The number of spanning trees in a planar graph

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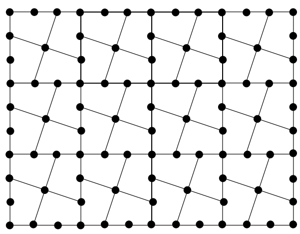
(joint work with Ares Ribó, Xuerong Yong)

Theorem 1. (1) *A planar graph with n vertices has at most* $5.33333333\ldots^n$ *spanning trees.*

(2) *A planar graph with n vertices and without a triangle has at most* $(\frac{4}{\sqrt[8]{e}})^n < 3.529988^n$ *spanning trees.*

(3) *A three-connected planar graph with n vertices and without a face cycle of length three or four has at most* $(\sqrt[3]{36}/e^{4/27})^n < 2.847263^n$ *spanning trees.*

Lower bounds that complement parts (1) and (2) come from the triangular and square grids [7], which have asymptotically $\approx 5.029545^n$ and $\approx 3.209912^n$ spanning trees, respectively, see also [6, (2.17–2.19)]. (The exact values are $\exp\big(\frac{3\sqrt{3}}{\pi}(1-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{11^2}+\frac{1}{13^2}-\cdots)\big)$ and $\exp\big(\frac{4}{\pi}(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\cdots)\big)$.) A large graph with pentagonal faces with the regular structure shown on the right is a candidate for the best construction in case (3). The asymptotic number of trees of this example can be calculated by the technique of Shrock and Wu [6], but we haven't done this.



Our motivation for studying this problem comes from the task of realizing 3-dimensional polytopes with (small) integral vertex coordinates. The combinatorial structure of a 3-polytope is specified by a three-connected planar graph. Such a graph always contains at least a triangular, a quadrilateral, or a pentagonal face; this is the reason why we did not continue after part (3) of Theorem 1.

To construct a 3-polytope with a given combinatorial structure, we follow the approach described in Richter-Gebert [5, Part IV]: we construct a planar equilibrium embedding for a specified self-stress and lift it to a polyhedral surface via the Maxwell-Cremona correspondence. The analysis of the determinant of the linear system of equations which is used to define the equilibrium embedding leads directly to the number of spanning trees of the graph, via the Matrix-Tree theorem.

With the improved bounds of Theorem 1 and some additional technique for graphs containing a quadrilateral face, we can improve the results of Richter-Gebert as follows:

Theorem 2. (1) *A 3-polytope P with n vertices can be realized with integral coordinates of absolute value less than* 2^{12n^2} *(or more precisely,* $n^{10n}\,2^{10n^2}$*).*

(2) *If P contains a quadrilateral face, the bound is reduced to* 156^n*.*

(3) *If the graph of P contains a triangle, the bound is reduced to* 29^n*.* □

In this abstract, we will only sketch the techniques for proving Theorem 1. Full details can be found in [4]. The proof of part (1) is rather simple: we add edges until we obtain a triangulated supergraph G; its dual graph G^* is 3-regular and has $2n-4$ vertices. Applying the upper bound for regular (not necessarily planar) graphs of McKay [3], and of Chung and Yau [1] yields our bound.

For parts (2) and (3) of Theorem 1, we introduce the *Outgoing Arc Approach.* We choose an arbitrary root vertex r. In the directed graph obtained by replacing every edge by two opposite directed arcs, we form a subset R of arcs by selecting one outgoing arc uniformly at random for each vertex different from the root.

If R does not contain cycles, it forms a spanning tree. Each tree is generated in exactly one way by this process. Multiplying the number of possibilities, which is the product of the vertex degrees $\prod_{v\in V-\{r\}} d_v$, by the "success probability" yields the following expression for the number T of spanning trees:

Lemma 1.

$$T = \prod_{v\in V-\{r\}} d_v \cdot \text{Prob}(R \text{ does not contain a cycle}) \qquad \square$$

From the product of vertex degrees $\prod d_v$ and the arithmetic-geometric mean inequality we already get an easy upper bound of 6^n for the number of spanning trees of planar graphs. We improve this by estimating the probability that some cycle appears. The probability that a particular cycle c appears can be easily calculated as the reciprocal of the product of the degrees. However, cycles do not appear independently. Cycles are independent if they have disjoint vertex sets, and hence we expect that "most" short cycles will be independent of each other. We use Suen's inequality for this case of controlled dependence. Suen's inequality uses the concept of a dependency graph. Let $\{X_i\}_{i\in\mathcal{I}}$ be a family of random variables. A *dependency graph* is a graph L with node set $\mathcal{I}$ such that if A and B are two disjoint subsets of $\mathcal{I}$ with no edge between A and B, then the families $\{X_i\}_{i\in A}$ and $\{X_i\}_{i\in B}$ are mutually independent. In particular, two variables X_i and X_j are independent unless there is an edge in L between i and j. If there exists such an edge, we write $i \sim j$. Suen's inequality is useful in cases in which there exists a sparse dependency graph. The expected value of a random variable X is denoted by $\mathbb{E}X$. The following theorem is a special case of Suen's inequality, see [2]:

Theorem 3. *Let I_i, $i \in \mathcal{I}$, be a finite family of Bernoulli random variables with success probability p_i, having a dependency graph L. Let $X = \sum_i I_i$ and $\lambda = \mathbb{E}X = \sum_i p_i$. Moreover, let $\Delta = \frac{1}{2}\sum_i \sum_{j:i\sim j} \mathbb{E}(I_i I_j)$ and $\zeta = \max_i \sum_{k\sim i} p_k$. Then*

$$\text{Prob}(X = 0) \le \exp(-\lambda + \Delta e^{2\zeta}).$$

In our case, the nodes of the dependency graph are all directed cycles in the graph that avoid r. We connect two cycles by an edge if they share some vertex. The independent choice of an outgoing arc for each vertex in R the ensures that this dependency graph is valid for our model.

Two directed cycles c and c' that share a vertex can never occur together in R, because every vertex has only one outgoing arc in R. Hence, $i \sim j$ implies that $\mathbb{E}(I_i I_j) = 0$, which means that $\Delta = 0$ in Theorem 3. Therefore, we have

$$\text{Prob}(R \text{ does not contain a cycle}) = \text{Prob}(X = 0) \le \exp(-\lambda),$$

where λ is the sum of probabilities for all directed cycles c that can appear in R:

$$(1)\quad \lambda = \sum_c (1/\prod_{v\in c} d_v) = \sum_{(i,j)\in\mathcal{C}_2} \frac{1}{d_i d_j} + \sum_{(i,j,k)\in\mathcal{C}_3} \frac{2}{d_i d_j d_k} + \sum_{(i,j,k,l)\in\mathcal{C}_4} \frac{2}{d_i d_j d_k d_l} + \cdots$$

Here $\mathcal{C}_b$ denotes the set of undirected cycles of length b that don't contain r. To prove an upper bound on $\prod d_v \cdot e^{-\lambda}$ we truncate the sum (1) after $\mathcal{C}_2$. We let the variable f_{ij}, with $i \le j$, stand for the number of edges connecting a vertex of degree i and a vertex of degree j. The logarithm of $\prod d_v \cdot e^{-\lambda}$ can then be written as a linear function in the variables f_{ij}:

$$Z = \sum_{v\in V} \ln d_v - \sum_{(i,j)\in E} \frac{1}{d_i d_j} = \sum_{i\le j} f_{ij} \left(\frac{\ln i}{i} + \frac{\ln j}{j} - \frac{1}{ij} \right)$$

We maximize Z under constraints that reflect the total number n of vertices and the total number of edges in a planar graph (at most $3n$):

$$(2)\qquad \sum_{i\le j} f_{ij} \left(\frac{1}{i} + \frac{1}{j} \right) = n, \quad \text{and} \quad \sum_{i\le j} f_{ij} \le 3n$$

The optimum $Z = \ln 6 - \frac{1}{12}$ with $e^Z \approx 5.5203$ is achieved when $f_{66} = 3n$ and all other $f_{ij} = 0$. However, this bound for part (1) is not as strong as the easy bound that comes from the dual graph. If we replace the edge bound $3n$ in (2) by $2n$ and $5n/3$, respectively, we obtain parts (2) and (3) of Theorem 1. The corresponding optimal solutions are $f_{44} = 2n$ (corresponding to the square grid), and $f_{33} = n/3$, $f_{34} = 4n/3$ (corresponding to the grid graph with pentagonal faces shown on the first page). Planarity enters this proof only via the bound on the number of edges.

As a next step, one can include in the sum (1) larger cycles $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_5$. If we consider only *face cycles* and introduce corresponding variables f_{ijk}, f_{ijkl}, f_{ijklm} for the number of faces with vertices of degree i, j, k, l, m, calculations indicate that this would reduce the bound in part (2) of Theorem 1 to 3.5026. (In this case, no cycles of length 3 appear.) However, this appears quite complicated to prove. Also, it appears that one cannot beat the current bound for part (1) with this technique, even if longer and longer cycles are included.

References

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