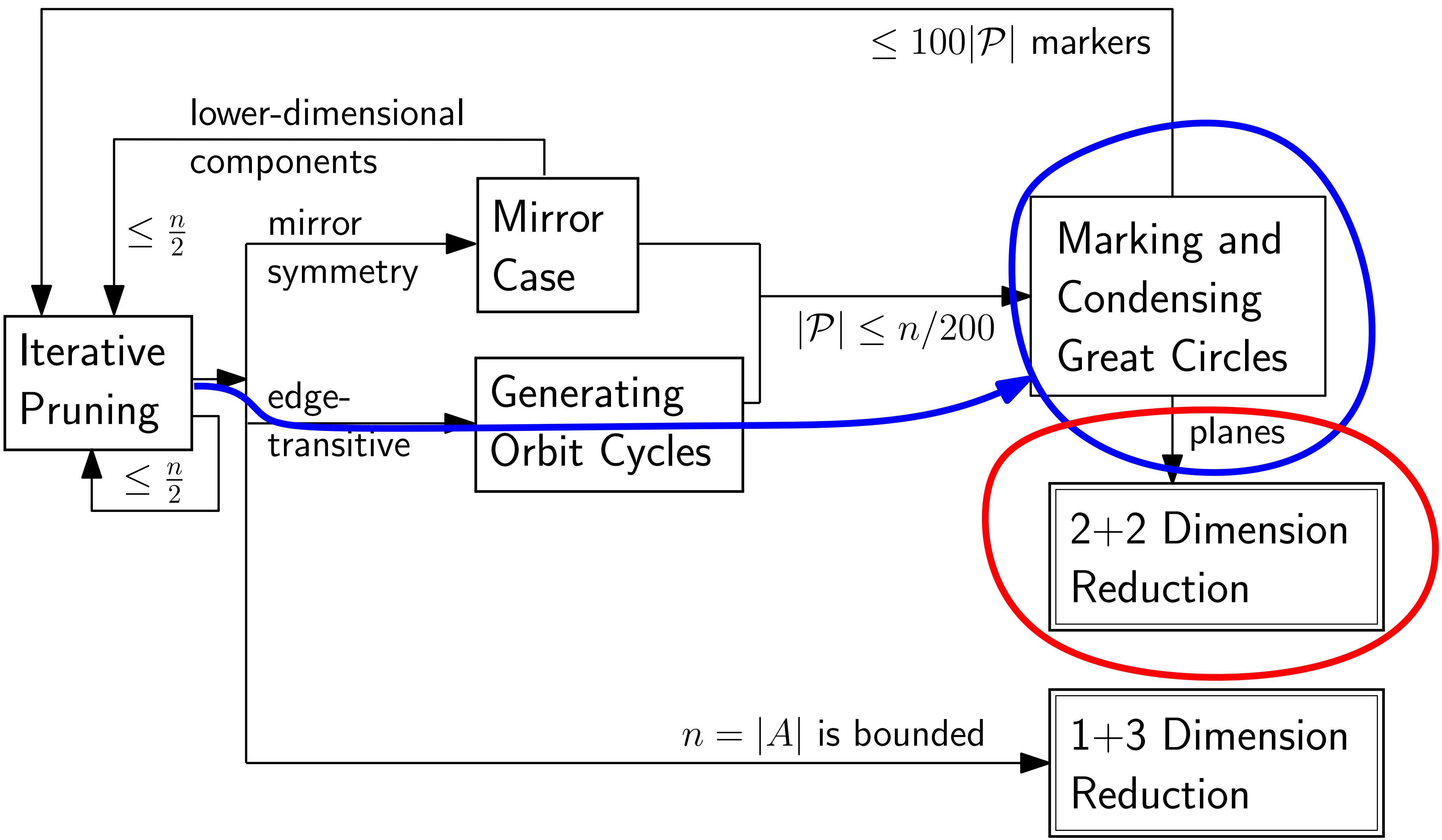
Termination:
All points have the same color and the same cell shape (a modular *lattice*)

ANY point is as good a representative as any other.

CANONICAL SET $c(A)$:
move (any) representative point to $(\varphi_1, \varphi_2) = (0, 0)$, or to $(x_1, 0, x_3, 0)$.

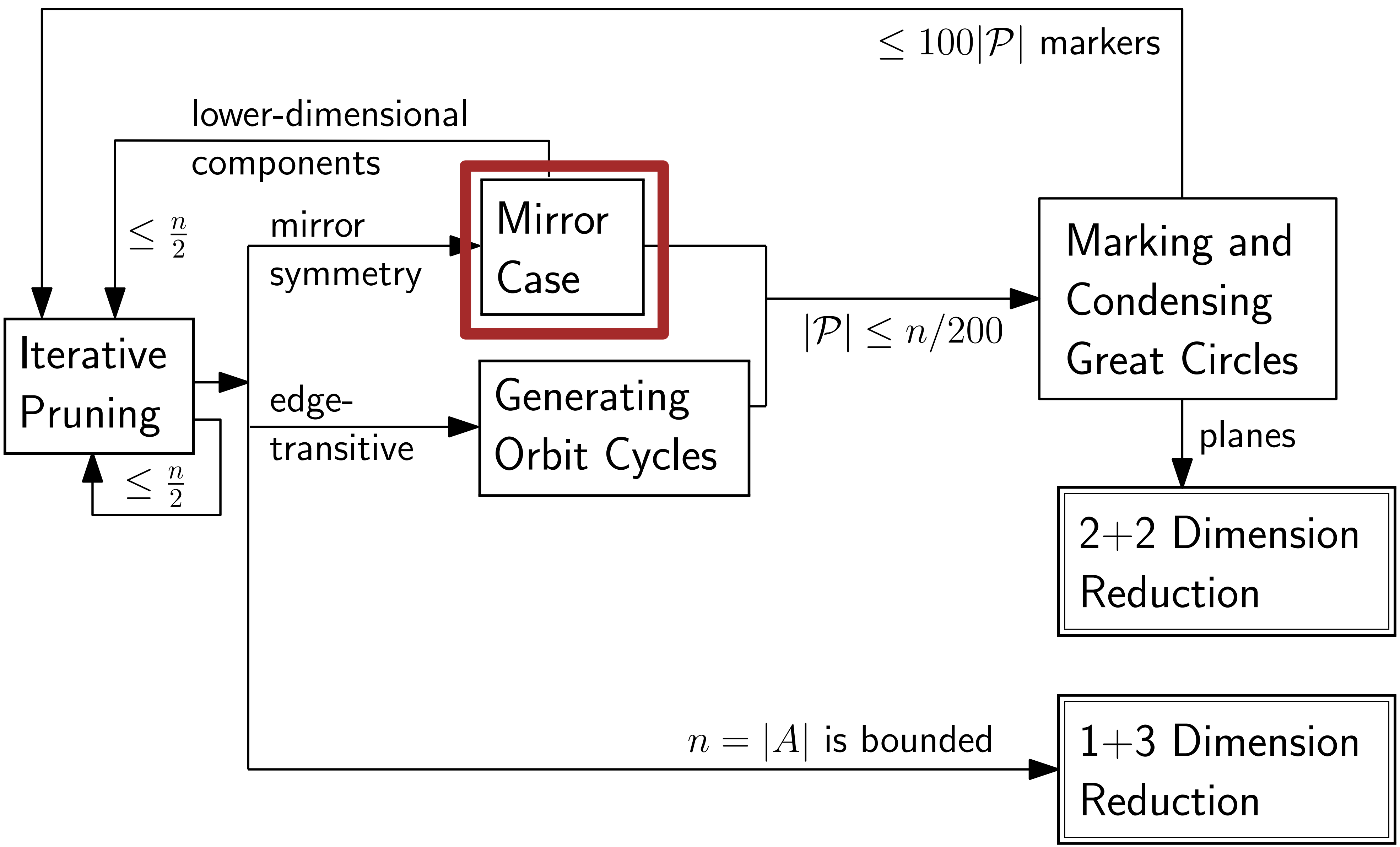
$$\exists T \text{ with } TP = P \text{ and } TA = B \iff c(A) = c(B)$$

Algorithm Overview



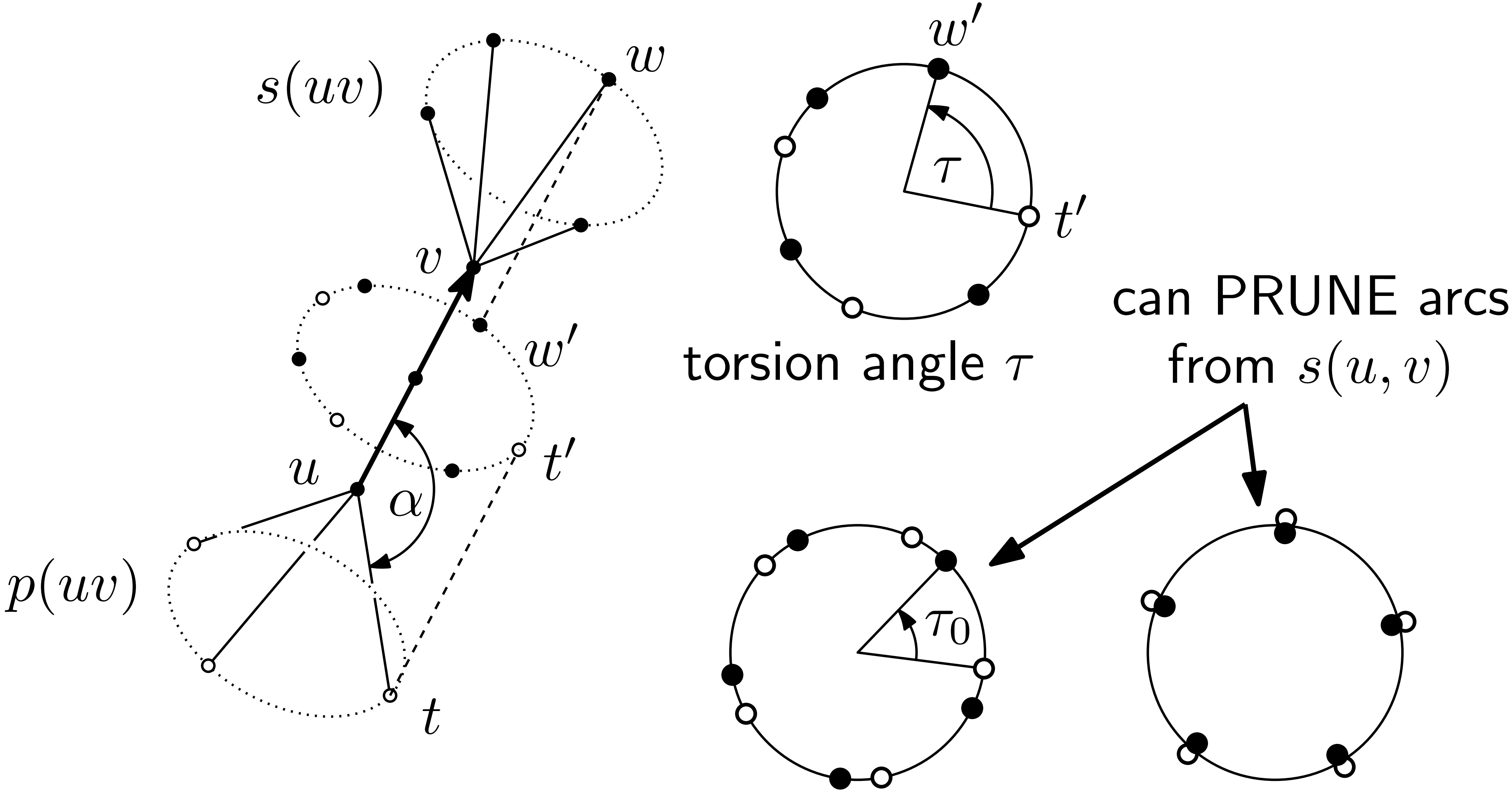
Algorithm Overview

Freie Universität Berlin



The Mirror Case

Pick some α. $\quad s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$



The Mirror Case

Every edge acts like a perfect mirror of the neighborhood.

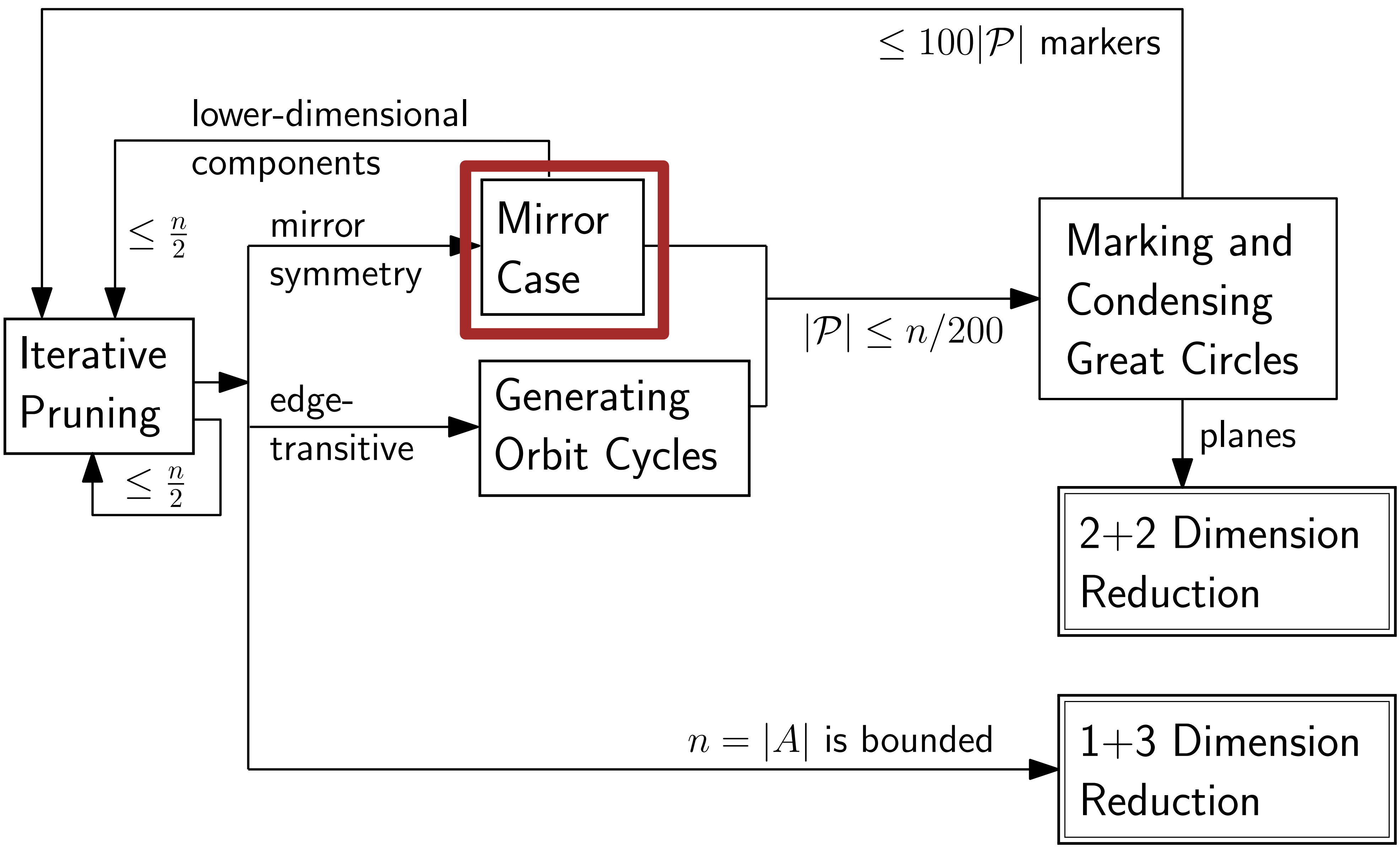
$\rightarrow$ Every connnected component is the orbit of a point under a group generated by reflections.

These groups have been classified. (Coxeter groups)

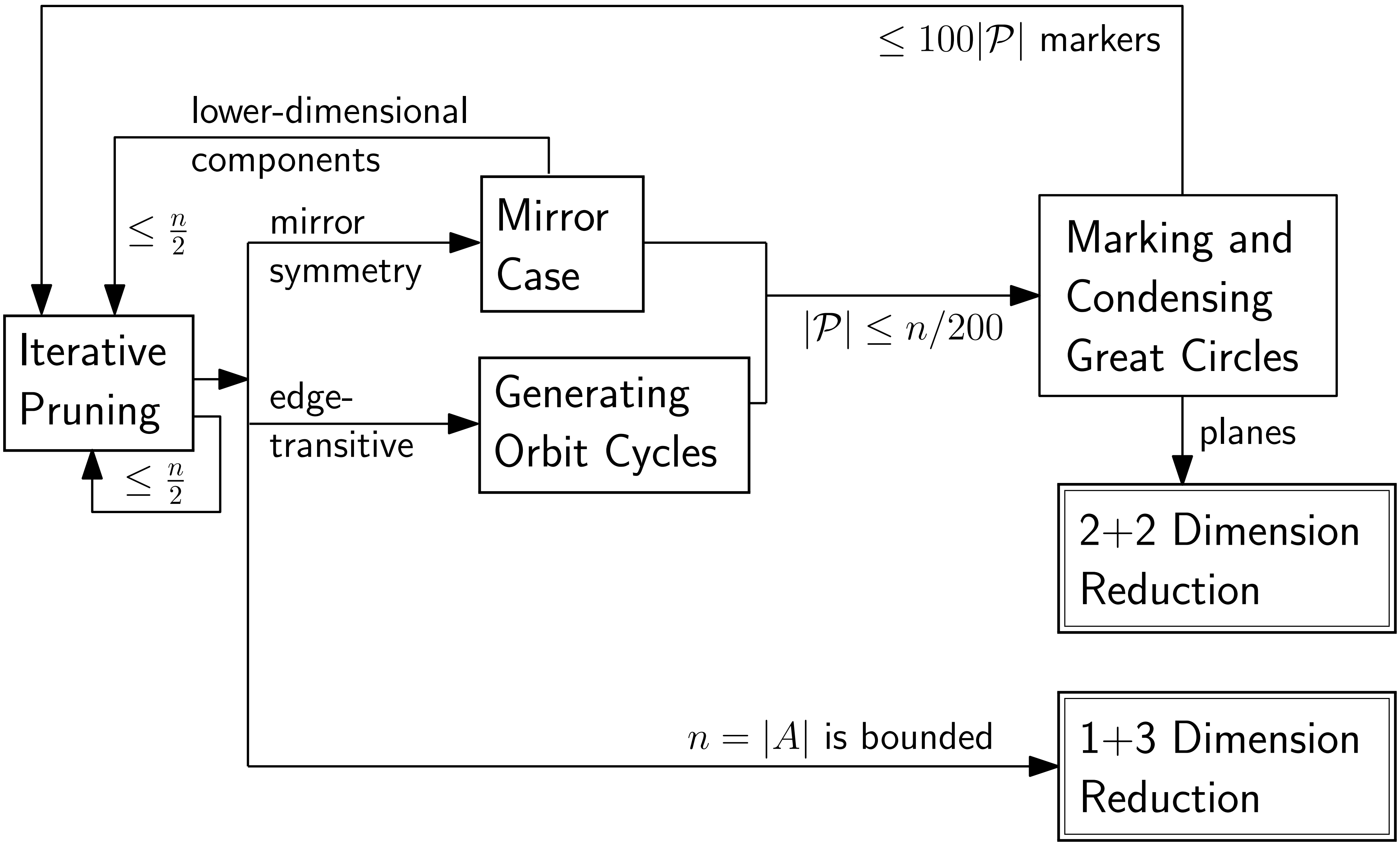
- "small" components
  $\rightarrow$ condensing
- Cartesian product of 2-dimensional groups (infinite family)
  $\rightarrow$ 2+2 dimension reduction
- "large" components (finite family)
  $\rightarrow |A| \le n_0$

Algorithm Overview

Freie Universität Berlin



Algorithm Overview



Afterthoughts

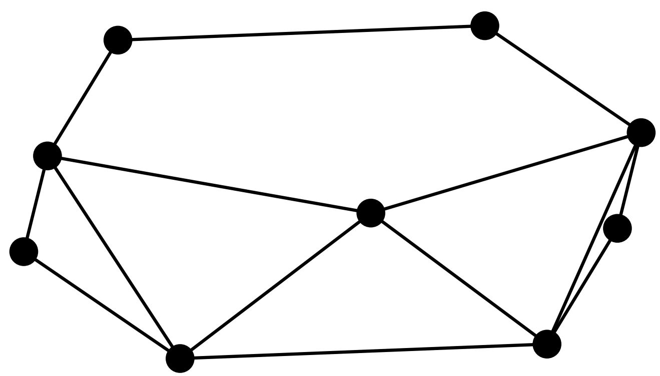
Freie Universität Berlin

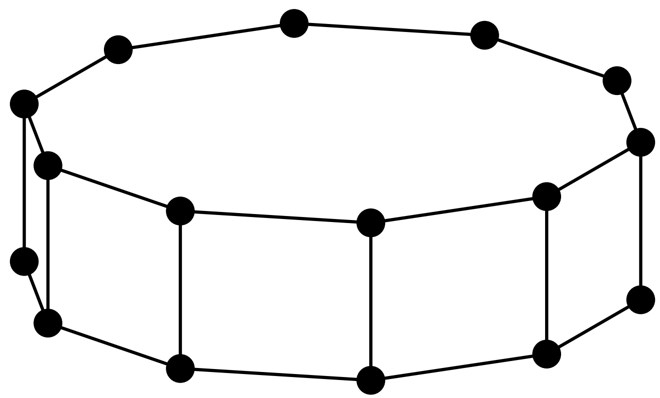
- 5 dimensions and higher
- terrible constants
- chimeras
- tolerances, $\leq \varepsilon$ versus $\geq 10\varepsilon$
- depth of construction ($\rightarrow$ degree of predicates)
- Plücker space
- point groups in 4 dimensions

Symmetry groups

COROLLARY. The symmetry group of a finite full-dimensional point set in 3-space ($= $ a discrete subgroup of $O(3)$) is

- the symmetry group of a Platonic solid,
- the symmetry group of a regular prism,
- or a subgroup of such a group.





The *point groups* (discrete subgroups of $O(3)$) are classified (Hessel's Theorem).

[F. Hessel 1830, M. L. Frankenheim 1826]

Point groups in higher dimensions

Bold and naive CONJECTURE:

¿The symmetry group of a finite full-dimensional point set in d-space ($=$ a discrete subgroup of $O(d)$) is

- the symmetry group of a regular d-dimensional polytope:
  - a regular simplex
  - * a hypercube (or its dual, the crosspolytope)
  - a regular n-gon in two dimensions
  - a dodecahedron (or its dual, the icosahedron) in 3 d.
  - a 24-cell, or a 120-cell (or its dual, the 600-cell) in 4 d.
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

Point groups in higher dimensions

Freie Universität Berlin

Bold and naive CONJECTURE:

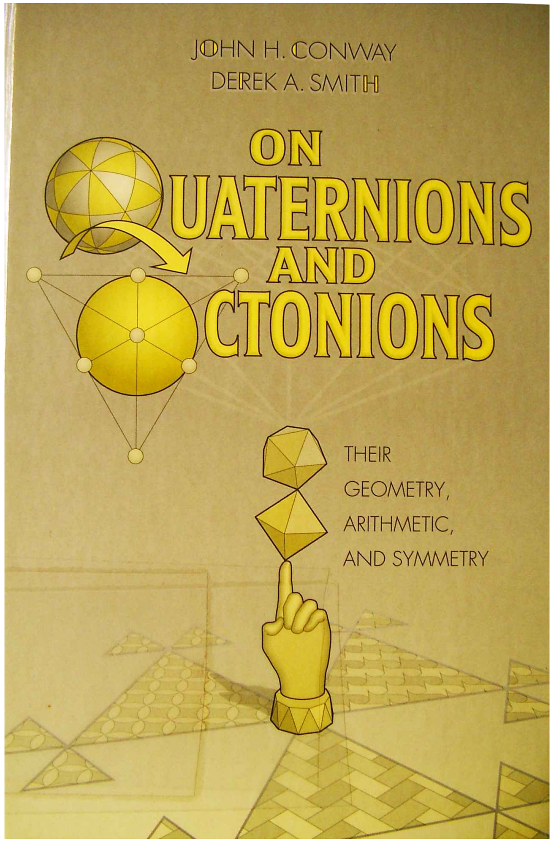
¿The symmetry group of a finite full-dimensional point set in d-space ($=$ a discrete subgroup of $O(d)$) is

- the symmetry group of a regular d-dimensional polytope:
  - a regular simplex
  - * a hypercube (or its dual, the crosspolytope)
  - a regular n-gon in two dimensions
  - a dodecahedron (or its dual, the icosahedron) in 3 d.
  - a 24-cell, or a 120-cell (or its dual, the 600-cell) in 4 d.
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

Counterexample (Paco Santos, by divisibility). The symmetry groups of the root systems E_6, E_7, E_8 in 6, 7, 8 dimensions.

The four-dimensional point groups

- [W. Threlfall and H. Seifert, Math. Annalen, 1931, 1933] enumerated discrete subgroups of $SO(4)$ (determinant $+1$)
- [J. Conway and D. Smith 2003] complete enumeration of point groups



4d-rotation $T \leftrightarrow$ pair (R, S) of 3d-rotations.
(for example, via quaternions)

Goursat's Lemma: [É. Goursat 1890]
Pairs of 3d point groups
$+$ additional information
$\rightarrow$ 4d point groups

- The groups generated by reflections (Coxeter groups) have been enumerated up to 8 dimensions.

[Norman Johnson, unpublished book manuscript]

The four-dimensional point groups

Group	Generators
$\pm[I \times O]$	$[i_I, 1], [\omega, 1], [1, i_O], [1, \omega]$;
$\pm[I \times T]$	$[i_I, 1], [\omega, 1], [1, i], [1, \omega]$;
$\pm[I \times D_{2n}]$	$[i_I, 1], [\omega, 1], [1, e_n], [1, j]$;
$\pm[I \times C_n]$	$[i_I, 1], [\omega, 1], [1, e_n]$;
$\pm[O \times T]$	$[i_O, 1], [\omega, 1], [1, i], [1, \omega]$;
$\pm[O \times D_{2n}]$	$[i_O, 1], [\omega, 1], [1, e_n], [1, j]$;
$\pm\frac{1}{2}[O \times D_{2n}]$	$[i, 1], [\omega, 1], [1, e_n]; [i_O, j]$
$\pm\frac{1}{2}[O \times \overline{D}_{4n}]$	$[i, 1], [\omega, 1], [1, e_n], [1, j]; [i_O, e_{2n}]$
$\pm\frac{1}{6}[O \times D_{6n}]$	$[i, 1], [j, 1], [1, e_n]; [i_O, j], [\omega, e_{3n}]$
$\pm[O \times C_n]$	$[i_O, 1], [\omega, 1], [1, e_n]$;
$\pm\frac{1}{2}[O \times C_{2n}]$	$[i, 1], [\omega, 1], [1, e_n]; [i_O, e_{2n}]$
$\pm[T \times D_{2n}]$	$[i, 1], [\omega, 1], [1, e_n], [1, j]$;
$\pm[T \times C_n]$	$[i, 1], [\omega, 1], [1, e_n]$;
$\pm\frac{1}{3}[T \times C_{3n}]$	$[i, 1], [1, e_n]; [\omega, e_{3n}]$
$\pm\frac{1}{2}[D_{2m} \times \overline{D}_{4n}]$	$[e_m, 1], [1, e_n], [1, j]; [j, e_{2n}]$
$\pm[D_{2m} \times C_n]$	$[e_m, 1], [j, 1], [1, e_n]$;
$\pm\frac{1}{2}[D_{2m} \times C_{2n}]$	$[e_m, 1], [1, e_n]; [j, e_{2n}]$
$+\frac{1}{2}[D_{2m} \times C_{2n}]$	− , − ; +
$\pm\frac{1}{2}[\overline{D}_{4m} \times C_{2n}]$	$[e_m, 1], [j, 1], [1, e_n]; [e_{2m}, e_{2n}]$

Table 4.1. Chiral groups, I. These are most of the "metachiral" groups—s
4.6—some others appear in the last few lines of Table 4.2.

Table 4.1. The *chiral* groups (groups of orientation-preserving orthogonal transformations)

[Conway and Smith 2003]

← both m and n must be odd.

The four-dimensional point groups

Table 4.2.
The *chiral* groups
(continued)

Group	Generators	Coxeter Name
$\pm[I\times I]$	$[i_I,1],[\omega,1],[1,i_I],[1,\omega];$	$[3,3,5]^+$
$\pm\frac{1}{60}[I\times I]$	$;[\omega,\omega],[i_I,i_I]$	$2.[3,5]^+$
$+\frac{1}{60}[I\times I]$	$;\ +\ ,\ +$	$[3,5]^+$
$\pm\frac{1}{60}[I\times\overline{I}]$	$;[\omega,\omega],[i_I,i'_I]$	$2.[3,3,3]^+$
$+\frac{1}{60}[I\times\overline{I}]$	$;\ +\ ,\ +$	$[3,3,3]^+$
$\pm[O\times O]$	$[i_O,1],[\omega,1],[1,i_O],[1,\omega];$	$[3,4,3]^+:2$
$\pm\frac{1}{2}[O\times O]$	$[i,1],[\omega,1],[1,i],[1,\omega];[i_O,i_O]$	$[3,4,3]^+$
$\pm\frac{1}{6}[O\times O]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\omega],[i_O,i_O]$	$[3,3,4]^+$
$\pm\frac{1}{24}[O\times O]$	$;[\omega,\omega],[i_O,i_O]$	$2.[3,4]^+$
$+\frac{1}{24}[O\times O]$	$;\ +\ ,\ +$	$[3,4]^+$
$+\frac{1}{24}[O\times\overline{O}]$	$;\ +\ ,\ -$	$[2,3,3]^+$
$\pm[T\times T]$	$[i,1],[\omega,1],[1,i],[1,\omega];$	$[^+3,4,3^+]$
$\pm\frac{1}{3}[T\times T]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\omega]$	$[^+3,3,4^+]$
$\cong\pm\frac{1}{3}[T\times\overline{T}]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\overline{\omega}]$	”
$\pm\frac{1}{12}[T\times T]$	$;[\omega,\omega],[i,i]$	$2.[3,3]^+$
$\cong\pm\frac{1}{12}[T\times\overline{T}]$	$;[\omega,\overline{\omega}],[i,-i]$	”
$+\frac{1}{12}[T\times T]$	$;\ +\ ,\ +$	$[3,3]^+$
$\cong+\frac{1}{12}[T\times\overline{T}]$	$;\ +\ ,\ +$	”
$\pm[D_{2m}\times D_{2n}]$	$[e_m,1],[j,1],[1,e_n],[1,j];$	
$\pm\frac{1}{2}[\overline{D}_{4m}\times\overline{D}_{4n}]$	$[e_m,1],[j,1],[1,e_n],[1,j];[e_{2m},e_{2n}]$	
$\pm\frac{1}{4}[D_{4m}\times\overline{D}_{4n}]$	$[e_m,1],[1,e_n];[e_{2m},j],[j,e_{2n}]$	Conditions
$+\frac{1}{4}[D_{4m}\times\overline{D}_{4n}]$	$-\ ,\ -\ ;\ +\ ,\ +$	m,n odd
$\pm\frac{1}{2f}[D_{2mf}\times D^{(s)}_{2nf}]$	$[e_m,1],[1,e_n];[e_{mf},e^s_{nf}],[j,j]$	$(s,f)=1$
$+\frac{1}{2f}[D_{2mf}\times D^{(s)}_{2nf}]$	$-\ ,\ -\ ;\ +\ ,\ +$	m,n odd, $(s,2f)=1$
$\pm\frac{1}{f}[C_{mf}\times C^{(s)}_{nf}]$	$[e_m,1],[1,e_n];[e_{mf},e^s_{nf}]$	$(s,f)=1$
$+\frac{1}{f}[C_{mf}\times C^{(s)}_{nf}]$	$-\ ,\ -\ ;\ +$	m,n odd, $(s,2f)=1$

Table 4.2. Chiral groups, II. These groups are mostly “orthochiral,” with a few “parachiral”

The four-dimensional point groups

Table 4.3.
The *achiral* groups

Group	Extending element	Coxeter Name
$\pm[I\times I]\cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I\times I]\cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I\times I]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I\times \bar{I}]\cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{60}[I\times \bar{I}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
$\pm[O\times O]\cdot 2$	$*$	$[3,4,3]:2$
$\pm\frac{1}{2}[O\times O]\cdot 2$ or $\bar{2}$	$*$ or $*[1,i_O]$	$[3,4,3]$ or $[3,4,3]^{+}\cdot 2$
$\pm\frac{1}{6}[O\times O]\cdot 2$	$*$	$[3,3,4]$
$\pm\frac{1}{24}[O\times O]\cdot 2$	$*$	$2.[3,4]$
$+\frac{1}{24}[O\times O]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,4]$ or $[3,4]^\circ$
$+\frac{1}{24}[O\times \bar{O}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[2,3,3]^\circ$ or $[2,3,3]$
$\pm[T\times T]\cdot 2$	$*$	$[3,4,3^+]$
$\pm\frac{1}{3}[T\times T]\cdot 2$	$*$	$[^+3,3,4]$
$\pm\frac{1}{3}[T\times \bar{T}]\cdot 2$	$*$	$[3,3,4^+]$
$\pm\frac{1}{12}[T\times T]\cdot 2$	$*$	$2.[^+3,4]$
$\pm\frac{1}{12}[T\times \bar{T}]\cdot 2$	$*$	$2.[3,3]$
$+\frac{1}{12}[T\times T]\cdot 2_3$ or 2_1	$*$ or $-*$	$[^+3,4]$ or $[^+3,4]^\circ$
$+\frac{1}{12}[T\times \bar{T}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3]^\circ$ or $[3,3]$
$\pm[D_{2n}\times D_{2n}]\cdot 2$	$*$	
$\pm\frac{1}{2}[\bar{D}_{4n}\times \bar{D}_{4n}]\cdot 2$ or $\bar{2}$	$*$ or $*[1,e_{2n}]$	
$\pm\frac{1}{4}[D_{4n}\times \bar{D}_{4n}]\cdot 2$	$*$	Conditions
$+\frac{1}{4}[D_{4n}\times \bar{D}_{4n}]\cdot 2_3$ or 2_1	$*$ or $-*$	n odd
$\pm\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\bar{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*See*
$+\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\bar{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*Text*
$\pm\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1,e^{\gamma(f,s+1)}_{2nf}]$	*in*
$+\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1,e^{\gamma(f,s+1)}_{2nf}]$	*Appendix*

Table 4.3. Achiral groups.

The four-dimensional point groups

Group	Extending element	Coxeter Name
$\pm[I\times I]\cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I\times I]\cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I\times I]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I\times \overline{I}]\cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{60}[I\times \overline{I}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
$\pm[O\times O]\cdot 2$	$*$	$[3,4,3]:2$
$\pm\frac{1}{2}[O\times O]\cdot 2$ or $\overline{2}$	$*$ or $*[1,i_O]$	$[3,4,3]$ or $[3,4,3]^{+}\cdot 2$
$\pm\frac{1}{6}[O\times O]\cdot 2$	$*$	$[3,3,4]$
$\pm\frac{1}{24}[O\times O]\cdot 2$	$*$	$2.[3,4]$
$+\frac{1}{24}[O\times O]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,4]$ or $[3,4]^\circ$
$+\frac{1}{24}[O\times \overline{O}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[2,3,3]^\circ$ or $[2,3,3]$
$\pm[T\times T]\cdot 2$	$*$	$[3,4,3^+]$
$\pm\frac{1}{3}[T\times T]\cdot 2$	$*$	$[^+3,3,4]$
$\pm\frac{1}{3}[T\times \overline{T}]\cdot 2$	$*$	$[3,3,4^+]$
$\pm\frac{1}{12}[T\times T]\cdot 2$	$*$	$2.[^+3,4]$
$\pm\frac{1}{12}[T\times \overline{T}]\cdot 2$	$*$	$2.[3,3]$
$+\frac{1}{12}[T\times T]\cdot 2_3$ or 2_1	$*$ or $-*$	$[^+3,4]$ or $[^+3,4]^\circ$
$+\frac{1}{12}[T\times \overline{T}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3]^\circ$ or $[3,3]$
$\pm[D_{2n}\times D_{2n}]\cdot 2$	$*$	
$\pm\frac{1}{2}[\overline{D}_{4n}\times \overline{D}_{4n}]\cdot 2$ or $\overline{2}$	$*$ or $*[1,e_{2n}]$	
$\pm\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2$	$*$	Conditions
$+\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2_3$ or 2_1	$*$ or $-*$	n odd
$\pm\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*See*
$+\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*Text*
$\pm\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1, e^{\gamma(f,s+1)}_{2nf}]$	*in*
$+\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1, e^{\gamma(f,s+1)}_{2nf}]$	*Appendix*

Table 4.3. Achiral groups.

Table 4.3.
The *achiral* groups

- Project: Visualize these groups: Schlegel diagram of a 4-polytope which has these symmetries.

Bold and naive CONJECTURE:

¿The symmetry group of a finite full-dimensional point set in 4-space ($=$ a discrete subgroup of $O(4)$) is

- the symmetry group of a regular d-dimensional polytope:
  - a regular simplex
  - a regular n-gon in two dimensions
  - a dodecahedron (or its dual, the icosahedron) in 3 d.
  - a 24-cell, or a 120-cell (or its dual, the 600-cell) in 4 d.
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

Point groups in four dimensions

Bold and naive CONJECTURE:

¿The symmetry group of a finite full-dimensional point set in 4-space ($=$ a discrete subgroup of $O(4)$) is

- the symmetry group of a regular d-dimensional polytope:
  - a regular simplex
  - a regular n-gon in two dimensions
  - a dodecahedron (or its dual, the icosahedron) in 3 d.
  - a 24-cell, or a 120-cell (or its dual, the 600-cell) in 4 d.
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

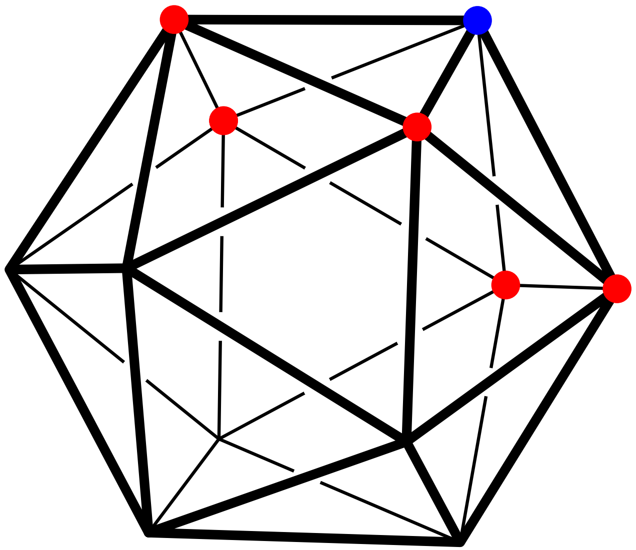
Counterexample: $I \times C_n$ (group-theoretic product, but not geometric Cartesian product)

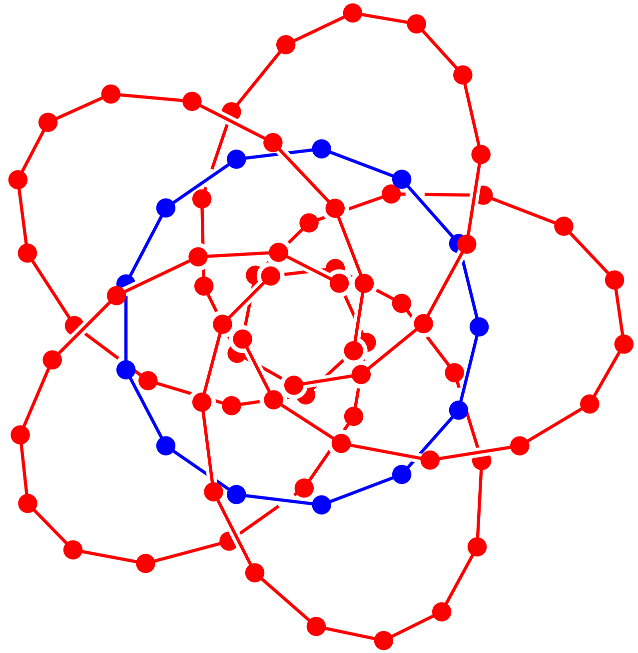
Icosahedron on $\mathbb{S}^2 \Rightarrow 12$ great circles with regular n-gons in $\mathbb{S}^3$

Point groups in four dimensions

Counterexample: $I \times C_n$ (group-theoretic product, but not geometric Cartesian product)

Icosahedron on $\mathbb{S}^2 \Rightarrow 12$ great circles with regular n-gons in $\mathbb{S}^3$





http://www.geom.uiuc.edu/~banchoff/script/b3d/hypertorus.html