

Straightening Polygonal Arcs and Convexifying Polygonal Cycles

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Abstract

Consider a planar linkage, consisting of disjoint polygonal arcs and cycles of rigid bars joined at incident endpoints (polygonal chains), with the property that no cycle surrounds another arc or cycle. We prove that the linkage can be continuously moved so that the arcs become straight, the cycles become convex, and no bars cross while preserving the bar lengths. Furthermore, our motion is piecewise-differentiable, does not decrease the distance between any pair of vertices, and preserves any symmetry present in the initial configuration. In particular, this result settles the well-studied carpenter's rule conjecture.

1 Introduction

A planar *polygonal arc* or *open polygonal chain* is a sequence of finitely many line segments in the plane connected in a path without self-intersections. It has been an outstanding question as to whether it is possible to continuously move a polygonal arc in such a way that each edge remains a fixed length, there are no self-intersections during the motion, and at the end of the motion the arc lies on a straight line. This has come to be known as the carpenter's rule problem. A related question is whether it is possible to continuously move a polygonal simple closed curve in the plane, often called a *closed polygonal chain* or *polygon*, again without creating self-intersections or changing the lengths of the edges, so that it ends up a convex closed curve (see Figure 1). We solve both problems here by showing that in both cases there is such a motion.

Physically, we think of a polygonal arc as a *linkage* or *framework* with hinges at its vertices, and rigid bars at its edges. The hinges can be folded as desired, but the bars must maintain their length and cannot cross. Motions of such linkages have been studied in

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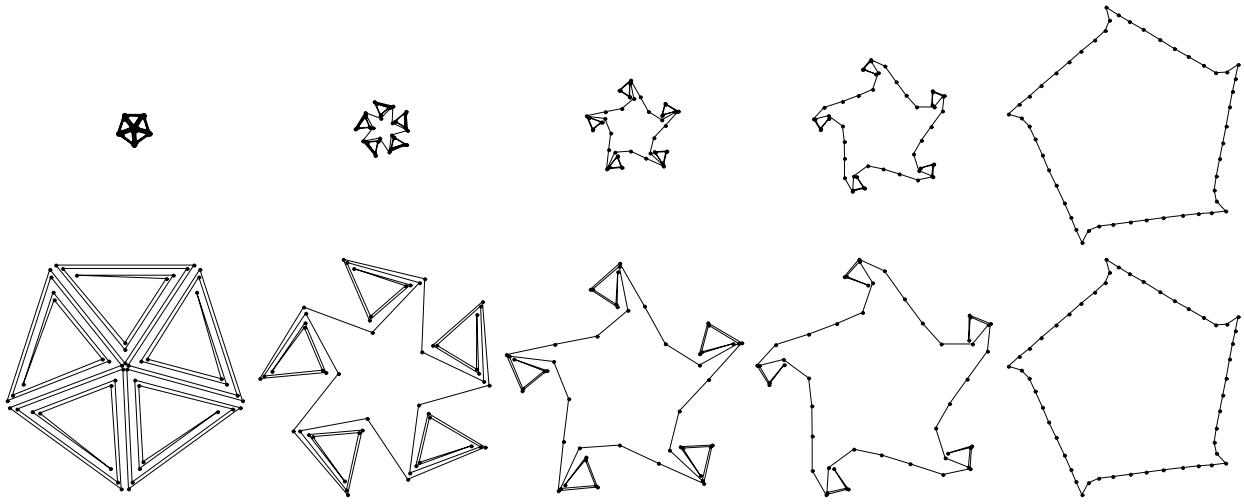


Figure 1: Two views of convexifying a polygon that comes from doubling each edge in a locked tree. The top snapshots are all scaled the same, and the bottom snapshots are scaled differently to improve visibility. More animations can be seen at the world-wide-web pages of the first author.²

2:01 discrete and computational geometry [BDD⁺01, Erd35, Grü95, LW95, Nag39, O’R98, Sal73,
 2:02 Tou99, Weg93, Weg96, Whi92b], in knot theory [CJ98, Mil94], and in molecular biology and
 2:03 polymer physics [FK97, MOS90, McM79, Mil94, SW88, SG72, Whi83]. Applications of this
 2:04 field include robotics, wire bending, hydraulic tube folding, and the study of macromolecule
 2:05 folding [O’R98, Tou99].

2:06 We say an arc is *straightened* by a motion if at the end of the motion it lies on a straight
 2:07 line. We say a polygonal simple closed curve (or *cycle*) is *convexified* by a motion if at the
 2:08 end of the motion it is a convex closed curve. All motions must be *proper* in the sense that
 2:09 no self-intersections are created, and each edge length is kept fixed. It is easy to see that
 2:10 if any cycle can be convexified by a motion, then any arc can be straightened by a motion:
 2:11 simply extend each arc to a cycle and convexify it. It is then easy to straighten the portion
 2:12 of the cycle that is the original arc.

2:13 It seems intuitively easy to straighten an entangled chain: just grab the ends and pull
 2:14 them apart. Similarly, a cycle might be opened by blowing air into it until it expands. But
 2:15 these methods have the difficulty that they might introduce singularities, where the arc or
 2:16 cycle intersects itself. Our approach is to use an *expansive* motion in which all distances
 2:17 between two vertices increase. We also show that the area of a polygon increases in such an
 2:18 expansive motion.

2:19 We consider the more general situation, which we call an *arc-and-cycle set* A , consisting
 2:20 of a finite number of polygonal arcs and polygonal cycles in the plane, with none of the arcs or
 2:21 cycles intersecting each other or having self-intersections. We say that A is in an *outer-convex*
 2:22 *configuration* if each component of A that is not contained in any cycle of A is either straight
 2:23 (when it is an arc) or convex (when it is a cycle). The best we can hope for, in general, is
 2:24 to bring an arc-and-cycle set to an outer-convex configuration, because components nested

2:25 ²Currently, <http://db.uwaterloo.ca/~eddemain/linkage/>

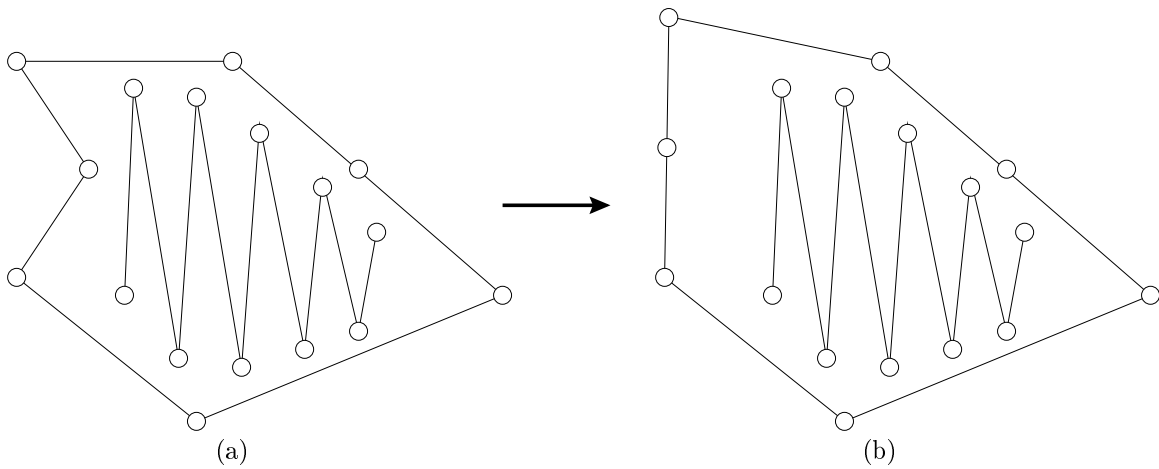


Figure 2: (a) The nested arc cannot be straightened because there is insufficient space in the containing cycle. (b) Once the containing cycle becomes convex, any expansive motion must move the arc and the cycle rigidly in unison.

3:01 within cycles cannot always be straightened or convexified; see Figure 2(a).

3:02 We say that a motion of an arc-and-cycle set A is *expansive* if for every pair of vertices
 3:03 of A the distance is monotonically nondecreasing over time, at all times either increasing
 3:04 or staying the same. In any expansive motion, once a cycle becomes convex, the cycle and
 3:05 any components it contains become a single rigid object; see Figure 2(b). (This fact is a
 3:06 consequence of Cauchy's Arm Lemma [Cau13, Cro97, SZ67].) Also, once two incident bars
 3:07 become collinear, they will remain so throughout any expansive motion, effectively acting as
 3:08 a single bar. We say that a motion is *strictly expansive* if the distance is constant between
 3:09 two vertices connected by a bar or straight chain of bars, and between two vertices on the
 3:10 boundary of or interior to a common convex cycle, but the distance between all other pairs
 3:11 of vertices monotonically strictly increases over time.

3:12 We say that the arc-and-cycle set A has *separated* if there is a line L in the plane such
 3:13 that L is disjoint from A and at least one component of A lies on each side of L . Our main
 3:14 result is the following:

3:15 **Theorem 1** *Every arc-and-cycle set has a piecewise-differentiable proper motion to an outer-*
 3:16 *convex configuration. Moreover, the motion is strictly expansive until the arc-and-cycle set*
 3:17 *becomes separated.*

3:18 We can also extend this result to insist that the motion be strictly expansive throughout
 3:19 the entire motion:

3:20 **Theorem 2** *Every arc-and-cycle set has a strictly expansive piecewise-differentiable proper*
 3:21 *motion to an outer-convex configuration.*

3:22 For Theorem 2, the definition of the motion is actually even simpler than the one we use
 3:23 for Theorem 1, but unfortunately the proof is a great deal more complex. Thus we focus on
 3:24 the proof of Theorem 1. Theorem 2 has a similar proof outline, but it relies on a Boundedness

4:01 Lemma (Lemma 10) whose proof is complicated. We include the proof of the Boundedness
 4:02 Lemma in Appendix A. Note that Theorems 1 and 2 only differ when there is more than one
 4:03 component in the arc-and-cycle set; the Boundedness Lemma is straightforward for a single
 4:04 arc or cycle.

4:05 In contrast to our results, in dimension three there are arcs that cannot be straightened
 4:06 and polygons that cannot be convexified [BDD⁺01, CJ98]. In four and higher dimensions,
 4:07 no arcs, cycles, or trees can be locked, i.e., all arcs, cycles, or trees can be straightened
 4:08 or convexified, respectively. This is true because there are enough degrees of freedom and
 4:09 one can easily avoid any impending self-intersection by “moving around” it. An explicit
 4:10 unlocking algorithm for arcs and cycles in four dimensions was given in [CO99].

4:11 In the plane, there are examples of trees embedded in the plane that are *locked* in the
 4:12 sense that they cannot be properly moved so that the vertices lie nearly on a line [BDD⁺98].
 4:13 In other words, there are two embeddings of the tree such that there is no proper motion from
 4:14 one configuration to the other. The important difference between trees and arc-and-cycle
 4:15 sets is that arc-and-cycle sets have maximum degree two. We have recently strengthened
 4:16 the example of [BDD⁺98] by constructing a locked tree with just one vertex of degree three
 4:17 and all other vertices of degree one or two [CDR02]. This shows that the restriction to
 4:18 arc-and-cycle sets in Theorem 1 is best possible.

	Arcs and Cycles	Trees
2-D	Not lockable [Theorem 1]	Lockable [BDD ⁺ 98]
3-D	Lockable [BDD ⁺ 01, CJ98]	Lockable [BDD ⁺ 01, CJ98]
4-D ⁺	Not lockable [CO99, CO01]	Not lockable [CO01]

Table 1: Summary of what types of linkages can be locked.

4:19 Whether every arc in the plane can be straightened, and whether every polygon in the
 4:20 plane can be convexified, have been outstanding open questions until now. The problems
 4:21 are natural, so they have arisen independently in a variety of fields, including topology,
 4:22 pattern recognition, and discrete geometry. We are probably not aware of all contexts in
 4:23 which the problem has appeared. To our knowledge, the first person to pose the problem
 4:24 of convexifying cycles was Stephen Schanuel. George Bergman learned of the problem from
 4:25 Schanuel during Bergman’s visit to the State University of New York at Buffalo in the
 4:26 early 1970’s, and suggested the simpler question of straightening arcs. As a consequence
 4:27 of this line of interest, the problems are included in Kirby’s *Problems in Low-Dimensional*
 4:28 *Topology* [Kir97, Problem 5.18].

4:29 During the period 1986–1989, Ulf Grenander and the members of the Pattern Theory
 4:30 Group at Brown University explored various problems involving the probabilistic structure
 4:31 when generators (e.g., points and line segments) were transformed by diffeomorphisms (e.g.,
 4:32 Euclidean transformations) subject to global constraints (e.g., avoiding intersections). For
 4:33 the purposes of Bayesian image understanding, they were interested in whether the process
 4:34 was *ergodic*, i.e., every configuration could be reached from any other. In particular, they
 4:35 proved this for polygonal cycles in which the roles of angles and lengths are reversed: *angles*
 4:36 are fixed but *lengths* may vary [GCK91, Appendix D, pp. 108–128]. Grenander posed the
 4:37 problems considered here in a seminar talk with the title “Can one understand shapes in

nature?” at Indiana University, on March 27, 1987, and probably also on earlier occasions (according to personal communication with Allan Edmonds).

In the discrete and computational community, the problems were independently posed by William Lenhart and Sue Whitesides in March 1991 and by Joseph Mitchell in December 1992 (according to personal communications with Sue Whitesides and Joseph Mitchell). Sue Whitesides first posed this problem in a talk in 1991 [LW91]. In this community the problems were first published in a technical report in 1993 [LW93] and in a journal in 1995 [LW95].

Solutions were already known for the special cases of monotone cycles [BDL⁺99] and star-shaped cycles [ELR⁺98], and for certain types of “externally visible” arcs [BDST99].

A fairly large group of people, mentioned in the acknowledgments, was involved in trying to construct and prove or disprove locked arcs and cycles, at various times over the past few years. Typically, someone in the group would distribute an example that s/he constructed or was given by a colleague. We would try various motions that did not work, and we would often try proving that the example was locked because it appeared so! For some examples, it took several months before we found an unlocking motion. The main difficulty was that “simple” motions that change a few vertex angles at once, while easiest to visualize, seemed to be insufficient for unlocking complex examples. Amazingly, it also seemed that nevertheless there was always a global unlocking motion, and furthermore it was felt that there was a driving principle permitting “blowing up” of the linkage. This notion was formalized by the third author with the idea that perhaps an arc could be straightened via an expansive motion.

The tools that are applied here for the first time come from the theory of mechanisms and rigid frameworks. Arcs and cycles can be regarded as frameworks. See [AR78, AR79, Con80, Con82, Con93, CW96, CW93, CW82, CW94, GSS93, RW81, Whi84a, Whi84b, Whi87, Whi88, Whi92a] for relevant information about this theory.

Our approach is to prove that for any configuration there is an infinitesimal motion that increases all distances. Because of the nature of the arc-and-cycle set, this implies that there is a motion that works at least for a small expansive perturbation. We then combine these local motions into one complete motion. These notions are described in the rest of this paper. Section 3 proves the existence of infinitesimal motions using the nonexistence of certain stresses, a notion dual to infinitesimal motions for the underlying framework. The analysis of these stresses uses a lifting theorem from the theory of rigidity that was known to James Clerk Maxwell and Luigi Cremona [CW82, CW93, Whi82] in the nineteenth century. Section 4 shows how to maneuver through the space of local motions to find a global motion with the desired properties.

A short version of this paper was presented at the 41st Annual Symposium on Foundations of Computer Science in November 2000 [CDR00].

2 Basics

A *linkage* or *bar framework* $G(\mathbf{p})$ is a finite graph $G = (V, E)$ without loops or multiple edges, together with a corresponding configuration $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ of n distinct points in the plane, where \mathbf{p}_i corresponds to vertex $i \in V$. (For convenience we assume $V = \{1, \dots, n\}$.) The edges of G constitute the set E and correspond to the bars in the framework, i.e., the links

6:01 of a linkage. Arc-and-cycle sets are a particular kind of bar framework in which the graph
 6:02 G is a disjoint union of paths and cycles.

6:03 A *flex* or *motion* of $G(\mathbf{p})$ is a set of continuous functions $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_n(t))$, defined
 6:04 for $0 \leq t \leq 1$, such that $\mathbf{p}(0) = \mathbf{p}$ and the Euclidean distance $\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ is constant
 6:05 for each $\{i, j\} \in E$. We are interested in finding a motion of the arc-and-cycle set with the
 6:06 additional property that it is strictly expansive.

6:07 2.1 Expansiveness

6:08 We begin with some basic properties of expansive motions. Namely, we will show that
 6:09 if a motion expands the distance between all pairs of vertices, it also expands the distance
 6:10 between all pairs of points on the arc-and-cycle framework. One consequence of this property
 6:11 is a key reason why we use expansive motions: they automatically avoid self-intersection.
 6:12 To prove the property, we need the following known basic geometric tool, which will also be
 6:13 useful later:

6:14 **Lemma 1** *In the plane, suppose that a point \mathbf{c} is contained in the closed triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$,*
 6:15 *and \mathbf{p}_1 and \mathbf{p}_2 are farther from another point \mathbf{q}_3 than from \mathbf{p}_3 , i.e.,*

$$6:16 \quad \|\mathbf{q}_3 - \mathbf{p}_2\| \geq \|\mathbf{p}_3 - \mathbf{p}_2\| \text{ and } \|\mathbf{q}_3 - \mathbf{p}_1\| \geq \|\mathbf{p}_3 - \mathbf{p}_1\|. \quad (1)$$

6:17 *Then \mathbf{c} is also farther from \mathbf{q}_3 than from \mathbf{p}_3 , i.e., $\|\mathbf{q}_3 - \mathbf{c}\| \geq \|\mathbf{p}_3 - \mathbf{c}\|$, with equality only if*
 6:18 *both inequalities of (1) are equalities.*

6:19 **Proof:** Refer to Figure 3. The circular disk C_0 centered at \mathbf{c} with radius $\|\mathbf{p}_3 - \mathbf{c}\|$ is
 6:20 contained in the union of the circular disks C_1 with center \mathbf{p}_1 and radius $\|\mathbf{p}_3 - \mathbf{p}_1\|$, and C_2
 6:21 with center \mathbf{p}_2 and radius $\|\mathbf{p}_3 - \mathbf{p}_2\|$. This implies the result, because \mathbf{q}_3 must be outside C_1
 6:22 and C_2 . See also [Con82] for a proof in terms of tensegrities. \square

6:23 **Corollary 1** *Any expansive motion of an arc-and-cycle set only increases the distance be-*
 6:24 *tween two points on the arc-and-cycle set (each either a vertex or on a bar). In particular,*
 6:25 *there can be no self-intersections.*

6:26 **Proof:** Refer to Figure 4. First, the result is obvious if the two points are both vertices of
 6:27 the arc-and-cycle set, by definition of expansiveness. Second, consider the distance between
 6:28 a vertex \mathbf{p}_3 of the arc-and-cycle set and a point \mathbf{c} on a bar $\mathbf{p}_1\mathbf{p}_2$ of the arc-and-cycle set.
 6:29 Expansiveness implies that \mathbf{p}_3 only gets farther from \mathbf{p}_1 and \mathbf{p}_2 , so by Lemma 1, \mathbf{p}_3 only
 6:30 gets farther from \mathbf{c} . Third, consider the distance between \mathbf{c} (again on the bar $\mathbf{p}_1\mathbf{p}_2$) and
 6:31 another point \mathbf{d} on a bar $\mathbf{p}_4\mathbf{p}_5$ of the arc-and-cycle set. Substituting \mathbf{p}_4 and \mathbf{p}_5 as options
 6:32 for \mathbf{p}_3 in the previous argument, we know that \mathbf{p}_4 and \mathbf{p}_5 only get farther from \mathbf{c} . Applying
 6:33 Lemma 1 with \mathbf{c} playing the role of \mathbf{p}_3 , we obtain that \mathbf{c} can only get farther from \mathbf{d} . \square

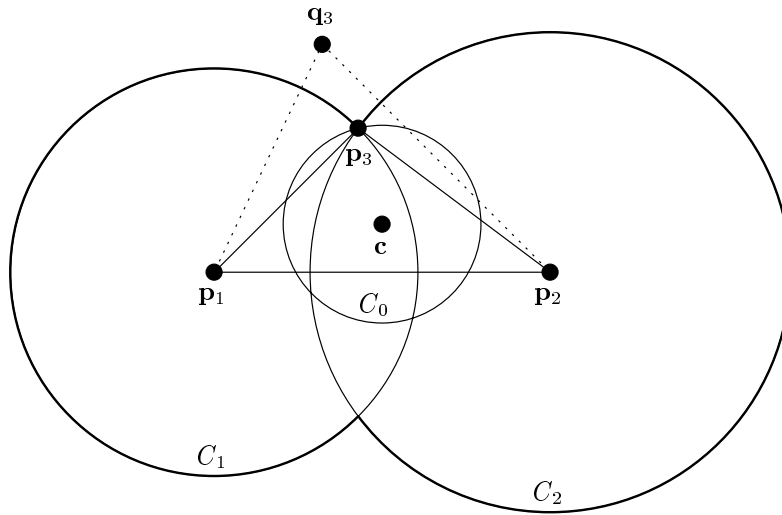


Figure 3: Illustration of Lemma 1.

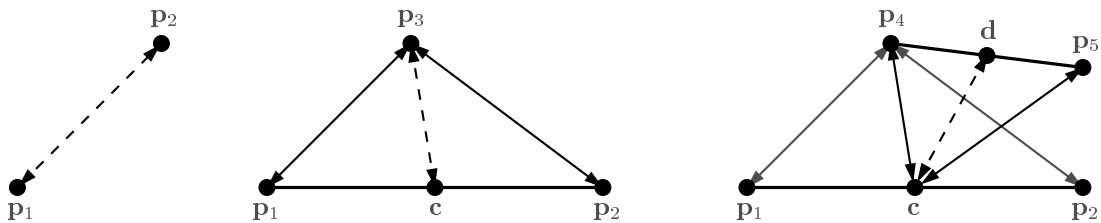


Figure 4: Illustration of the three cases of Corollary 1: (left) two vertices $\mathbf{p}_1, \mathbf{p}_2$; (middle) one vertex \mathbf{p}_3 and one point \mathbf{c} on a bar $\mathbf{p}_1\mathbf{p}_2$; (right) one point \mathbf{c} on a bar $\mathbf{p}_1\mathbf{p}_2$ and another point \mathbf{d} on a bar $\mathbf{p}_4\mathbf{p}_5$. Bold edges denote bars, and arrows denote expansion; dashed arrows are derived from solid arrows.

7:01 **2.2 The Framework $G_A(\mathbf{p})$**

7:02 Given an arc-and-cycle set A that we would like to move to an outer-convex configuration, we
 7:03 make four modifications to A . The first three modifications simplify the problem by removing
 7:04 a few special cases that are easy to deal with; see Figure 5. The fourth modification will
 7:05 bring the problem of finding a strictly expansive motion into the area of tensegrity theory.
 7:06 In the end we will have defined a new framework, $G_A(\mathbf{p})$, which we will use throughout the
 7:07 rest of the proof.

7:08 **Modification 1: Remove straight vertices.** First we show that our arc-and-cycle set
 7:09 can be assumed to have no straight vertices, i.e., vertices with angle π . Furthermore, if during
 7:10 an expansive motion of the arc-and-cycle set we find that a vertex becomes straight, we can
 7:11 proceed by induction. For once the arc-and-cycle set has a straight subarc of more than
 7:12 one bar, we can coalesce this subarc into a single bar, thereby preserving the straightness
 7:13 of the subarc throughout the motion once it becomes straight. This reduces the number of
 7:14 bars and the number of vertices in the framework. By induction, this reduced framework
 7:15 has a motion according to Theorem 1, and such a motion extends directly to the original

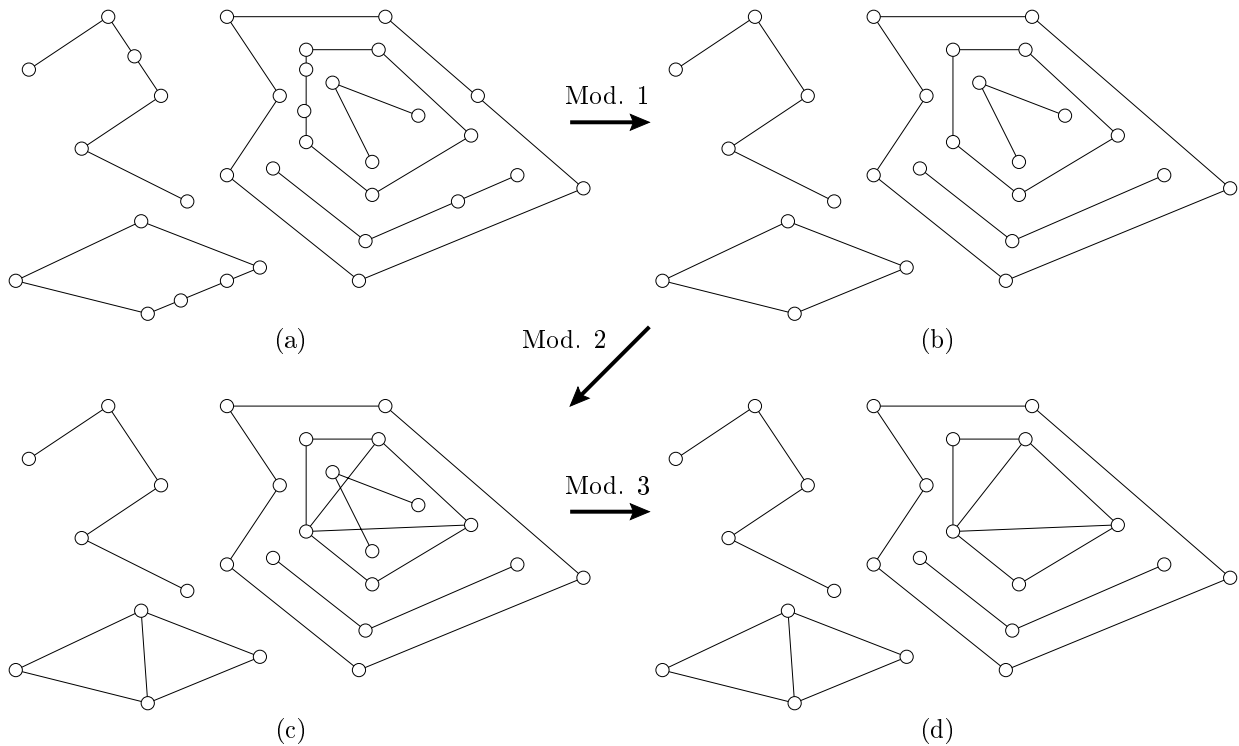


Figure 5: (a) Original arc-and-cycle framework. (b) With straight vertices removed. (c) With convex cycles rigidified. (d) With components nested within convex cycles removed.

8:01 framework. The resulting motion is also strictly expansive by Corollary 1.

8:02 **Modification 2: Rigidify convex polygons.** Once a cycle becomes convex, we no longer
 8:03 have to expand it, indeed it is impossible to expand, so we hold it rigid from that point on.
 8:04 Of course, we allow a convex cycle to translate or rotate in the plane, but its vertex angles
 8:05 are not allowed to change. This can be directly modeled in the bar framework by introducing
 8:06 bars in addition to the arc-and-set cycle. Specifically, we add the edges of a triangulation of
 8:07 a cycle once that cycle becomes convex. We deal with the contents of the cycle in the next
 8:08 modification.

8:09 **Modification 3: Remove components nested within convex cycles.** The previous
 8:10 modification did not address the fact that components can be nested within cycles. Once
 8:11 a cycle becomes convex, not only can we rigidify it, but we can also rigidify any nested
 8:12 components, and treat them as moving in synchrony with the convex cycle. We do this by
 8:13 removing from the framework any components nested within a convex cycle. Assuming there
 8:14 were some nested components to deal with, this results in a framework with fewer vertices and
 8:15 fewer bars. By induction, this reduced framework has a motion according to Theorem 1. This
 8:16 motion can be extended to apply to the original framework by defining nested components
 8:17 to follow the rigid motion of the containing convex cycle (rigid by Modification 2). By the
 8:18 following consequence of Lemma 1, the resulting motion is also strictly expansive.

9:01 **Lemma 2** *Extending a motion to apply to components nested within convex cycles preserves*
 9:02 *strict expansiveness.*

9:03 **Proof:** Consider some vertex \mathbf{c} on a component inside some convex cycle, and a vertex \mathbf{p}_3
 9:04 outside the cycle. We first consider the case that \mathbf{p}_3 does not lie inside another convex cycle.
 9:05 Extend the ray from $\mathbf{p}_3\mathbf{c}$ beyond \mathbf{c} , and let $\mathbf{p}_1\mathbf{p}_2$ be the edge through which this ray exits
 9:06 the cycle. Thus, \mathbf{c} is in the triangle $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, so Lemma 1 applies, and the distance $\mathbf{p}_3\mathbf{c}$
 9:07 increases.

9:08 For two points \mathbf{c}_1 and \mathbf{c}_2 in two different cycles C_1 and C_2 , we extend the ray $\mathbf{c}_1\mathbf{c}_2$ to
 9:09 identify the edge $\mathbf{p}_1\mathbf{p}_2$ on C_2 where the ray leaves C_2 . From the first part of the proof we
 9:10 conclude that $\mathbf{c}_1\mathbf{p}_1$ and $\mathbf{c}_1\mathbf{p}_2$ increase, and by Lemma 1, the distance $\mathbf{c}_1\mathbf{c}_2$ increases. \square

9:11 **Modification 4: Add struts.** In order to model the expansive property we need, we
 9:12 apply the theory of tensegrity frameworks, in which frameworks can consist of both bars
 9:13 and “struts.” In contrast to a bar, which must stay the same length throughout a motion,
 9:14 a *strut* is permitted to increase in length, or stay the same length, but cannot shorten. The
 9:15 last modification adds a strut between nearly every pair of vertices in the framework. The
 9:16 exceptions are those vertices already connected by a bar, and vertices on a common convex
 9:17 cycle, because in both cases we cannot hope to strictly increase the distance.

9:18 **Final framework:** $G_A(\mathbf{p})$. The above modifications define a tensegrity (bar-and-strut)
 9:19 framework $G_A(\mathbf{p})$ in terms of the arc-and-cycle set A . Specifically, assume that A has no
 9:20 straight vertices (Modification 1) and no components nested within convex cycles (Modi-
 9:21 fication 3). We call such an arc-and-cycle set *reduced*. We define the set of bars, B , to
 9:22 consist of the bars from the arc-and-cycle set together with the bars forming the rigidifying
 9:23 triangulation of each convex cycle (Modification 2). The set S of struts consists of all vertex
 9:24 pairs that are not connected by a bar in B and which do not belong to a common convex
 9:25 cycle (Modification 4). See Figure 6 for an example of A and the resulting bar-and-strut
 9:26 framework $G_A(\mathbf{p})$. (The rightmost framework $G'_A(\mathbf{p}')$ will be defined later.)

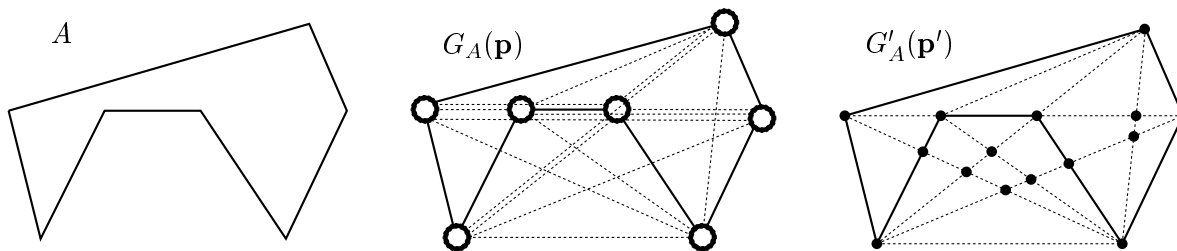


Figure 6: Construction of the frameworks $G_A(\mathbf{p})$ and $G'_A(\mathbf{p}')$. Solid lines denote bars, and dashed lines denote struts.

9:27 Our goal in the proof of Theorem 1 is to find a motion such that all bars maintain their
 9:28 length, while all struts strictly increase in length, in other words, a motion of $G_A(\mathbf{p})$ that is
 9:29 *strict* on all struts.

10:01 Thus, we want to find a motion $\mathbf{p}(t)$ for $0 \leq t \leq 1$ such that $\mathbf{p}(0) = \mathbf{p}$ and

$$10:02 \quad \begin{aligned} \frac{d}{dt} \|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| &= 0 && \text{for } \{i, j\} \in B, \\ \frac{d}{dt} \|\mathbf{p}_j(t) - \mathbf{p}_i(t)\| &> 0 && \text{for } \{i, j\} \in S. \end{aligned}$$

10:03 Differentiating the *squared* distances $\|\mathbf{p}_j(t) - \mathbf{p}_i(t)\|^2 = (\mathbf{p}_j(t) - \mathbf{p}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t))$ and
 10:04 denoting the velocity vectors by $\mathbf{v}_i(t) := \frac{d}{dt}\mathbf{p}_i(t)$, we obtain the following equivalent condi-
 10:05 tions.

$$10:06 \quad \begin{aligned} (\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t)) &= 0 && \text{for } \{i, j\} \in B, \\ (\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t)) &> 0 && \text{for } \{i, j\} \in S. \end{aligned}$$

10:07 Intuitively, the first-order change in the distance between vertex i and j is modeled by
 10:08 projecting the velocity vectors onto the line segment between the two vertices; see Figure 7.

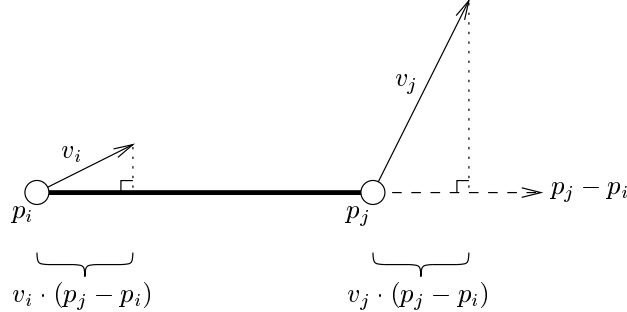


Figure 7: The dot product $(\mathbf{v}_j(t) - \mathbf{v}_i(t)) \cdot (\mathbf{p}_j(t) - \mathbf{p}_i(t))$ is zero if the distance between \mathbf{p}_i and \mathbf{p}_j stays the same to the first order, positive if the distance increases to the first order, and negative if the distance decreases to the first order.

10:09 2.3 Infinitesimal Motions

10:10 A *strict infinitesimal motion* or *strict infinitesimal flex* $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ specifies the first
 10:11 derivative of a strictly expansive motion at time 0. In other words, it assigns a velocity
 10:12 vector \mathbf{v}_i to each vertex i so that it preserves the length of the bars to the first order, and
 10:13 strictly increases the length of struts to the first order. More precisely, a strict infinitesimal
 10:14 motion must satisfy

$$10:15 \quad \begin{aligned} (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) &= 0 && \text{for } \{i, j\} \in B, \\ (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) &> 0 && \text{for } \{i, j\} \in S, \end{aligned} \tag{2}$$

10:16 where \mathbf{p}_i denotes the initial position of vertex i .

10:17 In the next section, we prove that such a strict infinitesimal motion always exists. In
 10:18 Section 4 we show how this leads to motions for small amounts of time. These motions are
 10:19 then shown to continue globally until the configuration reaches an outer-convex configuration.

10:20 3 Local Motion

10:21 Recall that an arc-and-cycle set is called *reduced* if adjacent collinear bars have been coa-
 10:22 lesced, and components nested within convex cycles have been removed. In this section, we

11:01 prove the following:

11:02 **Theorem 3** *For any reduced arc-and-cycle set A there is an infinitesimal motion \mathbf{v} of the*
11:03 *corresponding bar-and-strut framework $G_A(\mathbf{p})$ satisfying (2).*

11:04 The proof will go through a sequence of transformations from motions to stresses, and
11:05 from there to polyhedral terrains, to which geometric reasoning is finally applied. A second,
11:06 independent, but no less indirect proof of Theorem 3 follows from the results about the
11:07 structure of the *expansion cone* in [RSS02, Theorem 4.3].

11:08 3.1 Equilibrium Stresses

11:09 The equations and inequalities in (2) form a linear feasibility problem that is common
11:10 for tensegrity frameworks. In order to solve this problem, it is helpful to apply linear-
11:11 programming duality and consider the equivalent dual problem. We discuss the duality first
11:12 in terms of equilibrium stresses in tensegrity frameworks, and later reconnect it to linear-
11:13 programming duality.

11:14 A *stress* in a framework $G(\mathbf{p})$ is an assignment of a scalar $\omega_{ij} = \omega_{ji}$ to each edge $\{i, j\}$
11:15 of G (a bar or strut). A negative scalar means that the edge is pushing on its two endpoints
11:16 by an equal amount, a positive value means that the edge is pulling on its endpoints by an
11:17 equal amount, and zero means that the edge induces no force. The whole stress is denoted
11:18 by $\omega = (\dots, \omega_{ij}, \dots)$. We say that the stress ω is an *equilibrium stress* if each vertex i of G
11:19 is in equilibrium, i.e., stationary subject to the the forces from the incident edges:

$$11:20 \quad \sum_{j: \{i,j\} \in BUS} \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0 \quad (3)$$

11:21 We say that the stress ω is *proper* if furthermore, for all struts $\{i, j\}$, $\omega_{ij} \leq 0$. That is, struts
11:22 can carry only zero or negative stress. There is no sign condition for bars: they can carry
11:23 zero, positive, or negative stress. Thus, only bars can carry positive stress. (Terminology
11:24 and sign conditions have not always been uniform in the literature. An equilibrium stress
11:25 is also called a *self-stress* or simply a *stress*. All stresses that we deal with are equilibrium
11:26 stresses.)

11:27 A trivial example of an equilibrium stress is the *zero stress* in which every edge is assigned
11:28 a scalar of zero. All other stresses are called *nonzero*. To prove Theorem 3, we use the
11:29 following duality principle connecting nonzero equilibrium stresses and infinitesimal motions:

11:30 **Lemma 3** *If the only proper equilibrium stress in a bar-and-strut framework is the zero*
11:31 *stress, then the framework has an infinitesimal motion satisfying (2).*

11:32 This equivalence is a standard technique in the theory of rigidity. See [CW96, Theorem
11:33 2.3.2] for a similar statement. For completeness, we give a brief proof here based on linear
11:34 programming duality:

11:35 **Proof:** To make it easier to take the dual of the linear feasibility problem defined by (2),
11:36 we write a linear program in standard form. First we add an otherwise pointless objective

12:01 function of $0 = 0 \cdot \mathbf{v}$. Then we rescale the velocities \mathbf{v} in (2) to obtain the following equivalent
 12:02 linear program:

$$\begin{aligned}
 & \text{minimize} && 0 \cdot \mathbf{v} \\
 & \text{subject to} && (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } \{i, j\} \in B, \\
 & && (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) \geq 1 \text{ for } \{i, j\} \in S,
 \end{aligned} \tag{4}$$

12:04 We wish to show that the framework has an infinitesimal motion, which is equivalent to this
 12:05 linear program having a feasible solution, that is, an optimal solution of value 0. By linear-
 12:06 programming duality (the Farkas lemma), it suffices to show that the dual linear program

$$\begin{aligned}
 & \text{maximize} && \sum_{\{i,j\} \in S} \bar{\omega}_{ij} \\
 & \text{subject to} && \sum_{j: \{i,j\} \in B \cup S} \bar{\omega}_{ij} (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } i \in V, \\
 & && \bar{\omega}_{ij} = \bar{\omega}_{ji} && \text{for } \{i, j\} \in B \cup S, \\
 & && \bar{\omega}_{ij} \geq 0 && \text{for } \{i, j\} \in S,
 \end{aligned} \tag{5}$$

12:07 has an optimal solution of value 0. If we express the constraints of this linear program in
 12:08 terms of negated dual variables $\omega_{ij} = -\bar{\omega}_{ij}$, they specify precisely a proper equilibrium stress.
 12:09 Thus, it suffices to show that every proper equilibrium stress has $\omega_{ij} = 0$ for all $\{i, j\} \in S$.
 12:10 In particular, it suffices to show that every proper equilibrium stress is identically zero. \square

12:11 The important consequence of this lemma is that, in order to prove the desired Theorem 3,
 12:12 it suffices to prove the following:

12:13 **Theorem 4** *The framework $G_A(\mathbf{p})$ corresponding to a reduced arc-and-cycle set A has only*
 12:14 *the zero proper equilibrium stress.*

12:15 3.2 Planarization

12:16 To prove that only the zero equilibrium stress exists (i.e., to prove Theorem 4), we use
 12:17 another tool in rigidity called the Maxwell-Cremona theorem. Before we can apply this tool,
 12:18 we need to transform the framework $G_A(\mathbf{p})$ into a planar framework $G'_A(\mathbf{p}')$. (Refer to the
 12:19 framework on the right of Figure 6.) We introduce new vertices at all intersection points
 12:20 between edges of $G_A(\mathbf{p})$. We subdivide every bar and strut at every vertex through which
 12:21 it goes, be it an existing vertex of $G_A(\mathbf{p})$ or a newly added intersection vertex. Overlapping
 12:22 collinear edges will result in multiple edges; such multiple are merged into one edge. We
 12:23 define the resulting framework $G'_A(\mathbf{p}')$ to have bars precisely covering the bars of $G_A(\mathbf{p})$. All
 12:24 the other edges of $G'_A(\mathbf{p}')$ are struts. $G'_A(\mathbf{p}')$ is *planar* in the sense that two edges (bars or
 12:25 struts) intersect only at a common endpoint.

12:26 A natural concern is that the added vertices in this modification introduce additional
 12:27 freedom in finding infinitesimal motions, so they may not transfer directly to infinitesimal
 12:28 motions in the original framework. Nonetheless, the planar framework $G'_A(\mathbf{p}')$ is effectively
 12:29 equivalent to the original framework $G_A(\mathbf{p})$ in the sense of equilibrium stresses. Indeed, the

13:01 following stronger statement holds. We call a stress *outer-zero* if the only edges that carry a
 13:02 nonzero stress are edges of convex cycles and edges interior to convex cycles. Otherwise, an
 13:03 edge exterior to all convex cycles carries a nonzero stress, and we call the stress *outer-nonzero*.

13:04 **Lemma 4** *If $G_A(\mathbf{p})$ has a nonzero proper equilibrium stress ω , then $G'_A(\mathbf{p}')$ has an outer-*
 13:05 *nonzero proper equilibrium stress ω' .*

13:06 **Proof:** During the modifications to $G_A(\mathbf{p})$ that made $G'_A(\mathbf{p}')$, we modify ω to make ω' as
 13:07 follows. When we subdivide an edge $\{i, j\}$ with stress ω_{ij} , each edge $\{k, l\}$ of the subdivision
 13:08 gets the stress $\omega_{ij}\|p_i - p_j\|/\|p_k - p_l\|$. (The ratio of lengths is necessary because ω_{ij} is a
 13:09 weight, and the actual force comes from scaling by the length of the edge $\{i, j\}$; see (3).)
 13:10 When merging several edges, we add the corresponding stresses. The resulting stress is
 13:11 in equilibrium because edges meet in opposing pairs at the added vertices, and because
 13:12 summation preserves force. The stress is also proper because a strut in $G'_A(\mathbf{p}')$ is made up
 13:13 only of struts from $G_A(\mathbf{p})$, and the sum of nonnegative numbers is nonnegative. It only
 13:14 remains to check that positive and negative stresses do not completely cancel during the
 13:15 merging process, and that the stress is furthermore outer-nonzero.

13:16 First we prove that some strut $\{i, j\}$ of $G_A(\mathbf{p})$ carries a negative stress. In other words,
 13:17 $G_A(\mathbf{p})$ cannot be stressed only on its bars; in particular, a framework consisting exclusively
 13:18 of arcs, cycles, and triangulated convex cycles cannot carry a nonzero stress. This follows
 13:19 because, in any such bar framework, there is a vertex \mathbf{v} with degree at most two; in particular,
 13:20 every triangulated convex cycle has a degree-two vertex (an ear). Because the framework is
 13:21 reduced, the two bars incident to \mathbf{v} are not parallel, so these two bars cannot carry stress
 13:22 while satisfying equilibrium at \mathbf{v} . Removing them and proceeding inductively with the rest
 13:23 of the framework, we conclude that the stress is zero on the whole bar framework. Hence, the
 13:24 bars alone cannot carry a nonzero stress, so some strut $\{i, j\} \in G_A(\mathbf{p})$ must have a nonzero
 13:25 stress.

13:26 The conditions of Theorem 3 enforce that no angles at vertices of the arc-and-cycle set are
 13:27 π or 0: an angle of π would create a straight subarc of two bars (contradicting the assumption
 13:28 that the framework is reduced), and an angle of 0 would violate simplicity. Thus, no strut
 13:29 of $G_A(\mathbf{p})$ is completely covered by bars. Therefore, for the strut $\{i, j\}$ of $G_A(\mathbf{p})$ that carries
 13:30 a negative stress, some portion of it in $G'_A(\mathbf{p}')$ will also have a negative stress, because a
 13:31 negative stress can only be canceled by a stress on a bar. In particular, ω' must be nonzero.

13:32 Furthermore, if the strut $\{i, j\}$ is exterior to all convex cycles in A , we have that ω'
 13:33 is outer-nonzero. Now suppose that $\{i, j\}$ is partially interior to convex cycles in A (by
 13:34 construction, the strut cannot be entirely within convex cycles of A). Then there is a
 13:35 portion of $\{i, j\}$ with the property that it is incident to a convex cycle and exterior to all
 13:36 convex cycles in A . This portion must be uncovered by bars, because no bar in A has this
 13:37 property, and the only additional bars in $G_A(\mathbf{p})$ are interior to convex cycles. Hence, the
 13:38 corresponding strut in $G'_A(\mathbf{p}')$ carries a negative stress, so ω' is outer-nonzero in all cases. \square

13:39 Thus, to prove that the original framework $G_A(\mathbf{p})$ has only the zero proper equilibrium
 13:40 stress, it suffices to prove that the planar framework $G'_A(\mathbf{p}')$ has only outer-zero proper
 13:41 equilibrium stresses.

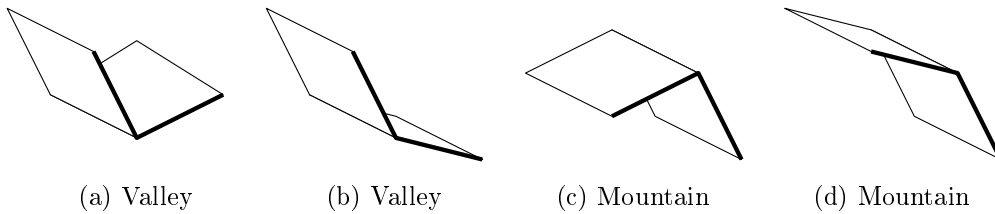


Figure 8: Valleys and mountains in a polyhedral terrain. The thick edges indicate the intersection with a vertical plane.

3.3 Maxwell-Cremona Theorem

To prove that only outer-zero equilibrium stresses exist, we employ the Maxwell-Cremona correspondence between equilibrium stresses in planar frameworks and three-dimensional polyhedral graphs that project onto these frameworks. When the vertices and edges of a planar framework are removed from the plane the resulting connected regions are called the *faces* of the framework. A *polyhedral graph* or *polyhedral terrain* Γ comes from lifting a planar framework into three dimensions—that is, assigning a z coordinate (positive or negative) to each vertex in the framework—such that the vertices of each face of the framework (including the exterior face) remain coplanar. Thus, each face of the framework lifts to a planar polygon in 3-space. The polyhedral surface Γ is then the graph of a piecewise-linear continuous function of two variables that is linear on the faces determined by $G'_A(\mathbf{p}')$.

Consider an edge $\{i, j\}$ in a planar framework, separating faces F and F' . We distinguish whether this edge lifts in Γ to a “valley,” “mountain,” or “flat” edge according to its dihedral angles; see Figure 8. More formally, let $z(\mathbf{p}) = \mathbf{a} \cdot \mathbf{p} + b$ for $\mathbf{p} \in F$ and $z(\mathbf{p}) = \mathbf{a}' \cdot \mathbf{p} + b'$ for $\mathbf{p} \in F'$ be the two linear functions specifying the graph Γ on F and F' . Thus, \mathbf{a} and \mathbf{a}' are vectors in (the dual space of) \mathbb{R}^2 , and b and b' are real numbers. A straightforward calculation reveals that the vector $\mathbf{a} - \mathbf{a}'$ in \mathbb{R}^2 must be perpendicular to the edge $\{i, j\}$:

$$\mathbf{a} - \mathbf{a}' = \omega_{ij} \mathbf{e}_{ij}^\perp \tag{6}$$

where \mathbf{e}_{ij}^\perp is a vector in \mathbb{R}^2 of the same length as the vector $\mathbf{p}_j - \mathbf{p}_i$, perpendicular to it, and pointing from F towards F' . We call the edge $\{i, j\}$ a *valley* if $\omega_{ij} < 0$, a *mountain* if $\omega_{ij} > 0$, and *flat* if $\omega_{ij} = 0$.

These definitions are illustrated in Figure 8. In particular, if two sides of an edge both “go up” in z , as in Figure 8(a), then the edge is a valley; however, the converse does not hold (so a valley might not carry water), as shown in Figure 8(b). Similarly, if the two sides of an edge both “go down” in z (Figure 8(c)), then the edge must be a mountain, but not all mountains have this property (Figure 8(d)).

Theorem 5 (Maxwell-Cremona Theorem) (i) *For every polyhedral graph Γ that projects to a planar bar framework $G(\mathbf{p})$, the stress ω defined by (6) forms an equilibrium stress on $G(\mathbf{p})$.*

(ii) *For every proper equilibrium stress ω in a planar framework $G(\mathbf{p})$, $G(\mathbf{p})$ can be lifted to a polyhedral graph Γ such that (6) holds for all edges. In particular, edges with positive stress lift to valleys, edges with negative stress lift to mountains, and edges with no stress lift to flat edges. Furthermore, Γ is unique up to addition of affine-linear functions.*

15:01 A proof of this result can be found in [Whi82, Glu74, HK92, CW94].

15:02 3.4 Main Argument

15:03 The zero equilibrium stress corresponds to a *trivial* polyhedral graph in which all faces are
15:04 coplanar (i.e., defined by a single linear function). More generally, an outer-zero equilibrium
15:05 stress corresponds to an *outer-flat* polyhedral graph that is flat on every edge exterior to
15:06 all convex cycles. Therefore, to prove that all equilibrium stresses of the planar framework
15:07 are outer-zero, and hence prove Theorem 4, it suffices to show that all polyhedral graphs
15:08 projecting to the planar framework are outer-flat.

15:09 More precisely, consider any polyhedral graph Γ that projects to the planar framework
15:10 $G'_A(\mathbf{p}')$ with the property that all struts are lifted to valleys or flat edges (because struts can
15:11 carry only negative or zero stress), and bars are lifted to valleys, mountains, or flat edges.
15:12 We need to show that nonflat edges can only appear within or on the boundary of convex
15:13 cycles. Because we may add an arbitrary affine-linear function, we may conveniently assume
15:14 that the exterior face of Γ is on the xy -plane. Thus the problem is to show that Γ does not
15:15 lift off the xy -plane any vertex of $G'_A(\mathbf{p}')$ except possibly vertices interior to convex cycles
15:16 of A .

15:17 One simple fact that we will need is the following:

15:18 **Lemma 5** *Any mountain in the polyhedral graph Γ projects to a bar in the planar framework*
15:19 *$G'_A(\mathbf{p}')$.*

15:20 **Proof:** A strut can only carry negative or zero stress, so by Theorem 5 it can only lift to a
15:21 valley or a flat edge. \square

15:22 We now come to the heart of our proof, the proof of Theorem 6. It is here that we finally
15:23 show that the stress must be outer-zero, by looking at the maximum of any Maxwell-Cremona
15:24 lift. The following statement immediately implies Theorem 4 and hence Theorem 3:

15:25 **Theorem 6** *Let M denote the region in the xy -plane where the z value attains its maximum*
15:26 *in the polyhedral graph Γ . Then M contains every face of the planar framework $G'_A(\mathbf{p}')$ that*
15:27 *is exterior to all convex cycles.*

15:28 M is a nonempty union of faces, edges, and vertices of the planar framework $G'_A(\mathbf{p}')$.
15:29 Consider the boundary ∂M , which may be empty if M fills the whole plane. Because points
15:30 in M lift to maximum height, all edges of ∂M must lift to mountains. Thus by Lemma 5,
15:31 all edges of ∂M must be bars in the framework. Hence, ∂M consists of disjoint vertices,
15:32 paths of edges, and complete cycles of the arc-and-cycle set, together with a subset of the
15:33 triangulations of the convex components. Figure 9 shows typical cases of all possibilities.
15:34 We will show that the only case in Figure 9 that can actually occur is (ℓ) , in which ∂M
15:35 includes a convex cycle and M includes the local exterior of that cycle (and possibly some
15:36 of its interior).

15:37 Our main technique for arriving at a contradiction in all cases except (ℓ) is that of *slicing*
15:38 the polyhedral graph. Consider a plane Π that is parallel to the xy -plane and just below
15:39 the maximum z coordinate of Γ . (By “just below” we mean that Π is above all vertices of Γ

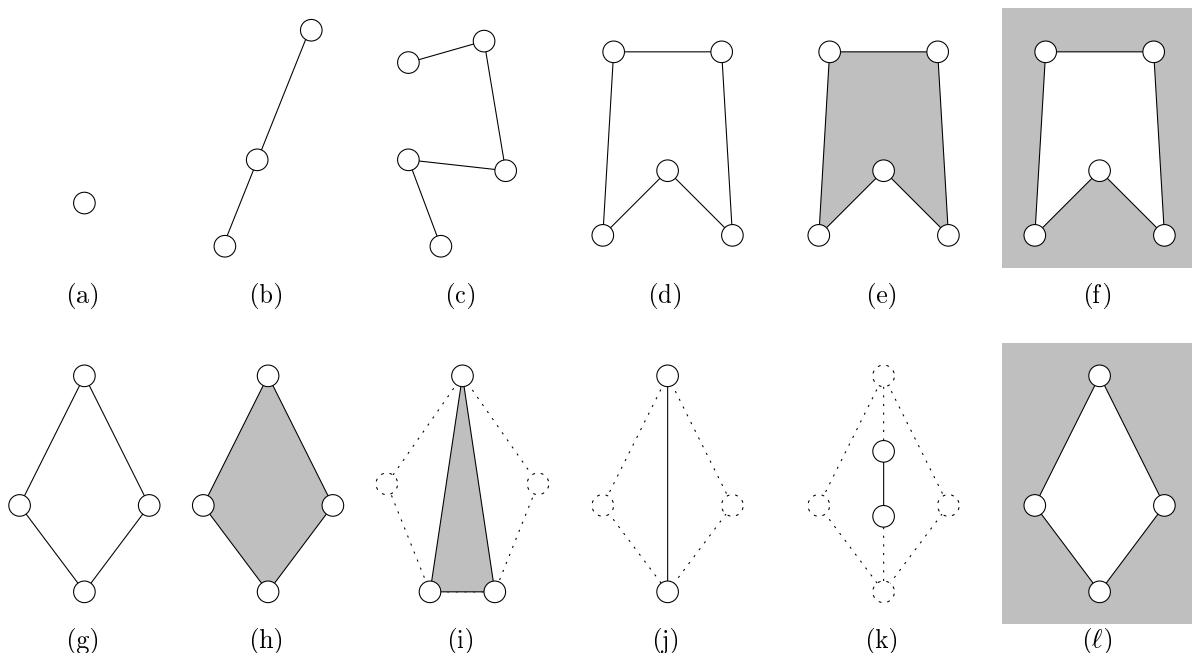


Figure 9: Hypothetical connected components of ∂M and their relation to M . Solid lines are edges of ∂M ; white regions are absent from M ; and shaded regions are present in M . (a) An isolated vertex. (b) A straight subarc. (c) A nonstraight subarc. (d) A nonconvex cycle. (e) A nonconvex cycle and its local interior. (f) A nonconvex cycle and its local exterior. (g–k) Various situations with a convex cycle. (ℓ) The only possible case: A convex cycle, its local exterior, and possibly some of its interior.

16:01 not at the maximum z coordinate.) Now take the intersection of Π with the surface Γ , and
 16:02 project this intersection to the xy -plane. The resulting set X is shown in Figure 10 for the
 16:03 various cases.

16:04 The set X captures several properties of the polyhedral graph Γ . First note that because
 16:05 X is the boundary of a small neighborhood of M in the plane, it is a disjoint union of
 16:06 cycles. It is also polygonal. Each edge of X corresponds to a face of Γ , and each vertex
 16:07 of X corresponds to an edge of Γ . The *angle* at a vertex of X (on the side interior to M)
 16:08 determines the type of edge corresponding to that vertex: the angle is π (straight) if the
 16:09 edge is flat, less than π (convex) if the edge is a mountain, and more than π (reflex) if the
 16:10 edge is a valley.

16:11 The basic idea is to show that X has “many” convex angles, and apply Lemma 5 to
 16:12 prove that the framework has “too many” bars. The key fact underlying the proof is that
 16:13 the original arc-and-cycle set has maximum bar-degree two: every vertex is incident to at
 16:14 most two bars. In the planar framework $G'_A(\mathbf{p}')$, only vertices \mathbf{v} of convex cycles can have
 16:15 bar-degrees greater than two, and these bars are contained in a convex wedge from \mathbf{v} .

16:16 Our proof deals with all cases at once. To illustrate the essence of the proof, we first
 16:17 describe it for a subcase of case (a) in which one component of ∂M is a single vertex \mathbf{v} that
 16:18 does not belong to a convex cycle. In this case, one component of X is a planar polygonal
 16:19 cycle P that is star-shaped around \mathbf{v} , that is, every point on the boundary of P is visible

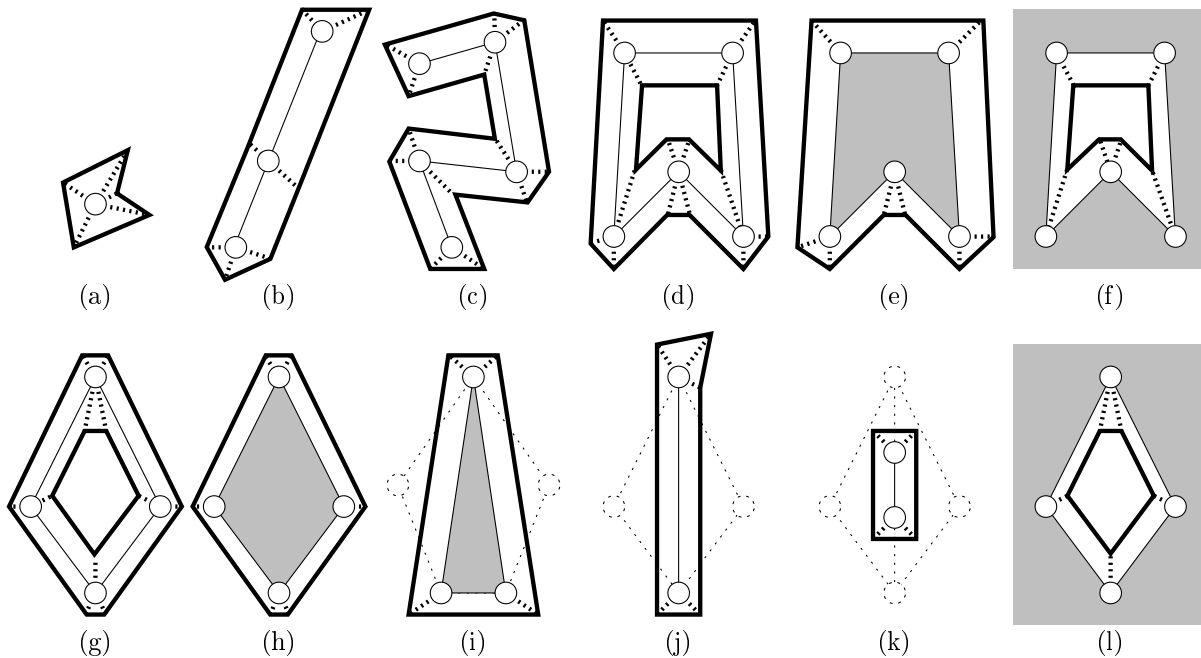


Figure 10: Slicing the polyhedral graph Γ just below the maximum z coordinate, in each case corresponding to those in Figure 9. Thick lines denote the slice intersection X , and thick dotted lines denote the corresponding edges in the polyhedral graph Γ .

17:01 from \mathbf{v} . In particular, P is a planar polygon with positive area and no self-intersections.
 17:02 Every such polygon has at least three convex vertices. (To see this, define the *turn angle* at
 17:03 a vertex to be π minus the interior angle, so it is positive for convex angles and negative for
 17:04 reflex angles, and always strictly between $-\pi$ and π . Because the turn angles of a planar
 17:05 polygon sum to 2π , and the maximum turn angle of a vertex is $< \pi$, every polygon has at
 17:06 least three vertices with positive turn angles.) These three convex vertices correspond to
 17:07 three mountains in Γ , all incident to a common vertex v . By Lemma 5, there are three bars
 17:08 incident to v , contradicting the maximum-degree-two property for vertices not on convex
 17:09 cycles. Therefore, this subcase of case (a) cannot exist.

17:10 The general reason that cases (a–k) cannot exist is the following:

17:11 **Lemma 6** *Let \mathbf{v} be a vertex on the boundary of M , and let b_1, \dots, b_k be the bars incident*
 17:12 *to \mathbf{v} in cyclic order. Consider a small disk D around \mathbf{v} .*

- 17:13 (1) *If there is an angle of at least π at \mathbf{v} between two consecutive bars, say b_i and b_{i+1} ,*
 17:14 *then the pie wedge P of D bounded by b_i and b_{i+1} belongs to M (see Figure 11).*
- 17:15 (2) *If there are no bars or only one bar incident to \mathbf{v} , i.e., $k \leq 1$, then the entire disk D*
 17:16 *belongs to M . (This can be viewed as a special case of (1).)*

17:17 **Proof:** (1) Because there are no bars in the pie wedge P , and hence no edges of ∂M in P ,
 17:18 P must be completely contained in or disjoint from M . Assume to the contrary that P is
 17:19 disjoint from M . Then the intersection of the slice X with the pie wedge P is a star-shaped

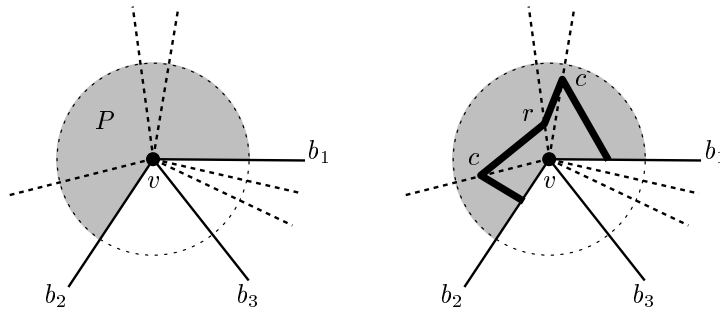


Figure 11: (Left) Illustration of Lemma 6: solid lines are bars, dotted lines are struts, and the shaded pie wedge P must be contained in M . (Right) Illustration of the proof; the thick lines form the portion of X inside P , and the symbols c and r denote convex and reflex vertices, respectively.

18:01 polygonal arc around \mathbf{v} starting from a point on b_i and ending at a point on b_{i+1} . By the
 18:02 properties of X , convex vertices on this arc correspond to mountains emanating from \mathbf{v} , and
 18:03 reflex vertices correspond to valleys emanating from \mathbf{v} . Because the angle of the pie wedge
 18:04 P is at least π , the arc must have at least one convex vertex in P . (The turn angles along
 18:05 the arc must sum to a positive number, so some vertex must have a positive turn angle.) By
 18:06 Lemma 5, there must be a bar in P , a contradiction.

18:07 (2) If $k = 1$, the bars b_i and b_{i+1} coincide, and the same proof applies. The star-shaped
 18:08 polygonal arc becomes a star-shaped polygonal cycle, which must have at least two convex
 18:09 vertices not on $b_i = b_{i+1}$. If $k = 0$, X also has a star-shaped polygonal cycle around \mathbf{v} , which
 18:10 must have at least three convex vertices, yet \mathbf{v} has no incident bars. \square

18:11 Note that this lemma applies to every vertex in our planar framework $G'_A(\mathbf{p}')$, because
 18:12 every vertex either has bar-degree at most two or is a vertex of a convex cycle, and in either
 18:13 case there is a nonconvex angle between two consecutive bars.

18:14 One can immediately verify that the examples in Figure 10(a–k) contradict Lemma 6. For
 18:15 example, applying the lemma to any vertex of ∂M shows that M should contain a positive
 18:16 two-dimensional area incident to that vertex. This immediately rules out cases (a–d), (g),
 18:17 (j), and (k).

18:18 A general proof is also easy with Lemma 6 in hand:

18:19 **Proof (Theorem 6):** Consider first a degree-0 or degree-1 vertex \mathbf{v} in ∂M . (Such a
 18:20 point would appear when M has a component that is an isolated point or an arc of bars.)
 18:21 Because Lemma 6 applies to every vertex of the framework, we know that some positive
 18:22 two-dimensional area in the vicinity of \mathbf{v} belongs to M , contradicting that \mathbf{v} has degree 0 or
 18:23 1 in ∂M . This rules out cases (a–c) and (j–k).

18:24 It follows that ∂M is a union of cycles. A component of ∂M can be of two kinds:

- 18:25 (1) If it is formed from the edges of a convex cycle and its triangulation, Lemma 6 ap-
 18:26 plies to any vertex in it, and we conclude that M contains the face of the framework
 18:27 immediately exterior to the cycle. This rules out cases (g–i).
- 18:28 (2) If it consists of a complete nonconvex cycle, we can apply Lemma 6 to some convex
 18:29 vertex and to some reflex vertex (they must both exist), and we conclude that M

19:01 contains both the face of the framework immediately interior and the face immediately
 19:02 exterior to the cycle. This rules out cases (d–f).

19:03 In the end, the only faces of the framework that can be missing from M are those interior to
 19:04 convex cycles (case (ℓ)). This completes the proof of Theorem 6 and of Theorems 4 and 3.
 19:05 □

19:06 4 Global Motion

19:07 In this section, we combine the infinitesimal motions into a global motion, thereby proving
 19:08 Theorem 1, the main theorem. In overview, Theorem 3 establishes the existence of *some*
 19:09 direction of motion \mathbf{v} . We select a unique vector $\mathbf{v} = f(\mathbf{p})$ for each configuration \mathbf{p} as the
 19:10 solution of a convex optimization problem (7–9). We then set up the differential equation

$$19:11 \quad \frac{d}{dt}\mathbf{p}(t) = f(\mathbf{p}(t)).$$

19:12 The solution of this differential equation moves the linkage to a configuration where an angle
 19:13 between two bars becomes straight. At this point we merge the two bars and continue
 19:14 with the reduced framework that has one vertex less. This procedure is iterated until the
 19:15 framework is outer-convex and no further expansive motion is possible.

19:16 It is convenient for the proof of Theorem 1 to effectively pin an edge in the configuration.
 19:17 Choose any edge, say $\{\mathbf{p}_1, \mathbf{p}_2\}$, that is a bar. During the motion we will arrange matters so
 19:18 that this bar is stationary.

19:19 We now go into the details of the proof. We use the following nonlinear minimization
 19:20 problem to define a unique direction \mathbf{v} for every configuration \mathbf{p} of a reduced arc-and-cycle
 19:21 set.

$$19:22 \quad \text{minimize} \quad \sum_{i \in V} \|\mathbf{v}_i\|^2 + \sum_{\{i,j\} \in S} \frac{1}{(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_j - \mathbf{p}_i\|} \quad (7)$$

$$19:23 \quad \text{subject to} \quad (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) > \|\mathbf{p}_j - \mathbf{p}_i\|, \quad \text{for } \{i, j\} \in S \quad (8)$$

$$19:24 \quad (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0, \quad \text{for } \{i, j\} \in B \quad (9)$$

$$19:25 \quad \mathbf{v}_1 = \mathbf{v}_2 = 0 \quad (10)$$

19:26 The restrictions (8) place a uniform constraint on the growth of the struts S : the derivative
 19:27 of the length of each strut must be larger than 1. Since the system (2) is homogeneous, the
 19:28 system (8–9) is feasible for any choice of right-hand sides in (8). This particular right-hand
 19:29 side has been chosen for convenience in the proof.

19:30 The objective function (7) includes the norm of \mathbf{v} as a quadratic term, plus a barrier-type
 19:31 penalty term that keeps the solution away from the boundary (8) of the feasible region. This
 19:32 penalty term is necessary to achieve a smooth dependence of the solution on the data. The
 19:33 objective function is strictly convex because it is a sum of strictly convex functions, of the
 19:34 form x^2 for a variable x , and convex functions, of the form $1/f(x_1, x_2, x_3, x_4)$ where f is
 19:35 an affine function in four variables that is guaranteed to be positive. Because the objective

20:01 function is strictly convex, and it goes to infinity if \mathbf{v} increases to infinity or approaches the
 20:02 boundary of condition (8), there is a unique solution \mathbf{v} for every \mathbf{p} ; we denote this solution
 20:03 by $f(\mathbf{p})$.

20:04 The function $f(\mathbf{p})$ is defined on an open set $U \subset \mathbb{R}^{2n}$ that is characterized by the
 20:05 conditions of Theorem 3: no angles are 0° or 180° , no vertex touches a bar, and at least one
 20:06 cycle is nonconvex or at least one open arc is not straight.

20:07 4.1 Smoothness

20:08 We will show that f is differentiable in the domain U . This follows from the stability theory
 20:09 of convex programming under equality constraints, as applied to parametric optimization
 20:10 problems of the type

$$20:11 \min\{g(p, x) : x \in \Omega(p) \subseteq \mathbb{R}^n, A(p)x = b(p)\} \quad (11)$$

20:12 where $A(p)$ is an $m \times n$ matrix and $b(p)$ is an m -vector. The objective function g , the domain
 20:13 $\Omega(p)$, and the linear constraints (A, b) depend on a parameter p that ranges over an open
 20:14 region $U \subseteq \mathbb{R}^k$.

20:15 For such an optimization problem, the following lemma establishes the smooth depen-
 20:16 dence of the solution vector on the problem-definition data $A(p)$ and $b(p)$.

20:17 **Lemma 7** *Suppose that the following conditions are satisfied in the optimization problem (11).*

- 20:18 (a) *The objective function $g(p, x)$ is twice continuously differentiable and strictly convex as*
 20:19 *a function of $x \in \Omega(p)$, with a positive definite Hessian H_g , for every $p \in U$.*
- 20:20 (b) *The domain $\Omega(p)$ is an open set, for every $p \in U$.*
- 20:21 (c) *The rows of the constraint matrix $A(p)$ are linearly independent, for every $p \in U$.*
- 20:22 (d) *The problem-definition data $A(p)$ and $b(p)$ and the gradient ∇g of g with respect to x*
 20:23 *are continuously differentiable in p , for $p \in U$.*
- 20:24 (e) *The optimum point $x^*(p)$ of the problem (11) exists for every $p \in U$ (and is unique,*
 20:25 *by (a)).*

20:26 *Then $x^*(p)$ is continuously differentiable in U .*

20:27 **Proof:** The proof is based on the implicit function theorem and follows the standard lines
 20:28 of the proofs in this area; cf. [BS74, in particular Section 4] or [Fia76, Theorem 2.1] for
 20:29 more general theorems where inequalities are also allowed. For the benefit of the reader,
 20:30 we sketch the proof here. From (a) and (e) it follows that x^* can be found as part of the
 20:31 unique solution (x^*, λ) of the system of equations $h(p, x, \lambda) = 0$ that represents the first-order
 20:32 necessary optimality conditions. Specifically, λ is a k -element vector of Lagrange multipliers,
 20:33 and $h: U \times \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ is given by

$$20:34 h = \begin{pmatrix} \nabla g - \lambda^\top A^\top \\ Ax - b \end{pmatrix}.$$

21:01 The implicit function theorem guarantees the local existence of $x(p)$ (and $\lambda(p)$) as a solution
 21:02 of $h(p, x(p), \lambda(p)) = 0$ in a neighborhood of x if the Jacobian $J = \partial h / \partial(x, \lambda)$ is an invertible
 21:03 matrix for every $p \in U$. Moreover, differentiability of $x(p)$ is ensured if h is continuously
 21:04 differentiable. The Jacobi matrix is given by

$$21:05 \quad J = \frac{\partial h(p, x, \lambda)}{\partial(x, \lambda)} = \begin{pmatrix} H_g & A^T \\ A & 0 \end{pmatrix}.$$

21:06 Differentiability of h follows from assumption (d); we only have to check that J is invertible.
 21:07 By assumption (a), H_g is positive definite and hence invertible. Thus

$$21:08 \quad \det J = \det H_g \cdot \det(-AH_g^{-1}A^T).$$

21:09 By assumption (c), A has full row rank, and because H_g is positive definite, so is $AH_g^{-1}A^T$.
 21:10 Therefore $\det J \neq 0$. □

21:11 **Lemma 8** *f is differentiable on U .*

21:12 **Proof:** The objective function is the sum of the quadratic function $\sum \|\mathbf{v}_i\|^2$, which has
 21:13 a positive definite (constant) Hessian, and additional smooth convex terms, and therefore
 21:14 assumption (a) of Lemma 7 holds, as well as the part of assumption (d) regarding g . The
 21:15 feasible domain Ω is defined by the inequalities (8), and because the inequalities are strict,
 21:16 Ω is an open set, so assumption (b) holds. The problem-definition data A and b are defined
 21:17 by the linear constraints (9) and (10). Both are clearly continuously differentiable, verifying
 21:18 the remaining half of assumption (d). Assumption (e) follows from the existence of an
 21:19 infinitesimal motion (Theorem 3).

21:20 It only remains to check assumption (c), the linear independence of the defining equations.
 21:21 Note that (10) implies (9) for the edge $\{1, 2\}$, so the latter equation is redundant and can
 21:22 be omitted without changing the problem. In order to show that the remaining equations
 21:23 of the system (9–10) are linearly independent, we check that they have a solution for any
 21:24 choice of right-hand sides. We prove this by induction on the number of vertices. We select
 21:25 a vertex $i \notin \{1, 2\}$ that is incident to at most two bars $\{i, j\}, \{i, k\}$. The existence of such
 21:26 a vertex follows from an extension of an argument in the proof of Lemma 4: a framework
 21:27 consisting exclusively of arcs, cycles, and triangulated convex cycles, contains at least two
 21:28 non-adjacent vertices with degree at most two (unless the framework consists of a single
 21:29 triangle; in that case there are three vertices of degree two).

21:30 If the vertex i is incident to two bars, they cannot be parallel. Thus, the corresponding
 21:31 unknown vector \mathbf{v}_i appears in at most two equations in which the scalar products with two
 21:32 vectors $\mathbf{p}_i - \mathbf{p}_j$ and $\mathbf{p}_i - \mathbf{p}_k$ are taken; because these vectors are not parallel, there is always
 21:33 a solution for \mathbf{v}_i regardless of the values of the other variables. It follows that any solution
 21:34 of the system without the variable \mathbf{v}_i can be extended to \mathbf{v}_i . The reduced system is of the
 21:35 same form as the original system, and therefore we can apply induction. □

4.2 Solving the Differential Equation and Proving Theorem 1

Differentiability of f on U is sufficient to ensure that the initial-value problem

$$\frac{d}{dt}\mathbf{p}(t) = f(\mathbf{p}(t)), \quad \mathbf{p}(0) = \mathbf{p}_0 \quad (12)$$

has a (unique) *maximal solution* $\mathbf{p}(t)$, $0 \leq t < T$, that cannot be extended beyond some positive bound $T \leq \infty$; see for example [Wal96, Section II.XXI]. This means that one of three cases occurs:

- (a) $\mathbf{p}(t)$ exists for all t , i.e., $T = \infty$.
- (b) T is finite, and $\mathbf{p}(t)$ becomes unbounded as $t \rightarrow T$.
- (c) T is finite, and $\mathbf{p}(t)$ approaches the boundary of U as $t \rightarrow T$.

The last case (c) is the case we want: at the boundary of U , some angle becomes straight, and we can reduce the linkage. The other two cases must be avoided: In case (a), the motion of the framework slows down and never reaches the limit of an outer-convex configuration. Case (b) can arise only with multiple disjoint components: the components could repel and fly away from each other faster than they straighten or convexify, thus never reaching an outer-convex configuration.

Case (a) can be excluded very easily. By assumption, the bar-and-strut framework $G_A(\mathbf{p})$ has some strut $\{i, j\}$ between two points in the same component of the bar framework; their distance increases at least with rate 1, by (8), but it is bounded from above because i and j are linked by a sequence of bars. It follows that the solution cannot exist indefinitely and T must be finite.

To deal with case (b), we apply the following observation.

Lemma 9 *If an arc-and-cycle set A is not separated, then its diameter is bounded by the total length of all edges of A .* \square

This can be proved easily by induction on the number of components of A . Since some vertices are pinned at the origin, case (b) implies that A must become separated by a line L . At this point, we stop the motion defined by (12) and treat the two parts into which L separates the components of A independently and recursively. Unfortunately, the guarantee for the expansive property between different members of the partition is lost.

We are left with case (c). We show that $\mathbf{p}(t)$ converges as $t \rightarrow T$. Observe that all pairwise distances of vertices $\mathbf{p}(t)$ are monotonically increasing, and by condition (10) \mathbf{p}_1 and \mathbf{p}_2 are fixed during the motion. Thus, all other vertices are determined up to reflection (through $\mathbf{p}_1\mathbf{p}_2$), so the whole configuration is determined up to reflection. Thus $\mathbf{p}(t) \rightarrow \mathbf{p}$ for some configuration \mathbf{p} as $t \rightarrow T$. The configuration \mathbf{p} is on the boundary of U and thus must have some vertex with a straight angle. Then we inductively continue with a simpler linkage. This completes the proof of Theorem 1. \square

In this proof, the easy exclusion of possibility (b), that the motion becomes unbounded, depends crucially on the fact that the diameter of A is bounded, and the motion is stopped as soon as there is a separating line. Boundedness is valid even without this precaution, as stated in the following lemma.

23:01 **Lemma 10 (Boundedness Lemma)** *Let $\mathbf{p}(t)$ be the motion given by the differential equa-*
 23:02 *tion (12), where $\mathbf{v} = f(\mathbf{p})$ is given as the solution of the optimization problem (7–9). Then*
 23:03 *the motion of every vertex i is bounded:*

$$23:04 \quad \|\mathbf{p}_i(t) - \mathbf{p}_i(0)\| \leq \int_0^t \|\mathbf{v}_i(t)\| dt \leq K_{B,S,\mathbf{p}_0}(t),$$

23:05 *where $K_{B,S,\mathbf{p}_0}(t)$ is an explicit function of t that depends only on the combinatorial structure*
 23:06 *of the arc-and-cycle set (B and S) and on the initial configuration \mathbf{p}_0 .*

23:07 Note that the definition of \mathbf{v} does not involve the pinning constraints (10). The lemma
 23:08 implies that it is not necessary to treat separated components separately. The proof of the
 23:09 lemma is complicated, and it is given in Appendix A.

23:10 4.3 Alternative Approaches

23:11 There are many ways to select a local motion \mathbf{v} among the feasible local motions whose
 23:12 existence is guaranteed by Theorem 3. We have chosen one possibility in Equations (7–10)
 23:13 that is most convenient for the proof.

23:14 As a possible alternative approach, we might consider a *linear* programming problem,
 23:15 with some arbitrary artificial linear objective function \mathbf{c} , and some linear normalization
 23:16 condition to ensure boundedness, pinning down some bar $(i_1, i_2) \in B$:

$$23:17 \quad \begin{aligned} & \text{minimize} && \sum_{i \in V} \mathbf{c}_i \cdot \mathbf{v}_i \\ & \text{subject to} && (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0 \text{ for } \{i, j\} \in B, \end{aligned} \quad (13)$$

$$23:19 \quad (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) \geq 0 \text{ for } \{i, j\} \in S, \quad (14)$$

$$23:20 \quad \sum_{i \in V} \mathbf{d}_i \cdot \mathbf{v}_i = 1, \quad (15)$$

$$23:21 \quad \mathbf{v}_{i_1} = \mathbf{v}_{i_2} = 0, \quad (16)$$

23:22 We have given up *strict* expansiveness in (14). The set of vectors given by (13), (14), and
 23:23 (16) forms a polyhedral cone C . Theorem 3 guarantees that there are nonzero solutions.
 23:24 One can check that the pinning constraints (16) ensure that the cone is pointed. The idea
 23:25 is now to use an extreme ray of the cone C for the motion. A vector \mathbf{d} can be found which
 23:26 ensures that the feasible set (13–16) is a bounded set. Any basic feasible solution of the linear
 23:27 program will correspond to an extreme ray of the cone C . It will have a few inequalities of
 23:28 (14) fulfilled with equality. The resulting framework obtained by inserting “artificial” bars
 23:29 corresponding to the nonbasic inequalities of (14), will have a unique vector of velocities \mathbf{v}
 23:30 subject to the normalization constraint (15). This means that the framework is a *mechanism*,
 23:31 allowing one degree of freedom; as the mechanism follows this forced motion, all nonfixed
 23:32 distances will increase, at least for some time.

23:33 So one follows the paradigm of parametric linear programming: The optimal basic feasible
 23:34 solution will continue to remain feasible as the coefficients \mathbf{p}_i in the constraints (14) change
 23:35 smoothly. At some point, one of these constraints will threaten to become violated: this is
 23:36 the time to make a *pivot*, exchanging one of the artificial bars for a new one which allows
 23:37 the motion to be continued.

24:01 The above discussion has ignored several issues, such as possible degeneracy of the linear
24:02 program. However, this approach might be more attractive from a conceptual, as well as
24:03 practical, point of view.

24:04 Streinu [Str00] has found a class of such mechanisms, so-called *pseudo-triangulations*.
24:05 These structures have several nice properties; for example, they form a planar framework
24:06 of bars. Streinu [Str00] has proved that a polygonal arc can be opened by a sequence of
24:07 at most $O(n^2)$ motions, where each motion is given by the mechanism of a single pseudo-
24:08 triangulation.

24:09 The polyhedral cone C mentioned above has been more thoroughly investigated in Rote,
24:10 Santos, and Streinu [RSS02]. In particular, they studied the so-called *expansion cone*, which
24:11 is simply defined by the equations (14) for *all* pairs of points i and j . The extreme rays of
24:12 this cone are closely related to the set of all pseudo-triangulations of a point set.

24:13 4.4 Comparison of Approaches

24:14 The approach based on mechanisms might avoid some of the numerical difficulties associated
24:15 with solving the optimization problem (7–9). For example, consider a spiral n -bar arc wind-
24:16 ing around a unit square in layers of thickness ε (Figure 12). Basically, a strut deep inside
24:17 the spiral cannot increase in length quickly before an outer strut increases significantly. But
24:18 in the solution of (8–9), the inner strut lengths must increase at unit speed; a rough estimate
24:19 shows that this causes the outermost vertex to move with an *exponential* speed of at least
24:20 $(1/\varepsilon)^{n/4}$, as $\varepsilon \rightarrow 0$. On the other hand, the “natural” solution of unwinding the spiral one
24:21 bar at a time fits nicely into the setup of mechanisms and the parametric linear program
24:22 approach.

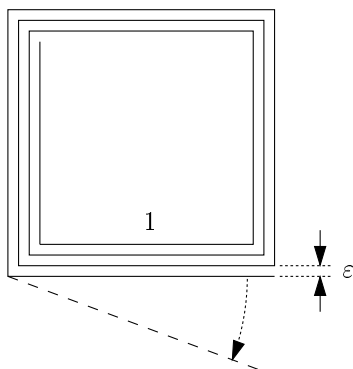


Figure 12: An arc that is numerically difficult to unfold.

24:23 Our proof has certain nonconstructive aspects: the direction \mathbf{v} of movement is specified
24:24 implicitly as the solution of an optimization problem, and the global motion arises as the
24:25 solution of a differential equation. Both of these items are numerically well-understood,
24:26 and our approach lends itself to a practical implementation. Indeed, we implemented our
24:27 approach to produce animations such as Figure 1. However, this does not necessarily lead to
24:28 a finite algorithm in the strict sense. The optimization problem (7–9), having an objective
24:29 function which is rational, can in principle be solved exactly by solving the system $h(p, x, \lambda) =$

25:01 0 of algebraic equations as in Lemma 7. The differential equation cannot be solved explicitly,
25:02 but it may be possible to bound the convergence and solve the differential equation up to a
25:03 given error bound.

25:04 Because the motions of a mechanism are described by algebraic equations, Streinu's
25:05 algorithm leads to a finite algorithm for a digital computer, at least in principle. It remains
25:06 to be seen how a practical implementation competes with our approach; in any case, as an
25:07 algorithm for a direct realization of the motion by a mechanical device, Streinu's algorithm
25:08 appears attractive.

25:09 On the other hand, our nonlinear programming approach might be preferable because it
25:10 produces a "canonical" movement. As a consequence of this, any symmetry of the starting
25:11 configuration is preserved (see Corollary 2 in the next section).

5 Additional Properties and Related Problems

5.1 Symmetry

25:12 We show that the deformation that we have defined in Section 4 preserves any symmetries
25:13 that the original configuration might have. We say that the arc-and-cycle set A has some
25:14 group H of congruences of the plane as *symmetry group* if each element of H permutes the
25:15 vertices and edges of A .
25:16

25:17 **Corollary 2** *If an arc-and-cycle set has a symmetry group H , then there is a piecewise-*
25:18 *differentiable proper motion to an outer-convex configuration; the motion is expansive until*
25:19 *the linkage becomes separated, and the symmetry group H is preserved during the motion.*

25:20 **Proof:** Because A has finitely many vertices and edges, H must be finite, so the Affine
25:21 Fixed Point Theorem implies that there must be a point fixed by all elements of H . Let this
25:22 point be the origin, and let \mathbf{p}_1 be any vertex of the configuration distinct from the origin.
25:23 Consider the infinitesimal motion defined by the conditions (7), (8), and (9) but not (10).
25:24 There is a unique solution \mathbf{v} to this minimization problem. This solution must be symmetric
25:25 with respect to the symmetry group H . If not, then the action of some element of H takes
25:26 \mathbf{v} to a distinct solution, contradicting the uniqueness of the solution. There is now a unique
25:27 infinitesimal rotation that we can add to \mathbf{v} so that \mathbf{p}_1 and \mathbf{v}_1 are parallel. This still maintains
25:28 the symmetry of the infinitesimal motion \mathbf{v} . Now it is clear that the limit exists as before
25:29 in the proof of Theorem 1, and the symmetry of H is preserved. \square

5.2 Increasing Area

25:30 A natural question is whether every expansive motion increases the area bounded by each
25:31 polygonal cycle. The answer turns out to be yes, but the proof is difficult from elementary
25:32 methods. A simple example that helps motivate why this problem is nontrivial is an obtuse
25:33 triangle: if the base edge increases in length (as a strut) and the others remain the same
25:34 length (as bars), then the area decreases. The cycle of bars in a polygonal cycle is therefore
25:35 crucial but difficult to exploit except with our theory of expansive motions.
25:36

26:01 First we show how to extend any given expansive infinitesimal motion to any point in
 26:02 the plane, which is of interest in its own right.

26:03 **Lemma 11** Consider an infinitesimal motion \mathbf{v} on points $\mathbf{p}_1, \dots, \mathbf{p}_n$ in \mathbb{R}^d , and suppose
 26:04 that the motion is expansive, i.e., $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{v}_i - \mathbf{v}_j) \geq 0$ for all i, j . Then the infinitesimal
 26:05 motion \mathbf{v} can be extended to another point \mathbf{p}_0 in \mathbb{R}^d and remain expansive. Furthermore,
 26:06 the new expansiveness inequalities are all strict unless \mathbf{p}_0 is in the convex hull of a subset of
 26:07 points on which the original infinitesimal motion is trivial, i.e., describes a rigid motion.

26:08 **Proof:** We have two proofs of this statement: the first proof is a calculation and the second
 26:09 proof uses Helly's Theorem.

26:10 We first consider the case where we want to prove strict expansiveness. By the Farkas
 26:11 lemma, the desired inequalities $(\mathbf{v}_0 - \mathbf{v}_i) \cdot (\mathbf{p}_0 - \mathbf{p}_i) > 0$, can be fulfilled by some unknown
 26:12 vector \mathbf{v}_0 if and only if the dual system

$$26:13 \quad \sum_{i=1}^n \lambda_i (\mathbf{p}_0 - \mathbf{p}_i) = 0 \quad (17)$$

$$26:14 \quad \sum_{i=1}^n \lambda_i \mathbf{v}_i \cdot (\mathbf{p}_0 - \mathbf{p}_i) \geq 0 \quad (18)$$

$$26:15 \quad \lambda_i \geq 0$$

26:16 has no solution except the trivial solution $\lambda \equiv 0$. In order to find a contradiction, suppose
 26:17 that a nontrivial solution λ exists. Without loss of generality, we may assume

$$26:18 \quad \sum_{i=1}^n \lambda_i = 1.$$

26:19 Then we get from (17) a representation of \mathbf{p}_0 as a convex combination

$$26:20 \quad \mathbf{p}_0 = \sum_{i=1}^n \lambda_i \mathbf{p}_i. \quad (19)$$

26:21 Substituting this into (18) yields

$$26:22 \quad \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j (\mathbf{v}_i \cdot \mathbf{p}_j) - \sum_{i=1}^n \lambda_i (\mathbf{v}_i \cdot \mathbf{p}_i) \geq 0. \quad (20)$$

26:23 On the other hand, multiplying the given inequalities

$$26:24 \quad \mathbf{v}_i \cdot \mathbf{p}_i - \mathbf{v}_i \cdot \mathbf{p}_j - \mathbf{v}_j \cdot \mathbf{p}_i + \mathbf{v}_j \cdot \mathbf{p}_j \geq 0 \quad (21)$$

26:25 by $-\lambda_j \lambda_j / 2$ and summing them over $i, j = 1, \dots, n$ (including the trivial cases for $i = j$)
 26:26 yields

$$26:27 \quad \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j (\mathbf{v}_i \cdot \mathbf{p}_j) - \sum_{i=1}^n \lambda_i (\mathbf{v}_i \cdot \mathbf{p}_i) \leq 0. \quad (22)$$

27:01 By the assumption of the lemma, we have $\lambda_i > 0$ in (19) for at least two points \mathbf{p}_i and \mathbf{p}_j
 27:02 whose distance expands strictly. This means that the corresponding strict inequality in (21)
 27:03 will hold in (22) too, a contradiction to (20). This finishes the case when \mathbf{p}_0 does not lie in
 27:04 the convex hull of some points which move rigidly.

27:05 In the other case, nonstrict expansiveness can be shown by a variation of the above
 27:06 argument. Alternatively, we can appeal to Lemma 2 (or its higher-dimensional extension)
 27:07 and let the point \mathbf{p}_0 move rigidly with the rigid point set in whose convex hull it lies. The
 27:08 resulting motion is expansive; the distance from \mathbf{p}_0 to the other points will expand strictly,
 27:09 with the obvious exception of the points with which it moves rigidly. This concludes the first
 27:10 proof of the lemma.

27:11 The other proof establishes a connection to tensegrity frameworks and is more intuitive.
 27:12 However, we have to deal with a few extra cases to reduce the statement of the lemma to the
 27:13 basic case that the points $\mathbf{p}_1, \dots, \mathbf{p}_n$ form the vertices of a simplex that contains the point
 27:14 \mathbf{p}_0 in its interior.

27:15 We proceed by induction on the dimension d . There is nothing to prove in case $d = 0$.
 27:16 So we assume the statement for $0, \dots, d - 1$ and $d \geq 1$. The desired inequalities $(\mathbf{v}_0 - \mathbf{v}_i) \cdot$
 27:17 $(\mathbf{p}_0 - \mathbf{p}_i) > 0$ define open half spaces

$$27:18 \quad H_i = \{ \mathbf{v}_0 \mid \mathbf{v}_0 \cdot (\mathbf{p}_0 - \mathbf{p}_i) > \mathbf{v}_i \cdot (\mathbf{p}_0 - \mathbf{p}_i) \}.$$

27:19 This finite collection of half spaces is nonempty precisely if every set of at most $d + 1$ of them
 27:20 is nonempty by Helly's theorem [DGK63]. So we consider any subset of $k \leq d + 1$ points
 27:21 $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k$ of \mathbf{p} . If \mathbf{p}_0 is outside the convex hull σ of these points, then simply choose \mathbf{v}_0
 27:22 in a direction along a normal to a hyperplane separating \mathbf{p}_0 and σ , pointing away from σ . If
 27:23 the magnitude of \mathbf{v}_0 is large enough, then the desired inequalities will be satisfied.

27:24 If \mathbf{p}_0 lies in the convex hull of the given points, we first consider the “general” case that
 27:25 there are $k = d + 1$ affine-independent points, forming the vertices of the d -dimensional
 27:26 simplex σ in \mathbb{R}^d , and \mathbf{p}_0 is interior to σ . Suppose that the inequalities defined above do not
 27:27 have a solution. Then if we look at the complementary half-spaces defined by

$$27:28 \quad H_i^- = \{ \mathbf{v}_0 \mid \mathbf{v}_0 \cdot (\mathbf{p}_0 - \mathbf{p}_i) \leq \mathbf{v}_i \cdot (\mathbf{p}_0 - \mathbf{p}_i) \},$$

27:29 they do have a solution. Let $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1})$ be a solution to those new inequalities.
 27:30 Now \mathbf{v} is an infinitesimal motion of the tensegrity that is obtained by having *cables* from
 27:31 \mathbf{p}_0 (whose lengths can only shrink) and the rest all struts as before. But it is easy to
 27:32 show that this tensegrity has no infinitesimal motion in \mathbb{R}^d . (For example, apply Theorem
 27:33 5.2(c) of [RW81] observing that the underlying bar framework has no non-trivial infinitesimal
 27:34 motion, and there must be a proper stress that is nonzero on all struts and cables. An
 27:35 explicit calculation of the proper stress is given in [BC99]. The calculation is similar to the
 27:36 calculations (19)–(22) in the first part of the proof, with $\lambda_i \lambda_j$ being interpreted as stress.)
 27:37 So $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}$ must be a trivial infinitesimal motion, which can only happen if the
 27:38 motion is trivial on all of σ .

27:39 There are two remaining cases:

- 27:40 (a) The points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k$ lie in a hyperplane S . This includes the cases when $k < d + 1$
 27:41 and when the points are affine-dependent.

28:01 (b) There are $k = d + 1$ points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{d+1}$ forming a d -dimensional simplex σ in \mathbb{R}^d ,
 28:02 and \mathbf{p}_0 lies on the boundary of σ .

28:03 In case (a), we know that $\mathbf{p}_0 \in S$, because otherwise it would lie outside the convex hull σ .
 28:04 We decompose each \mathbf{v}_i into a component \mathbf{v}_i^\parallel parallel to S and a component \mathbf{v}_i^\perp perpendicular
 28:05 to S . By the induction hypothesis, there is a vector \mathbf{v}_0^\parallel parallel to S such that it together
 28:06 with the other vertices is infinitesimally expansive with respect to the projected \mathbf{v}_i^\parallel and hence
 28:07 the \mathbf{v}_i themselves. It is strict unless \mathbf{p}_0 is in the convex hull of a subset of the points where
 28:08 the infinitesimal motion is trivial.

28:09 In case (b), there is a hyperplane S containing \mathbf{p}_0 and all of the points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_{d+1}$
 28:10 except one, say \mathbf{p}_{d+1} . We apply the construction of case (a) to the points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_d$,
 28:11 yielding a vector \mathbf{v}_0^\parallel which is infinitesimally expansive with respect to these vertices. To get
 28:12 expansiveness also for \mathbf{p}_{d+1} , we add to \mathbf{v}_0^\parallel a sufficiently large vector \mathbf{v}_0^\perp perpendicular to S ,
 28:13 pointing into the halfspace of S which does not contain \mathbf{p}_{d+1} . Since the added vector \mathbf{v}_0^\perp is
 28:14 perpendicular to S , this does not affect expansiveness with respect to the points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_d$
 28:15 in S .

28:16 Thus in any case we see that there is a strict solution to the inequalities for $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \dots,$
 28:17 \mathbf{p}_{d+1} , or \mathbf{p}_0 is in the convex hull of points where the infinitesimal motion acts as bars for all
 28:18 surrounding points. This concludes the second proof of the lemma. \square

28:19 Now we apply this lemma to prove that the area of a polygonal cycle increases by any
 28:20 expansive motion. Using the lemma inductively, we can extend an expansive motion to any
 28:21 finite set of points. Specifically, we apply Lemma 11 to the vertices of an appropriately chosen
 28:22 triangulation of the region bounded by a polygonal cycle. (The triangulation introduces new
 28:23 vertices in addition to the vertices of the polygonal cycle.) The following result can be found
 28:24 in [BGR88]. (See also [BMR95, Epp97] for faster algorithms.)

28:25 **Lemma 12** *Any simple closed polygonal curve in the plane can be triangulated, introducing*
 28:26 *extra vertices, such that all the triangles are nonobtuse, i.e., every angle is at most $\pi/2$.*

28:27 There has been some interest in providing acute triangulations and subdivisions (in con-
 28:28 trast to nonobtuse triangulations) of various planar polygonal objects. For example, the
 28:29 column of Martin Gardner [Gar60] (see also [Gar95] and [Man60]) asks for a dissection of a
 28:30 right triangle into acute triangles. but we do not know of a result guaranteeing an acute trian-
 28:31 gulation for a general polygon. Fortunately, the following is sufficient for the area-expanding
 28:32 property that we need:

28:33 **Lemma 13** *Let $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ be an infinitesimal motion of a nonobtuse triangle $\mathbf{p} =$*
 28:34 *$(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ such that for $i \neq j$,*

$$28:35 \quad (\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) \geq 0. \quad (23)$$

28:36 *Then the infinitesimal change in the area of triangle \mathbf{p} is always nonnegative. Furthermore,*
 28:37 *the infinitesimal change in the area is positive except in the following two cases:*

28:38 (a) *The infinitesimal flex \mathbf{v} is trivial, i.e., no inequality in (23) is strict.*

29:01 (b) \mathbf{p} is a right triangle and only the hypotenuse has a strict inequality in (23).

29:02 **Proof:** Let the lengths of the sides of the triangle be denoted by a, b, c , and let the area of
29:03 the triangle be denoted by A . If we differentiate Heron's formula

$$29:04 \quad 16A^2 = 2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$$

29:05 and rearrange terms, denoting derivatives by a', b', c', A' , we get

$$29:06 \quad 8AA' = (b^2 + c^2 - a^2)aa' + (a^2 + c^2 - b^2)bb' + (a^2 + b^2 - c^2)cc'. \quad (24)$$

29:07 We can regard aa', bb', cc' as the left hand side of (23). Each of the terms in parentheses in
29:08 (24) is nonnegative because \mathbf{p} is nonobtuse. Thus $A' \geq 0$.

29:09 If \mathbf{p} is an acute triangle and at least one of $a' > 0$ or $b' > 0$ or $c' > 0$, then (24) is positive,
29:10 and thus $A' > 0$. Suppose \mathbf{p} is a right triangle and c represents the length of the hypotenuse.
29:11 If at least one of $a' > 0$ or $b' > 0$, then (24) is positive, so $A' > 0$. \square

29:12 Note that with a right triangle it is possible for the first derivative of the length of the
29:13 hypotenuse to be positive while the first derivative of the length of the two legs is 0, and in
29:14 this case the first derivative of the area will still be 0. This is the reason for the condition
29:15 on the legs of the triangle.

29:16 **Theorem 7** *Any smooth expansive noncongruent motion of a simple closed polygonal curve*
29:17 *C in the plane, fixing the lengths of its edges, must increase the area of the interior of C*
29:18 *during the motion.*

29:19 **Proof:** Consider the vector field \mathbf{v}_t , $0 \leq t \leq 1$ defined as the derivative at each vertex of
29:20 C at time t . Apply Lemma 12 to find a triangulation T of the area bounded by C with all
29:21 triangles nonobtuse. Apply Lemma 11 to extend the vector field to the vertices of T .

29:22 To get a strictly increasing area, we have to show that the triangulation T has an acute
29:23 triangle with an edge interior to C , or a right triangle with a leg interior to C . Otherwise,
29:24 T would be a single triangle, or it would exclusively consist of right triangles with both
29:25 legs on C , hence it would be a convex quadrilateral with two opposite corners having right
29:26 angles. These cases are excluded because a triangle or a convex quadrilateral (or any convex
29:27 polygon) does not have an expansive noncongruent motion.

29:28 So we have established that T must have an acute triangle with an edge interior to C or
29:29 a right triangle with a leg interior to C . Because the motion is expansive and the derivative
29:30 of at least one of those lengths must be positive for all but a finite number of times, the
29:31 derivative of the area of at least one of those triangles must be strictly positive, and they all
29:32 are nonnegative by Lemma 13. So the derivative of the area bounded by C must be positive
29:33 for all but a finite number of times $0 \leq t \leq 1$. Thus the area must strictly increase. \square

29:34 5.3 Topology of Configuration Spaces

29:35 It is natural to ask more about the structure of the *configuration space* of an arc-and-cycle
29:36 set. Let $X(G, L)$ denote the space of all configurations of embeddings in the plane of a

30:01 bar graph G consisting of a finite number arcs and cycles, without self-intersections, where
30:02 the edge lengths are determined by $L = (\dots, \ell_{ij}, \dots)$. This inherits a natural topology from
30:03 considering all the coordinates of all the vertices as part of a large dimensional Euclidean
30:04 space. Let $X_0(G, L) \subset X(G, L)$ denote the subspace of outer-convex configurations. We
30:05 assume that L is chosen so that there is at least one realization in the plane. We mention
30:06 some results without proof.

30:07 **Theorem 8** *The space of outer-convex realizations $X_0(G)$ is a strong deformation retract*
30:08 *of $X(G)$.*

30:09 The main point to remember is that the limit in Theorem 1 depends continuously on the
30:10 initial starting configuration. The following is a natural consequence of Theorem 8.

30:11 **Corollary 3** *If the underlying graph G is a single arc or a single cycle, then $X(G, L)$ modulo*
30:12 *congruences (including orientation reversing ones) is contractible.*

30:13 Here the main task is to show that the space of convex realizations is contractible.

30:14 It is interesting to compare $X(G, L)$, as we have defined it, to the space of realizations
30:15 of an arc or cycle in the plane with fixed edge lengths, but where crossings are allowed. See
30:16 e.g. [LW95, KM99, KM95a, KM95b, KM96] for results about this space.

30:17 5.4 Open Problems

30:18 Another direction is to explore what happens when the arc-and-cycle set is allowed to touch
30:19 but not cross:

30:20 **Conjecture 1** *If G is a single arc or a single cycle, then the closure of $X(G, L)$ modulo*
30:21 *congruences is contractible.*

30:22 We conjecture that motions can be realized by a sequence of relatively simple motions:

30:23 **Conjecture 2** *If A is an arc-and-cycle set in the plane, then there is a motion that takes it*
30:24 *to an outer-convex configuration, by a finite sequence of motions, where each motion changes*
30:25 *at most four vertex angles.*

30:26 It also remains open precisely how many such moves are needed.

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A Appendix: Expansive Motion for Several Components

The motion described so far is expansive in each component, but may not expand the distance between vertices on different components once those components separate. In this final section we prove the following stronger version of Theorem 1 that guarantees a strictly expansive motion for multiple components even after the components separate. This result is attractive because it allows us to specify a canonical motion all the way to completion. Unfortunately, the proof is long.

Theorem 9 *Every arc-and-cycle set has a piecewise differentiable proper motion to an outer-convex configuration that is strictly expansive throughout the motion.*

Proof: As mentioned in Section 4.2, the motion is defined in the same way as for Theorem 1, except that the explicit pinning constraint (10) is removed from the definition of $\mathbf{v} = f(\mathbf{p})$.

As in the case of Theorem 1, we know that the initial value problem (12) has a unique maximal solution $\mathbf{p}(t)$, $0 \leq t < T$, that cannot be extended beyond T , and we have $T < \infty$. We have to show that, for $t \rightarrow T$, the configuration $\mathbf{p}(t)$ converges to a configuration on the boundary of U . From the Boundedness Lemma (Lemma 10 in Section 4.2), whose proof is given below, we conclude that the monotone limit

$$K_i := \lim_{t \rightarrow T} \int_0^t \|\mathbf{v}_i\| dt$$

exists. Therefore, for all $t_2 \geq t_1 \geq t_0$, we have

$$\|\mathbf{p}_i(t_2) - \mathbf{p}_i(t_1)\| \leq \int_{t_1}^{t_2} \|\mathbf{v}_i(t)\| dt \leq K_i - \int_0^{t_0} \|\mathbf{v}_i(t)\| dt.$$

The last bound goes to zero as $t_0 \rightarrow T$, and therefore $\mathbf{p}_i(t)$ converges. The limit configuration must lie on the boundary of the domain U because otherwise the solution of (12) could be extended beyond T . \square

The rest of this appendix proves the only remaining aspect: the Boundedness Lemma (Lemma 10 in Section 4.2). That is, we would like to show that, under the expansive motion given by the optimization problem (7–9), the motion of individual points is limited.

Before embarking on the proof, let us consider a possibly difficult situation, which the proof has to handle. This example may aid the reader in appreciating the arguments of the proof.

We know that the distances change monotonically, and therefore their total change is bounded if the distances remain bounded. However, it is conceivable that the points move back and forth without increasing the distances too much. Consider three collinear points \mathbf{p}_1 , \mathbf{p} and \mathbf{p}_2 on a horizontal line; see Figure 13. If the point \mathbf{p} wiggles vertically up and down with a small and decreasing amplitude, one can maintain an expansive motion where all distances remain bounded and yet \mathbf{p} moves on a path of infinite length. More precisely, we move \mathbf{p} repeatedly up and down, each time returning to its original location, with the

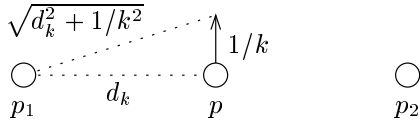


Figure 13: Wiggling point \mathbf{p} vertically up and down by harmonic distances allows \mathbf{p} to travel infinitely while pushing \mathbf{p}_1 and \mathbf{p}_2 away by only a bounded amount.

38:01 amount of vertical motion decreasing harmonically: $1, 1/2, 1/3, 1/4$, etc. Suppose the
 38:02 distance between \mathbf{p} and both \mathbf{p}_1 and \mathbf{p}_2 is initially $d_1 = 1$, and after $k - 1$ wiggles is $d_k \geq 1$.
 38:03 Then when \mathbf{p} moves up $1/k$ units, the distance between \mathbf{p} and \mathbf{p}_i increases from d_k to

$$38:04 \quad d_{k+1} = \sqrt{d_k^2 + 1/k^2} = d_k + \frac{1}{2d_k k^2} + O\left(\frac{1}{d_k^3 k^4}\right) \leq d_k + \frac{1}{k^2} + O\left(\frac{1}{k^4}\right).$$

38:05 When \mathbf{p} moves back down $1/k$ units, we move \mathbf{p}_1 left and move \mathbf{p}_2 right by this amount, to
 38:06 keep the motion expansive. After infinitely many wiggles, the distance between \mathbf{p} and \mathbf{p}_i is
 38:07 at most

$$38:08 \quad \sum_{k=1}^{\infty} \left(\sqrt{d_k^2 + 1/k^2} - d_k \right) \leq \sum_{k=1}^{\infty} \left(\frac{1}{k^2} + O\left(\frac{1}{k^4}\right) \right) = O(1)$$

38:09 while the total motion of \mathbf{p} is

$$38:10 \quad \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

38:11 This conceivable situation must be avoided.

38:12 We may get arbitrarily close to this situation if all points are very nearly collinear. We
 38:13 can however get a quantitative estimate of the motion by placing bounds on the minimum
 38:14 width and on the diameter of the point set. If the angles in the triangle $\mathbf{p}_1\mathbf{p}\mathbf{p}_2$ are not
 38:15 arbitrarily small, the motion of \mathbf{p} relative to \mathbf{p}_1 and \mathbf{p}_2 can be bounded in terms of the
 38:16 increase of pairwise distances.

38:17 A.1 Bounding the motion in terms of angles and distance changes

38:18 We give two geometric lemmas that allow us to bound the motions in terms of quantities
 38:19 that can only change monotonically.

38:20 **Lemma 14** *Let $\mathbf{p}_1, \dots, \mathbf{p}_n$ be the sequence of vertices of a component with the edges of fixed*
 38:21 *length $\|\mathbf{p}_j - \mathbf{p}_{j+1}\| = \ell_{j,j+1}$ for $j = 1, \dots, n - 1$. (These constraints do not include the edge*
 38:22 *from \mathbf{p}_n to \mathbf{p}_1 in the case of a polygonal cycle.) Say \mathbf{p}_1 and \mathbf{p}_2 are pinned, i.e., $\mathbf{v}_1 = \mathbf{v}_2 = 0$.*
 38:23 *Let θ_j be the internal angle at vertex j ($0 < \theta_j < \pi$; we disregard the orientation of the*
 38:24 *angles). For any motion that preserves the lengths $\ell_{j,j+1}$, let θ'_j denote the derivative of θ_j*
 38:25 *with respect to time, and let \mathbf{v}_k denote the derivative of \mathbf{p}_k with respect to time. Then, for*
 38:26 *$k = 2, \dots, n$,*

$$38:27 \quad \|\mathbf{v}_k\| \leq (\ell_{1,2} + \dots + \ell_{k-1,k}) \cdot (\theta'_2 + \dots + \theta'_{k-1})$$

39:01 **Proof:** Using complex notation,

$$39:02 \quad \frac{\mathbf{p}_{j+1} - \mathbf{p}_j}{\ell_{j,j+1}} = \frac{\mathbf{p}_{j-1} - \mathbf{p}_j}{\ell_{j-1,j}} \cdot \exp(\pm\theta_j i).$$

39:03 Taking derivatives gives

$$39:04 \quad \frac{\mathbf{v}_{j+1} - \mathbf{v}_j}{\ell_{j,j+1}} = \frac{\mathbf{v}_{j-1} - \mathbf{v}_j}{\ell_{j-1,j}} \cdot \exp(\pm\theta_j i) + \frac{\mathbf{p}_{j-1} - \mathbf{p}_j}{\ell_{j-1,j}} \cdot \exp(\pm\theta_j i) \cdot (\pm\theta'_j i).$$

39:05 Taking the norm of both sides and using the triangle inequality yields

$$39:06 \quad \frac{\|\mathbf{v}_{j+1} - \mathbf{v}_j\|}{\ell_{j,j+1}} \leq \frac{\|\mathbf{v}_{j-1} - \mathbf{v}_j\|}{\ell_{j-1,j}} \cdot \underbrace{|\exp(\pm\theta_j i)|}_{=1} + \underbrace{\frac{\|\mathbf{p}_{j-1} - \mathbf{p}_j\|}{\ell_{j-1,j}}}_{=1} \cdot \underbrace{|\exp(\pm\theta_j i)|}_{=1} \cdot |\theta'_j i| = \frac{\|\mathbf{v}_{j-1} - \mathbf{v}_j\|}{\ell_{j-1,j}} + |\theta'_j|.$$

39:07 Applying this fact recursively,

$$39:08 \quad \frac{\|\mathbf{v}_{j+1} - \mathbf{v}_j\|}{\ell_{j,j+1}} \leq \frac{\|\mathbf{v}_{j-1} - \mathbf{v}_j\|}{\ell_{j-1,j}} + |\theta'_j| \leq \dots \leq |\theta'_2| + \dots + |\theta'_j|,$$

39:09 and thus

$$39:10 \quad \|\mathbf{v}_{j+1} - \mathbf{v}_j\| \leq \ell_{j,j+1} \cdot (|\theta'_2| + \dots + |\theta'_j|) \leq \ell_{j,j+1} \cdot (|\theta'_2| + \dots + |\theta'_{k-1}|),$$

39:11 for $j \leq k$. Applying induction we obtain the desired result. \square

39:12 The above lemma bounds the motion for a single component. For disconnected compo-
 39:13 nents, we want to bound the speed of motion \mathbf{v} in terms of distance changes. However, it
 39:14 is always possible to move the point set as a whole without changing any distances at all.
 39:15 Therefore we will consider a modified motion in which one vertex \mathbf{p}_1 is fixed (“pinned”).
 39:16 Another vertex \mathbf{p}_2 remains on a fixed line through \mathbf{p}_1 , preventing the complete set from
 39:17 spinning. This modification corresponds to adding a rigid motion of the point set \mathbf{p} to the
 39:18 given velocities \mathbf{v} . It does not change the lengths ℓ_{ij} or their derivatives, $(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j)$.

39:19 The *width* of a set of points is the minimum width of a strip (infinitely long rectangle)
 39:20 containing the set.

39:21 **Lemma 15** Consider a set $\mathbf{p}_1, \dots, \mathbf{p}_n$ of $n \geq 3$ points with distances $\ell_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|$, moving
 39:22 with velocities $\mathbf{v}_i = \mathbf{p}_i'$ under an expansive motion: $\ell'_{ij} \geq 0$. Denote the initial width by w .
 39:23 Without loss of generality, we assume that the diameter D is equal to ℓ_{12} . We normalize the
 39:24 motion by pinning \mathbf{p}_1 and the direction of the axis from \mathbf{p}_1 to \mathbf{p}_2 . In other words, we assume
 39:25 that $\mathbf{v}_1 = 0$ and that \mathbf{v}_2 is parallel to $\mathbf{p}_2 - \mathbf{p}_1$. Then, for every $k \in \{1, \dots, n\}$,

$$39:26 \quad \|\mathbf{v}_k\| \leq C_0 \cdot \sum_{i < j} \ell'_{ij},$$

39:27 where $C_0 = 666D^3/w^3$.

39:28 The proof of this lemma is tedious, and delayed to Section A.5 at the end.

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A.2 Constant Quantities

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We define the following constants. We denote by D_0 the maximum sum of the bar lengths in a single component of the arc-and-cycle set. This is an upper bound on the diameter of a component. As we shall see, $D_0 6^{n-1}$, denoted henceforth as D_{\max} , is an upper bound on the distance between two points in the same cluster of components (a notion defined in the next subsection). Let m_0 ($0 < m_0 \leq 1$) be a lower bound on the distance between any two different components throughout the motion (achieved either as a distance between two vertices or as a distance from a vertex to an edge). Let m_1 ($0 < m_1 \leq 1$) be a lower bound on the distance between any two vertices (in the same component or in different components) throughout the motion. Because these distances can only increase during the expansive motion, we just need to choose m_0 and m_1 smaller than the minimum distance between two different components or between two vertices in the initial configuration.

The constant $C = 5328D_{\max}^3/m_0^3 = 8 \cdot 666D_{\max}^3/m_0^3$ will play an important role in Lemma 19 and subsequently.

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A.3 Combining components into clusters

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In the proof we distinguish between components that are “close” to each other or even entangled within each other and components that are separated. Namely, we call two nonempty sets of points A and B *well-separated* if they can be separated by a line, and moreover, the two inner common tangents separating A and B form an angle less than 60° . See Figure 14(a).

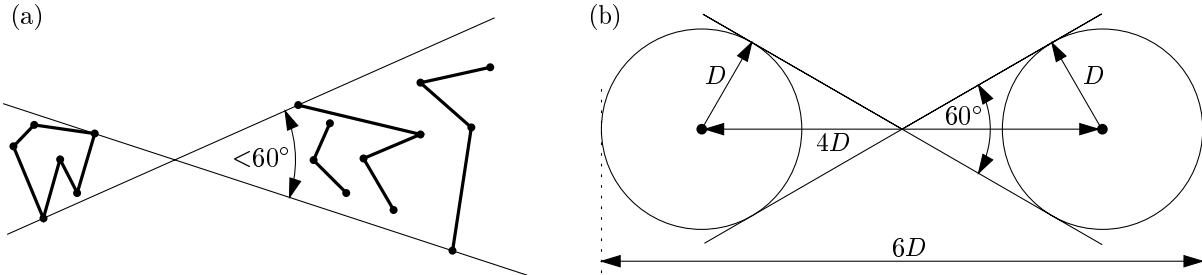


Figure 14: (a) Two well-separated clusters. (b) Inductive proof of Lemma 16. $D = D_0 6^{k-2}$ denotes the bound on the diameter of the two subclusters.

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We now define a clustering procedure that will combine the components into clusters. (This procedure is similar to bottom-up hierarchical clustering [Eve93, HJ97], except that we pay no attention to the order in which we merge clusters.) We start with each component as a separate cluster. Whenever two clusters are not well-separated, we merge them into a single cluster. We repeat this until we end up with a single cluster or with a set of clusters that are pairwise well-separated (because otherwise we would perform another merge). The order in which we merge clusters has no effect on the outcome, because merging clusters may destroy well-separated pairs but never creates them, and hence only creates mergeable pairs and never destroys them. However, the clustering may change over time as the framework moves.

40:30

Lemma 16 *A cluster with k components has diameter at most $D_0 6^{k-1}$.*

41:01 **Proof:** By induction on k . Because a single component is connected, the case $k = 1$
41:02 follows. A cluster with $k \geq 2$ components results from merging two clusters of at most $k - 1$
41:03 components that are not well-separated. By induction, these two clusters are contained in
41:04 two disks of radius at most $D_0 6^{k-2}$. If the distance between the disk centers were bigger
41:05 than $4 \cdot D_0 6^{k-2}$, the two sets would be well-separated; see Figure 14(b). Hence the union of
41:06 the two circles, which contains the cluster, has diameter at most $(2 + 4) \cdot D_0 6^{k-2} = D_0 6^{k-1}$.
41:07 \square

41:08 So the quantity $D_{\max} = D_0 6^{n-1}$, which has already been introduced, is indeed an upper
41:09 bound on the maximum possible distance between two points in the same cluster.

41:10 **Lemma 17** *A cluster with at least two components has width at least $m_0/2$.*

41:11 **Proof:** Assume for contradiction that a cluster with at least two components lies in a strip
41:12 of width less than $m_0/2$, which we can without loss of generality assume to be horizontal,
41:13 as in Figure 15. Consider any two components in the cluster. By the definition of m_0 ,
41:14 the distance between these two components is at least $m_0 > m_0/2$. Hence, there must
41:15 be a vertical line separating the two components. Consider one of the two inner common
41:16 tangents. The segment between the tangent points has length at least m_0 and lies in the
41:17 strip. Therefore the angle between the tangent and the horizontal axis is less than 30° , and
41:18 the same is true for the other tangent. It follows that any two components in the strip are
41:19 well-separated, and therefore they do not belong to the same cluster. \square

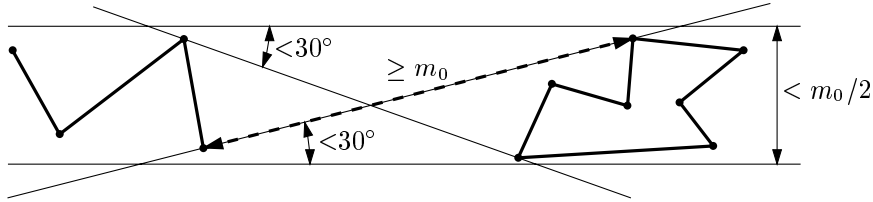


Figure 15: Proof of Lemma 17: two components in a thin strip.

41:20 **Lemma 18** *Let $\mathbf{p}_1 \in V_1$ and $\mathbf{p}_2 \in V_2$ be points in two well-separated clusters V_1 and V_2 , and*
41:21 *let $\bar{\mathbf{c}}^{(1)}$ and $\bar{\mathbf{c}}^{(2)}$ denote the center of gravity of the vertices in V_1 and V_2 , respectively. Then*
41:22 *the angle between the vectors $\mathbf{p}_2 - \mathbf{p}_1$ and $\bar{\mathbf{c}}^{(2)} - \bar{\mathbf{c}}^{(1)}$ is at most 60° .*

41:23 **Proof:** Immediate from the definition of well-separated, because the center of gravity of a
41:24 cluster is within the convex hull of the cluster. \square

41:25 A.4 Rounding up the proof

41:26 Using Lemmas 14 and 15, we can bound all velocities \mathbf{v}_i in terms of the sum of ℓ'_{ij} over all pairs
41:27 $(\mathbf{p}_i, \mathbf{p}_j)$ in the *same* cluster (plus some θ' terms to accommodate single-component clusters).
41:28 These quantities cannot grow indefinitely: we have an upper bound on the diameter of a
41:29 cluster.

42:01 **Lemma 19** Let V_1 denote the set of vertices of a cluster and $T_1 \subseteq V_1$ the set of degree-2
42:02 vertices, with angles θ_j for $j \in T_1$, $0 < \theta_j < \pi$. Let ℓ_{ij} denote the distance between vertices
42:03 $i, j \in V_1$. Given an expansive motion of all vertices, let θ'_j denote the derivative of θ_j with
42:04 respect to time, $j \in T_1$, let ℓ'_{ij} denote the derivative of ℓ_{ij} with respect to time, and let \mathbf{v}_j
42:05 denote the derivative of a vertex \mathbf{p}_j with respect to time, $j \in V_1$. If the motion is pinned as in
42:06 Lemma 14 (for the case of a single-component cluster) or Lemma 15 (for a multi-component
42:07 cluster), then, for every $k \in V_1$,

$$\|\mathbf{v}_k\| \leq C \cdot \sum_{i,j \in V_1} \ell'_{ij} + D_0 \cdot \sum_{i \in T_1} \theta'_i$$

42:08 where $C = 5328D_{\max}^3/m_0^3 = 8 \cdot 666D_{\max}^3/m_0^3$.

42:09 **Proof:** The first term handles a cluster with at least two components by Lemma 15. We
42:10 have width $w \geq m_0/2$ by Lemma 17 and diameter $D \leq D_{\max}$ by Lemma 16. The second
42:11 term accounts for a single-component cluster and follows from Lemma 14. \square
42:12

42:13 Everything so far has been generic to expansive motions. Now we will use the precise
42:14 form of (7–9).

42:15 **Lemma 20** For an arc-and-cycle set, let $T \subseteq V$ denote the set of degree-two vertices, and let
42:16 θ_j denote the angle at vertex $j \in T$, $0 < \theta_j < \pi$. For a motion $\mathbf{p}_j(t)$ of this arc-and-cycle set,
42:17 let $\ell_{ij} = \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\|$ denote distances between vertices, let $D(t) = \max\{1, \max_{i,j \in V} \ell_{ij}(t)\}$
42:18 denote the diameter at time t raised to be at least one, and let $\bar{\ell}_{ij}(t) = \min\{\ell_{ij}(t), D_{\max}\}$
42:19 denote the “truncated” distances. Then the solution \mathbf{v}_i of the minimization problem (7–9)
42:20 (defining a motion of the arc-and-cycle set) is bounded by

$$\|\mathbf{v}_k\| \leq \frac{6n}{m_0 m_1} \left(C \sum_{i,j \in V} \bar{\ell}'_{ij} + D_0 \sum_{i \in T} \theta'_i + 2 \right) \cdot D(t)$$

42:21 **Proof:** The strategy of the proof is as follows:

- 42:22 1. Start with \mathbf{v}_i given by the optimization problem (7–9).
- 42:23 2. In each cluster separately, “normalize” (“pin”) \mathbf{v}_i as indicated in Lemma 19, yielding
42:24 “well-bounded” \mathbf{v}_i ’s for each cluster.
42:25
- 42:26 3. Combine these pinned motions into an overall expansive motion by adding a vector to
42:27 each cluster that ensures that the clusters will fly away from each other. We have con-
42:28 trol over those vectors, and we guarantee that the overall motion is still well-bounded
42:29 (bounded in terms of the sum of ℓ'_{ij} over all pairs i, j in the same cluster).
- 42:30 4. Thus we have shown the existence of a well-bounded solution. Using the minimizing
42:31 property of the optimization problem we conclude that its solution is also well-bounded.

43:01 Here are the details. Assume that there are K clusters with vertex sets V_1, \dots, V_K . We first
 43:02 consider the motion $\hat{\mathbf{v}}^{(r)}$ that is obtained from \mathbf{v} by restricting it to a cluster V_r and pinning
 43:03 it as in Lemma 14 or 15. We know that

$$43:04 \quad \|\hat{\mathbf{v}}_k^{(r)}\| \leq C \cdot \underbrace{\sum_{i,j \in V_r} \bar{\ell}_{ij} + D_0 \cdot \sum_{i \in T_r} \theta'_i}_{\text{denoted by } H^{(r)}} \quad (25)$$

43:05 for all $k \in V_r$.

43:06 By construction, all distance increases $\|\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(r)}\|$ satisfy the conditions (8–9) for i, j
 43:07 in the same cluster V_r . We will now add a vector to the motion of each cluster in order to
 43:08 ensure that the expansion (8) also holds for points in different clusters. Intuitively, we want
 43:09 the clusters to “fly apart,” but in a controlled manner.

43:10 Let $\bar{\mathbf{c}}^{(r)}$ denote the center of gravity of cluster r . We set

$$43:11 \quad \bar{\mathbf{v}}_k := 2 \cdot \hat{\mathbf{v}}_k^{(r)} + E \cdot (\bar{\mathbf{c}}^{(r)} - \mathbf{p}_1) \quad (26)$$

43:12 for $k \in V_r$, with an arbitrarily chosen fixed vertex \mathbf{p}_1 and where E is defined by

$$43:13 \quad E = 2 (H^{(1)} + H^{(2)} + \dots + H^{(K)} + 2) / m_0 > 4.$$

43:14 Note that $m_0 E / 2$ is the value in parentheses in the bound of Lemma 20 that we want to
 43:15 prove. For vertices in the same cluster, we get $\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j = 2(\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(r)})$, and hence the
 43:16 constraint

$$43:17 \quad (\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) \geq \|\mathbf{p}_i - \mathbf{p}_j\| \quad (8)$$

43:18 is “twice over-fulfilled”:

$$43:19 \quad (\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) \geq 2 \cdot \|\mathbf{p}_i - \mathbf{p}_j\|$$

43:20 So we get the penalty term

$$43:21 \quad [(\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_i - \mathbf{p}_j\|]^{-1} \leq [\|\mathbf{p}_i - \mathbf{p}_j\|]^{-1} \leq 1/m_1. \quad (27)$$

43:22 We claim that this inequality also holds for points i and j in two different clusters V_r and V_s .
 43:23 We have

$$43:24 \quad \begin{aligned} (\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) &= (\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(s)}) \cdot (\mathbf{p}_i - \mathbf{p}_j) + E (\bar{\mathbf{c}}^{(r)} - \bar{\mathbf{c}}^{(s)}) \cdot (\mathbf{p}_i - \mathbf{p}_j) \\ 43:25 &\geq |E (\bar{\mathbf{c}}^{(r)} - \bar{\mathbf{c}}^{(s)}) \cdot (\mathbf{p}_i - \mathbf{p}_j)| - \left| (\hat{\mathbf{v}}_i^{(r)} - \hat{\mathbf{v}}_j^{(s)}) \cdot (\mathbf{p}_i - \mathbf{p}_j) \right| \\ 43:26 &\geq E \cdot \underbrace{\frac{1}{2} \|\bar{\mathbf{c}}^{(r)} - \bar{\mathbf{c}}^{(s)}\| \cdot \|\mathbf{p}_i - \mathbf{p}_j\|}_{\text{by Lemma 18}} - (H^{(r)} + H^{(s)}) \cdot \|\mathbf{p}_i - \mathbf{p}_j\| \\ &\geq \|\mathbf{p}_i - \mathbf{p}_j\| [Em_0/2 - (H^{(r)} + H^{(s)})] \\ 43:27 &\geq 2 \cdot \|\mathbf{p}_i - \mathbf{p}_j\| \end{aligned}$$

43:29 by the definition of E . So $\bar{\mathbf{v}}$ is indeed a feasible solution to (8–9), and (27) holds for all
 43:30 struts $\{i, j\} \in S$. From (25) and $\|\bar{\mathbf{c}}^{(r)} - \mathbf{p}_1\| \leq D(t)$ we obtain the bound

$$43:31 \quad \|\bar{\mathbf{v}}_k\| \leq 2H^{(r)} + E \cdot D(t) \leq Em_0 + E \cdot D(t) \leq 2E \cdot D(t)$$

44:01 because we defined $D(t)$ to be at least 1.

44:02 From this and from (27), the objective function (7) for the solution $\bar{\mathbf{v}}$ that we have
 44:03 constructed satisfies

$$44:04 \sum_{i \in V} \|\bar{\mathbf{v}}_i\|^2 + \sum_{\{i,j\} \in S} [(\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_j - \mathbf{p}_i\|]^{-1} \leq n \cdot (2E \cdot D(t))^2 + n^2/m_1$$

$$44:05 \leq n^2 \cdot (2E \cdot D(t))^2 \cdot \frac{1}{m_1} + \frac{n^2}{m_1} \cdot E^2 \cdot D(t)^2 = \frac{5n^2}{m_1} \cdot E^2 \cdot D(t)^2 \leq \left(\frac{3n}{m_1} \cdot E \cdot D(t) \right)^2. \quad (28)$$

44:06 It follows that the optimal solution \mathbf{v} , from which we started, must also satisfy this inequality.
 44:07 Therefore we obtain,

$$44:08 \sum_{i \in V} \|\mathbf{v}_i\|^2 \leq \sum_{i \in V} \|\bar{\mathbf{v}}_i\|^2 + \sum_{\{i,j\} \in S} [(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) - \|\mathbf{p}_j - \mathbf{p}_i\|]^{-1} \leq \left(\frac{3n}{m_1} \cdot E \cdot D(t) \right)^2.$$

44:09 and hence, for every $k \in V$,

$$44:10 \|\mathbf{v}_k\| \leq \sqrt{\sum_{i \in V} \|\mathbf{v}_i\|^2} \leq \frac{3n}{m_1} \cdot E \cdot D(t) \quad \square$$

44:11 **Corollary 4** $D(t) \leq D(0)e^{F(t)}$ where $F(t) = \frac{12n}{m_0 m_1} (Cn^2 D_{\max} + D_0 n \pi + 2t)$.

44:12 **Proof:** By noting the relation $D'(t) \leq 2 \max_{k \in V} \|\mathbf{v}_k\|$, we can apply the previous lemma to
 44:13 obtain an upper bound on $D'(t)/D(t) = \frac{d}{dt} \log D(t)$:

$$44:14 \frac{D'(t)}{D(t)} \leq \frac{6nE}{m_1} \leq \frac{12n}{m_0 m_1} \left(C \sum_{i,j \in V} \bar{\ell}'_{ij} + D_0 \sum_{i \in T} \theta'_i + 2 \right).$$

44:15 Integrating and noting the bounds $\bar{\ell}'_{ij} \leq D_{\max}$ and $\theta_i \leq \pi$, we obtain

$$44:16 \log D(t) - \log D(0) = \int_0^t \frac{D'(t)}{D(t)} dt \leq \frac{12n}{m_0 m_1} (Cn^2 D_{\max} + D_0 n \pi + 2t). \quad \square$$

44:17 We finally establish the result of the Boundedness Lemma (Lemma 10).

44:18 **Corollary 5** *The path length of a vertex \mathbf{p}_k until time T is bounded by*

$$44:19 \int_{t=0}^T \|\mathbf{v}_k\| dt \leq \frac{D(0)F(T)}{2} e^{F(T)}$$

44:20 **Proof:** This follows from Lemma 20 and the previous corollary:

$$44:21 \int_0^T \|\mathbf{v}_k(t)\| dt \leq \int_0^T \frac{3nE}{m_1} D(t) dt \leq \int_0^T \frac{3nE}{m_1} dt \cdot D(T) \leq \frac{F(T)}{2} \cdot D(0) \cdot e^{F(T)} \quad \square$$

44:22 **Remark:** By a different choice of shifting vectors in (26) instead of $E \cdot \bar{\mathbf{c}}^{(r)}$ for each
 44:23 cluster it is possible to avoid the exponential “blow-up” in the above bounds.

45:01

A.5 Proof of Lemma 15

45:02

Unfortunately, the proof of Lemma 15 is quite elaborate. It is based on the special case of three points, which is elementary.

45:03

45:04

Lemma 21 Consider a triangle with sides a, b, c bounded by $m \leq a, b, c \leq D$ and angles α, β, γ that are bounded by $\sin \alpha, \sin \beta, \sin \gamma \geq \mu > 0$. If the sides of the triangle are expanded with velocities $a', b', c' \geq 0$, then the change of the angles is bounded by

45:05

45:06

$$|\alpha'|, |\beta'|, |\gamma'| \leq \frac{2D}{m^2\mu} \cdot (a' + b' + c')$$

45:07

45:08

Proof: The cosine law gives $-\cos \alpha = -(b^2 + c^2 - a^2)/(2bc) = \frac{1}{2}[a^2/(bc) - b/c - c/b]$. Taking the derivative and omitting negative terms on the right-hand side leads to

45:09

45:10

$$\alpha' \sin \alpha \leq \frac{1}{2} \left(\frac{2aa'}{bc} + \frac{bc'}{c^2} + \frac{cb'}{b^2} \right) \leq \frac{1}{2} (2a' + c' + b') \frac{D}{m^2} \leq (a' + c' + b') \frac{D}{m^2}$$

45:11

and thus, $\alpha' \leq (a' + b' + c') \cdot D/(m^2\mu)$. We get the same upper bound for β' and γ' . To obtain a lower bound, note that $\alpha + \beta + \gamma = \pi$ and hence $\alpha' = -\beta' - \gamma' \geq -(a' + b' + c') \cdot 2D/(m^2\mu)$. By symmetry, the same bound holds for the derivatives of the other two angles. \square

45:12

45:13

45:14

Proof (Lemma 15): Let \mathbf{p}_3 be the point whose distance h from the segment $\mathbf{p}_1\mathbf{p}_2$ is largest. By assumption, $w/2 \leq h \leq \sqrt{3/4} \cdot D$. So the points are contained in a rectangle K of length D and width $2h$, see Figure 16. Without loss of generality, we draw $\mathbf{p}_1\mathbf{p}_2$ horizontal from left to right and \mathbf{p}_3 above $\mathbf{p}_1\mathbf{p}_2$.

45:15

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45:17

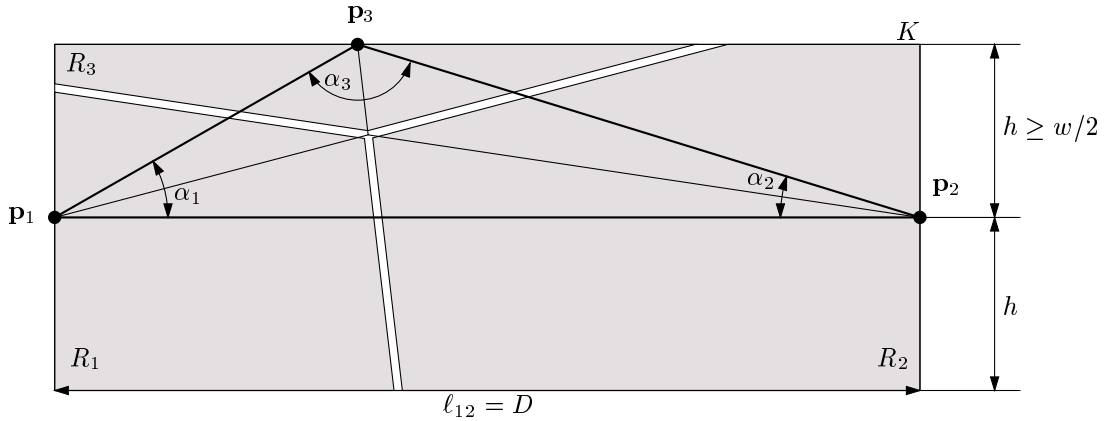


Figure 16: Proof of Lemma 15.

45:18

We use the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ with angles $\alpha_1, \alpha_2, \alpha_3$ as a “reference frame,” and we measure the motion of the other points with respect to this frame. In particular, every point will be assigned to one of the sides of the reference triangle: We partition the rectangle K into

45:19

45:20

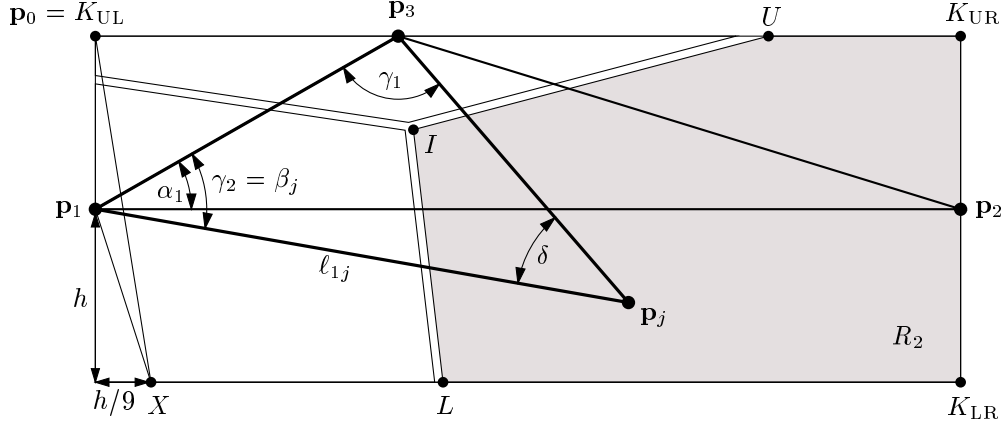


Figure 17: The triangle $\mathbf{p}_3\mathbf{p}_1\mathbf{p}_j$ associated with a point $\mathbf{p}_j \in R_2$ in the proof of Lemma 22.

46:01 three regions R_1, R_2, R_3 by extending the three angular bisectors from the incenter in the
 46:02 direction away from the vertices from which they emanate. R_i is the region containing the
 46:03 point \mathbf{p}_i , for $i = 1, 2, 3$. Now, every point \mathbf{p}_j is assigned to the triangle side which lies
 46:04 *opposite* to the region in which it lies. For example, a point \mathbf{p}_j in region R_1 is assigned to
 46:05 $\mathbf{p}_2\mathbf{p}_3$. We use this side as a *basis* of the triangle $\mathbf{p}_2\mathbf{p}_3\mathbf{p}_j$ that is associated with \mathbf{p}_j , and we
 46:06 select one of the angles at the basis as the *reference angle* β_j for \mathbf{p}_j . If \mathbf{p}_j lies in R_1 , we select
 46:07 the angle at \mathbf{p}_2 in the triangle $\mathbf{p}_2\mathbf{p}_3\mathbf{p}_j$. If \mathbf{p}_j lies in R_2 , we take the angle at \mathbf{p}_1 in the triangle
 46:08 $\mathbf{p}_3\mathbf{p}_1\mathbf{p}_j$ as the reference angle β_j for \mathbf{p}_j , see Figure 17; and for \mathbf{p}_j in R_3 , β_j is the angle at
 46:09 \mathbf{p}_1 in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_j$. Note that these assignments are not completely symmetric. The
 46:10 rationale is to select a reference angle incident to \mathbf{p}_1 or \mathbf{p}_2 .

46:11 The idea is that the triangle associated with each point cannot have arbitrarily small
 46:12 angles, and hence produces a “firm grip,” tying the point to its base: The point cannot move
 46:13 an unlimited distance without increasing the pairwise distances ℓ_{ij} by a noticeable amount.

46:14 **Lemma 22** *The reference triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ satisfies the assumptions of Lemma 21: The sines*
 46:15 *of the angles are at least $\mu := w/(3D)$ and the sides are at least $m := w/2$.*

46:16 *The triangle associated with each point satisfies the assumptions of Lemma 21: The sines*
 46:17 *of the angles are at least $\mu := w/(20D)$ and the sides are at least $m := w/4$.*

46:18 This lemma is not difficult to see but the precise proof is a bit technical and it is given
 46:19 at the end. Lemmas 21 and 22 allow us to conclude:

46:20
$$|\alpha'_1|, |\alpha'_3|, |\alpha'_3| \leq \frac{2D}{(w/2)^2 \cdot w/(3D)} \cdot (\ell'_{12} + \ell'_{23} + \ell'_{13}) \leq \frac{24D^2}{w^3} \cdot L,$$

46:21 with $L := \sum_{i < j} \ell'_{ij}$, and for every point \mathbf{p}_j :

46:22
$$|\beta'_j| \leq \frac{2D}{(w/4)^2 \cdot w/(20D)} \cdot L \leq \frac{640D^2}{w^3} \cdot L.$$

47:01 Now we can bound the motion of \mathbf{p}_j as follows. For $\mathbf{p}_j \in R_3$, the angle β_j between the
 47:02 fixed ray $\mathbf{p}_1\mathbf{p}_2$ and the ray $\mathbf{p}_1\mathbf{p}_j$, and the length ℓ_{1j} determine the position of \mathbf{p}_j like polar
 47:03 coordinates. We therefore have

$$47:04 \quad \|\mathbf{p}'_j\| = \sqrt{(\ell_{1j}\beta'_j)^2 + (\ell'_{1j})^2} \leq \ell_{1j}|\beta'_j| + \ell'_{1j}$$

47:05 For $\mathbf{p}_j \in R_2$, the situation is similar, except that the angle between the “ x -axis” $\mathbf{p}_1\mathbf{p}_2$ and
 47:06 $\mathbf{p}_1\mathbf{p}_j$ is given by $\alpha_1 - \beta_j$, see Figure 17:

$$47:07 \quad \|\mathbf{p}'_j\| \leq \ell_{1j}(|\alpha'_1| + |\beta'_j|) + \ell'_{1j}$$

47:08 For $\mathbf{p}_j \in R_1$, the situation still more complicated, because the “origin” \mathbf{p}_2 of the coordinate
 47:09 system moves at a speed of ℓ'_{12} . The angle between the “negative x -axis” $\mathbf{p}_2\mathbf{p}_1$ and $\mathbf{p}_2\mathbf{p}_j$ is
 47:10 given by $\alpha_2 - \beta_j$.

$$47:11 \quad \|\mathbf{p}'_j\| \leq \|\mathbf{p}'_2\| + \|\mathbf{p}'_1 - \mathbf{p}'_2\| \leq \ell'_{12} + \ell_{2j}(|\alpha'_2| + |\beta'_j|) + \ell'_{2j}$$

47:12 Putting everything together, we get for every j ,

$$47:13 \quad \|\mathbf{v}_j\| = \|\mathbf{p}'_j\| \leq L + D \cdot \left(\frac{24D^2}{w^3} \cdot L + \frac{640D^2}{w^3} \cdot L \right) + L = \left(2 + \frac{664D^3}{w^3} \right) \cdot L \leq \frac{666D^3}{w^3} \cdot L,$$

47:14 using $D \geq w$. This concludes the proof of Lemma 15. \square

47:15 Finally, we have to prove Lemma 22. An elementary geometric fact is needed.

47:16 **Lemma 23** *In a triangle ABC with height h_C through C , the distance of C from the angular*
 47:17 *bisector g of the angle α at A is at least $h_C/2$.*

47:18 **Proof:** Let C' be the foot of h_C , and let D be the point on g closest to C , see Figure 18.
 47:19 Then C' and D lie on the circle over the diameter AC . The chords CD and $C'D$ are equal
 47:20 since their peripheral angles at A are both equal to $\alpha/2$. (If $\alpha > \pi/2$, the peripheral angle
 47:21 over $C'D$ is not $\alpha/2$ but $\pi - \alpha/2$, and for $\alpha = \pi/2$ the equality of the chords follows directly.)
 47:22 From the isosceles triangle $CC'D$ we get $CD \geq h_C/2$. \square

47:23 **Proof (Lemma 22):** We first prove the lower bound on the lengths of the edges. The
 47:24 basis edges $\mathbf{p}_1\mathbf{p}_2$, $\mathbf{p}_2\mathbf{p}_3$, and $\mathbf{p}_3\mathbf{p}_1$ all have lengths at least $h \geq w/2$. By the construction of
 47:25 the three regions, the point \mathbf{p}_j is separated from the point \mathbf{p}_i on its base segment ($i = 1, 2$,
 47:26 or 3) by the angular bisector of the angle at \mathbf{p}_i in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. From Lemma 23 it
 47:27 follows that the distance $\mathbf{p}_j\mathbf{p}_i$ is at least $1/2$ times one of the three heights of the triangle
 47:28 $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. The smallest height of this triangle is opposite to the longest edge, which is $\mathbf{p}_1\mathbf{p}_2$.
 47:29 This height is h , and we obtain $\ell_{ij} \geq h/2 \geq w/4$.

47:30 Now let us consider the angles. Note that it suffices to check the condition of Lemma 21
 47:31 for the smallest angle: if α is the smallest angle in a triangle with angles α, β, γ , then
 47:32 $\alpha \leq \beta, \gamma \leq \pi - \alpha$, and hence $\sin \beta, \sin \gamma \geq \sin \alpha$.

47:33 The angle α_1 in the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$ is bigger than the angle between $\mathbf{p}_1\mathbf{p}_2$ and the ray
 47:34 from \mathbf{p}_1 to the upper right corner K_{UR} of the rectangle K :

$$47:35 \quad \sin \alpha_1 \geq \frac{h}{\sqrt{D^2 + h^2}} \geq \frac{h}{\sqrt{2D^2}} \geq \frac{w/2}{\sqrt{2} \cdot D} \geq \frac{w}{3D}$$

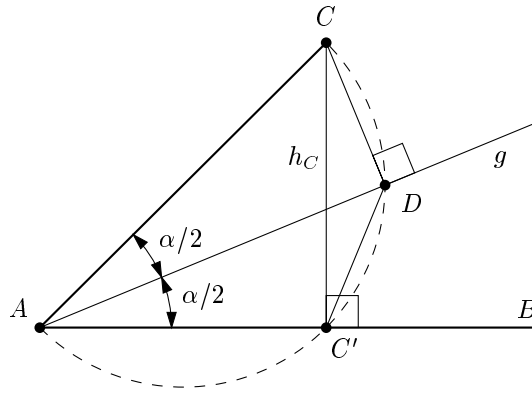


Figure 18: Proof of Lemma 23.

48:01 and similarly for α_2 . The angle α_3 lies opposite to the longest edge of $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$, and hence it
 48:02 is the largest angle. So we have shown the length and angle bounds for the reference triangle
 48:03 $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$.

48:04 Let us now consider the triangle associated with a point \mathbf{p}_j . We denote the basis angles
 48:05 of the triangle by γ_1 and γ_2 and the angle at the apex \mathbf{p}_j by δ , see Figure 17. By construction
 48:06 of the angular bisectors, an angle γ_j at the base is at least $1/2$ times the respective angle α_i
 48:07 in the reference triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$. So we have $\sin \gamma_1, \sin \gamma_2 \geq \sin \frac{\alpha_i}{2} \geq (\sin \alpha_i)/2 \geq w/(6D)$.

48:08 So the only angle which remains to be checked is the third angle δ at \mathbf{p}_j . If \mathbf{p}_j lies in R_3 ,
 48:09 δ lies opposite to the longest side of the triangle $\mathbf{p}_1\mathbf{p}_2\mathbf{p}_j$, and we are done.

48:10 In the remainder of the proof we show rather tediously that $\sin \delta \geq w/(20D)$ when
 48:11 $\mathbf{p}_j \in R_1$ or $\mathbf{p}_j \in R_2$. Let us consider a triangle $\mathbf{p}_3\mathbf{p}_1\mathbf{p}_j$ with $\mathbf{p}_j \in R_2$, see Figure 17. (The
 48:12 case $\mathbf{p}_j \in R_1$ is symmetric.) Keeping \mathbf{p}_j and \mathbf{p}_1 fixed, δ is smallest when \mathbf{p}_3 moves on the
 48:13 upper side of the rectangle K to the upper left corner $\mathbf{p}_0 = K_{UL}$. (This position of \mathbf{p}_3
 48:14 is hypothetical because ℓ_{23} would be larger than D .) Keeping \mathbf{p}_1 and \mathbf{p}_0 fixed, the angle
 48:15 $\delta = \angle \mathbf{p}_1\mathbf{p}\mathbf{p}_0$ is a quasi-concave function of $\mathbf{p} = \mathbf{p}_j$ as \mathbf{p} moves inside K . (This means that
 48:16 the region in K where δ is bigger than some given threshold is always convex.) Hence the
 48:17 minimum value of δ on any line segment occurs at one of its endpoints, and it suffices to
 48:18 check the lower bound for δ for the corners of the region R_2 . More specifically, it is sufficient
 48:19 to check it for three points: (i) $\mathbf{p} = K_{UR}$, the upper right corner of the rectangle K ; (ii)
 48:20 $\mathbf{p} = K_{LR}$, the lower right corner of K ; and (iii) $\mathbf{p} = X$, a point on the lower edge of K at
 48:21 distance $h/9$ from the lower left corner.

48:22 The other corners of R_2 can be treated as follows: If \mathbf{p} is the intersection U of the
 48:23 bisector of α_1 with the upper edge of K , then δ decreases as \mathbf{p} moves right towards K_{UR} .
 48:24 If the bisector of α_1 intersects the boundary of K in the right edge, then the intersection
 48:25 point lies on the segment between K_{UR} and K_{LR} , and hence it is dominated by one of these
 48:26 points. (This situation occurs for region R_1 in Figure 17.) The incenter I can be treated in
 48:27 the same way as U , by moving \mathbf{p} horizontally to the right until it hits the right edge.

48:28 Another corner is the intersection L of the bisector of α_3 with the lower edge of K .
 48:29 We cover this case by showing that the point X lies in region R_1 . Therefore L lies on the
 48:30 segment between X and K_{LR} , and hence it is dominated by these two points. If the bisector
 48:31 of α_3 intersects the boundary of K in the right edge, it is dominated by K_{UR} and K_{LR} . The

49:01 bisector cannot intersect the boundary of K in the left edge because of the point X .

49:02 Let us now consider the points $\mathbf{p} = K_{\text{UR}}$ and $\mathbf{p} = K_{\text{LR}}$. By symmetry, $\mathbf{p} = K_{\text{UR}}$ yields
 49:03 the same angle $\delta = \angle \mathbf{p}_0 \mathbf{p} \mathbf{p}_1$ as $\mathbf{p} = \mathbf{p}_2$ and that cannot be the minimum because it lies
 49:04 between K_{UR} and K_{LR} . For $\mathbf{p} = K_{\text{LR}}$, the sine law for the triangle $\mathbf{p}_0 \mathbf{p} \mathbf{p}_1$ with the angle
 49:05 $\delta = \angle \mathbf{p}_0 \mathbf{p} \mathbf{p}_1$ gives

$$49:06 \quad \frac{\sin \delta}{h} = \frac{\sin \angle \mathbf{p}_0 \mathbf{p}_1 \mathbf{p}}{\mathbf{p}_0 \mathbf{p}} = \frac{D/\sqrt{D^2 + h^2}}{\sqrt{D^2 + (2h)^2}} \geq \frac{D}{\sqrt{2D^2} \cdot \sqrt{5D^2}},$$

49:07 using $h \leq D$. So we get

$$49:08 \quad \sin \delta \geq \frac{1}{\sqrt{10}} \cdot \frac{h}{D} \geq \frac{1}{2\sqrt{10}} \cdot \frac{w}{D},$$

49:09 and the lemma holds in this case.

49:10 Let us finally consider the point X . Since $h \leq \sqrt{3/4} \cdot D$, the point X lies on the
 49:11 lower edge of K . We show that the distance of X from the ray $\mathbf{p}_3 \mathbf{p}_1$ is smaller than the
 49:12 distance from the ray $\mathbf{p}_3 \mathbf{p}_2$, and therefore X lies in R_1 . The distance of X from the ray
 49:13 $\mathbf{p}_3 \mathbf{p}_1$ is at most the distance $X \mathbf{p}_1 = \sqrt{1 + 1/81} \cdot h < 1.0062 h$. The point on the ray $\mathbf{p}_3 \mathbf{p}_2$
 49:14 which is closest to X lies either on the segment $\mathbf{p}_3 \mathbf{p}_2$ or in the region to the right of the
 49:15 rectangle. In the latter case, the distance is at least $D - h/9 \geq (\sqrt{4/3} - 1/9)h > 1.043 h$,
 49:16 which is larger than $X \mathbf{p}_1$. In the former case, the distance between X and the segment
 49:17 $\mathbf{p}_3 \mathbf{p}_2$ decreases if \mathbf{p}_3 or \mathbf{p}_2 are moved horizontally to the left. In the extreme cases, when
 49:18 $\mathbf{p}_3 = K_{\text{UL}} = \mathbf{p}_0$ and $\ell_{12} = D = \sqrt{4/3} \cdot h$, this distance can be calculated as $> 1.439 h$,
 49:19 which is also larger than $X \mathbf{p}_1$. A final calculation show that the angle $\delta = \angle \mathbf{p}_0 X \mathbf{p}_1$ satisfies
 49:20 $\sin \delta > 0.055 > 1/20 \geq \frac{w}{20D}$, since $w \leq D$, and the proof is complete. \square