Optimal strategies in fractional games: vertex cover and domination

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Abstract

In a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E}$, a real-valued function $f : V \to [0, 1]$ is a fractional transversal if $\sum_{v \in E} f(v) \geq 1$ holds for every $E \in \mathcal{E}$. Its size is $|f| := \sum_{v \in V} f(v)$, and the fractional transversal number $\tau^*(\mathcal{H})$ is the smallest possible $|f|$.

We consider a game scenario where two players have opposite goals, one of them trying to minimize and the other to maximize the size of a fractional transversal constructed incrementally. We prove that both players have strategies to achieve their common optimum, and they can reach their goals using rational weights.

Keywords: fractional vertex cover, fractional transversal game, fractional domination game.

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1 Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph, where V is the finite vertex set and $\mathcal{E}$ is the edge set, a set system over the underlying set V. We assume that every edge contains at least one vertex; that is, $\mathcal{E} \subseteq 2^V \setminus \{\emptyset\}$. A hypergraph is *$k$-uniform* if $|E| = k$ holds for all $E \in \mathcal{E}$. A set $T \subseteq V$ is a *transversal* [1] of $\mathcal{H}$ if every edge is covered by a vertex of T, which formally means that $T \cap E \neq \emptyset$ holds for all $E \in \mathcal{E}$. Its real relaxation, called *fractional transversal*, is a function $f\colon V \to [0,1]$ such that $\sum_{v\in E} f(v) \geq 1$ holds for every $E \in \mathcal{E}$. The size of f is defined as $|f| := \sum_{v\in V} f(v)$. The *transversal number* $\tau(\mathcal{H})$ and the *fractional transversal number* $\tau^*(\mathcal{H})$ of $\mathcal{H}$ are the minimum cardinality $|T|$ of a transversal and minimum value $|f|$ of a fractional transversal, respectively.

The *transversal game* is a competitive optimization version of hypergraph transversals, which was introduced in [9] and studied further in [10]. It is played on a hypergraph $\mathcal{H}$ by two players called *Edge-hitter* and *Staller*. They take turns choosing a vertex. The game is over when all edges are covered, and the length of the game is the number of vertices chosen by the players until the end of the game. Edge-hitter wants to finish the game as soon as possible, while Staller wants to delay the end. To prevent Staller from making completely useless moves, we stipulate that the chosen vertex must be contained in at least one previously uncovered edge.

Assuming that both players play optimally [2] and Edge-hitter starts, the length of the game on $\mathcal{H}$ is uniquely determined. It is called the *game transversal number* of $\mathcal{H}$ and is denoted by $\tau_g(\mathcal{H})$. Analogously, the *Staller-start game transversal number* of $\mathcal{H}$, denoted by $\tau_g'(\mathcal{H})$, is the length of the game under the same rules when Staller makes the first move. Among other results, it was proved in [9] that $|\tau_g(\mathcal{H}) - \tau_g'(\mathcal{H})| \leq 1$ always holds. We further recall that, denoting by $n(\mathcal{H})$ and $m(\mathcal{H})$ the number of vertices and edges in $\mathcal{H}$ respectively, $\frac{4}{11}(n(\mathcal{H}) + m(\mathcal{H}))$ is a (sharp) upper bound on $\tau_g(\mathcal{H})$ if $\mathcal{H}$ does not contain one-element edges and it is not isomorphic to the cycle C_4.

Below we shall refer to this game as the *integer game*, as opposed to its fractional version which we will introduce in the next section.

[1] In various areas of discrete mathematics and computer science, a transversal is called vertex cover, or hitting set, or blocking set. It is also equivalent to the set cover in the dual hypergraph.

[2] A *strategy* of a player means that every possible state of the game is associated with a move he/she will play if that situation arises. From Edge-hitter's point of view, the value $v_E(S)$ of a strategy S is the smallest integer k such that, if Edge-hitter plays according to S, the transversal game always finishes in at most k moves (no matter which strategy is applied by Staller). We say that S is an *optimal strategy* for Edge-hitter, if $v_E(S)$ is the possible smallest value over the family of all strategies. Similarly, from Staller's point of view, a strategy S can be associated with the value $v_S(S)$ that is the largest integer k such that, if Staller follows strategy S, the length of the game is always at least k; further S is an *optimal strategy* for Staller, if $v_S(S)$ is the largest value over the family of all strategies. The reader may find more about optimal strategies and the uniqueness of the corresponding parameters in [5, Section 1.2].

The important motivation of this approach are the *domination game* [7] and the *total domination game* [17], where in fact the transversal game is played on the 'closed neighborhood hypergraph' and on the 'open neighborhood hypergraph' of a graph, respectively.[3] Further variants studied so far include the disjoint domination [14], connected domination [2], and fractional domination [15] games on graphs, and the domination games on hypergraphs [13]. Some of the most recent results can be found in [3, 4, 8, 11, 12, 16, 19, 20, 21, 22, 23, 24]. For a thorough survey and list of further references see the book [5].

Our results

In Section 2 we introduce the rules of the game and prove that its value is well-defined. We present some examples, showing that it makes a difference whether Edge-hitter or Staller starts. Moreover, edges that are supersets of other edges of the hypergraph may influence the game value, in contrast to the standard non-game version of the transversal number.

In Section 3 we compare the game transversal number with other related parameters, and prove a monotonicity property, implying that changing the starting player can affect the value of the game by at most 1.

The rules of the game allow the players to split their moves into infinitely many submoves. In Section 4 and 5 we give some structural results showing that the full generality of the moves allowed by our rules is not needed. Namely, any infinite move is equivalent to a finite move, and Edge-hitter can restrict his strategy to moves in which every permutation of submoves is equally good.

In Section 6 we prove that the game can be modeled in a way that leads to an optimization problem solvable via the theory of piecewise linear continuous rational functions. From this, we derive that the game value is rational for every finite hypergraph; moreover both players can achieve their goals using rational submoves.

Consequences concerning domination games and several conjectures are given in the concluding Section 7.

2 Fractional transversal game

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. In the context of the fractional transversal game, we will consider a *cover function* $t\colon V \to [0,1]$ that is updated after each move during the game. We denote by $|t|$ the sum $\sum_{v\in V} t(v)$. Given a cover function t, the

[3]Recall from the literature that the closed and open neighborhood hypergraphs of a graph G are defined on the same vertex set V as G, and the closed (resp. open) neighborhood hypergraph consists of edges corresponding to the closed (resp. open) neighborhoods of vertices in G.

corresponding *load function* is $\ell\colon \mathcal{E} \to [0,1]$ defined by the rule

$$\ell(E) = \ell(E,t) = \min\Big\{1, \sum_{v\in E} t(v)\Big\}$$

for every $E \in \mathcal{E}$. If $\ell \equiv 1$, we say that $\mathcal{H}$ is fully covered. We shall write t_i and ℓ_i for the cover and load functions after the i^{th} move.

The game begins with $t_0 \equiv 0$ and therefore with $\ell_0 \equiv 0$. It is finished when the hypergraph becomes fully covered. Edge-hitter and Staller take turns making moves under the following rules. As long as $\ell \not\equiv 1$, the next player performs a *move*, which is a sequence $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ of arbitrary length (possibly infinite). It consists of the *submoves* (v_{i_k}, w_k), $k = 1, 2, \dots$, where $v_{i_1}, v_{i_2}, \dots$ are vertices of $\mathcal{H}$ with any number of repetitions allowed, and the *weights* $w_1, w_2, \dots$ are real numbers from $[0,1]$.

We say that a submove (v_{i_k}, w_k) is *legal* if it increases the load of some edge by w_k. In a *legal move*, a player makes a series of legal submoves such that the sum of the weights equals 1 or the move completes the game, whichever comes first. Formally, the i^{th} move $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ is legal, if the following conditions hold:

($*$) For every $k \geq 1$ there exists an edge $E \in \mathcal{E}$ such that $v_{i_k} \in E$ and

$$\ell_{i-1}(E) + \left(\sum_{\substack{v_{i_s} \in E \\ 1 \leq s \leq k-1}} w_s \right) + w_k \leq 1\,.$$

($**$) The *total weight constraint*: $\sum_{k\geq 1} w_k \leq 1$, and if the move does not end the game, then $\sum_{k\geq 1} w_k = 1$.

The cover function t_i can gradually be reached from t_{i-1} by adding the weight w_k to $t(v_{i_k})$ after each submove; this process converts also the corresponding load function from ℓ_{i-1} to ℓ_i.

Suppose that a fractional transversal game $\mathcal{G}$ finishes with the q^{th} move. The *value* $|\mathcal{G}|$ *of the game* is defined as the value $|t_q|$ of the cover function obtained at the end, that is the sum of the weights that have been spent during the game. The goal of Edge-hitter is to achieve a value $|\mathcal{G}|$ as small as possible, while Staller wants a large $|\mathcal{G}|$.

Assuming that Edge-hitter starts the fractional transversal game on $\mathcal{H}$, we consider the set of upper bounds,

$$U_{\mathcal{H}} = \{a \in \mathbb{R} : \text{Edge-hitter has a strategy that ensures } |\mathcal{G}| \leq a\}$$

and the set of lower bounds,

$$L_{\mathcal{H}} = \{b \in \mathbb{R} : \text{Staller has a strategy that ensures } |\mathcal{G}| \geq b\}.$$

Formally the *game fractional transversal number* $\tau_g^*(\mathcal{H})$ is defined as

$$\tau_g^*(\mathcal{H}) = \inf(U_{\mathcal{H}}).$$

The Staller-start game fractional transversal number $\tau_g^{*\prime}(\mathcal{H})$ is defined similarly, under the condition that the first move is made by Staller.

The following assertion shows that $\tau_g^*(\mathcal{H})$ is also equal to $\sup(L_{\mathcal{H}})$, and the situation is similar if Staller starts the game. The proof is essentially the same as the one for the game fractional domination number in [15].

Proposition 1. *For every hypergraph $\mathcal{H}$ we have $\inf(U_{\mathcal{H}}) = \sup(L_{\mathcal{H}})$, and the analogous equality holds for the Staller-start game, too.*

Proof. First, assume that $\inf(U_{\mathcal{H}}) < \sup(L_{\mathcal{H}})$ and consequently, there exist two reals x and y satisfying $\inf(U_{\mathcal{H}}) < x < y < \sup(L_{\mathcal{H}})$. By definition, $x \in U_{\mathcal{H}}$ and, therefore, Edge-hitter can ensure that, under every strategy of Staller, the value of the game is at most x. Similarly, $y \in L_{\mathcal{H}}$ and Staller has a strategy that ensures $|\mathcal{G}| \geq y$ whatever strategy is followed by Edge-hitter. This is a contradiction that establishes $\inf(U_{\mathcal{H}}) \geq \sup(L_{\mathcal{H}})$.

Now, we prove the reverse inequality. By definition, $z < \inf(U_{\mathcal{H}})$ implies that Edge-hitter does not have a strategy to achieve $|\mathcal{G}| \leq z$. That is, against each strategy of Edge-hitter there is a strategy of Staller which results in $|\mathcal{G}| > z$. We may infer that $z \in L_{\mathcal{H}}$ and therefore $z \leq \sup(L_{\mathcal{H}})$. Since it holds for every $z < \inf(U_{\mathcal{H}})$, we conclude $\inf(U_{\mathcal{H}}) \leq \sup(L_{\mathcal{H}})$. This completes the proof of the proposition. □

Later, in Section 6, we will show that $\inf(U_{\mathcal{H}}) = \min(U_{\mathcal{H}})$ and $\sup(L_{\mathcal{H}}) = \max(L_{\mathcal{H}})$. Therefore, Edge-hitter and Staller have optimal strategies under which, respectively, $|\mathcal{G}| \leq \tau_g^*(\mathcal{H})$ and $|\mathcal{G}| \geq \tau_g^*(\mathcal{H})$ are achieved.

2.1 Examples for the fractional transversal game

(1) Our first example is the 4-cycle $C_4 = v_1v_2v_3v_4v_1$, which can also be considered as a 2-uniform hypergraph. It is easy to check that $\tau^*(C_4) = 2$, while $\tau_g(C_4) = 3$ and $\tau_g'(C_4) = 2$ were proved for the integer games [9]. Now we prove that $\tau_g^*(C_4) = 5/2$.

- For the upper bound, the following strategy of Edge-hitter ensures that the sum of the weights spent during the game is at most $5/2$. His first move is $(v_1, \frac{1}{4}), (v_2, \frac{1}{4}), (v_3, \frac{1}{4}), (v_4, \frac{1}{4})$; it results in $\ell_1 \equiv \frac{1}{2}$ and $\sum_{E \in \mathcal{E}} \ell_1(E) = 2$. Then, for the first $\frac{1}{2}$ of the weight spent by Staller, each of her submoves necessarily increases the load of both incident edges; and each of her remaining submoves increases the load of at least one edge. Therefore, the total load increases by at least $\frac{1}{2} \times 2 + \frac{1}{2} = \frac{3}{2}$ to $\sum_{E \in \mathcal{E}} \ell_2(E) \geq \frac{7}{2}$, and Edge-hitter can achieve $\sum_{E \in \mathcal{E}} \ell_3(E) = 4$ by spending at most $\frac{1}{2}$ in the final move. This proves $\tau_g^*(C_4) \leq 5/2$.

- Let us show the reverse inequality. We note that the first move of the game has the same effect as a sequence $(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4)$ of submoves with $\sum_{i=1}^4 w_i = 1$.[4] After this move of Edge-hitter, $\sum_{E \in \mathcal{E}} \ell_1(E) = 2$ and hence, there is an edge E with $\ell_1(E) \le \frac{1}{2}$. By symmetry, we may assume that $\ell_1(v_1v_2) \le \frac{1}{2}$. Let Staller play the move $(v_3, w_1 + w_4), (v_4, w_2 + w_3)$. The move is legal as $\ell_1(v_2v_3) = w_2 + w_3 = 1 - (w_1 + w_4)$ and $\ell_1(v_4v_1) = w_1 + w_4 = 1 - (w_2 + w_3)$. We then have $\ell_2(v_1v_2) = \ell_1(v_1v_2) \le \frac{1}{2}$ and Edge-hitter needs to spend at least $\frac{1}{2}$ to finish the game. This strategy of Staller shows $\tau_g^*(C_4) \ge 5/2$.

If Staller starts the fractional transversal game on C_4 with the move $(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4)$, then Edge-hitter can ensure $|\mathcal{G}| = 2$ by playing $(v_1, w_3), (v_2, w_4), (v_3, w_1), (v_4, w_2)$. Indeed, ℓ_2 assigns $w_1 + w_2 + w_3 + w_4 = 1$ to every edge of the graph. Therefore, $\tau_g^{*\prime}(C_4) \le 2$. Since $\tau_g^{*\prime}(C_4) \ge \tau^*(C_4) = 2$ also holds[5], we get $\tau_g^{*\prime}(C_4) = 2$.

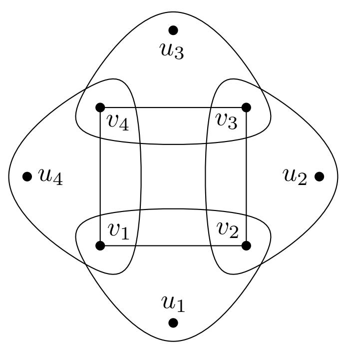

Figure 1: A hypergraph $\mathcal{H}$ with nested edges

(2) Now we modify the previous example C_4 by adding four new vertices $u_1, \dots, u_4$ and four new edges $\{v_1, v_2, u_1\}, \dots, \{v_4, v_1, u_4\}$ to get the hypergraph $\mathcal{H}$ shown in Figure 1. When the fractional (or integer) transversal number is considered, each edge that is a superset of another edge can be deleted, which implies $\tau^*(\mathcal{H}) = \tau^*(C_4) = 2$. We show that the situation is different for the fractional transversal game on $\mathcal{H}$, that is $\tau_g^*(\mathcal{H}) = 3 \neq \tau_g^*(C_4)$. Suppose that Edge-hitter starts the fractional transversal game $\mathcal{G}$ on $\mathcal{H}$.

- We first show that Edge-hitter can ensure that the value $|\mathcal{G}|$ of the game is at most 3. An optimal first move for him is $(v_1, \frac{1}{4}), (v_2, \frac{1}{4}), (v_3, \frac{1}{4}), (v_4, \frac{1}{4})$. Then, independently of Staller's reply, Edge-hitter plays $(v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4)$ as his second move, where w_i equals $\frac{1}{4}$ or, if $(v_i, \frac{1}{4})$ is not a legal submove, w_i

[4] Theorem 6 and Lemma 7 will give conditions for this replacement property in general. For the first move of the game, it is much easier to see as, for each edge E, its load $\ell_1(E)$ equals the sum of the weights assigned to the vertices of E.

[5] See the proof of Proposition 2 (i) for a simple explanation.

is the maximum legal weight for v_i. After this move, $\ell_3 \equiv 1$ and we may infer that $|\mathcal{G}| \le 3$. This proves $\tau_g^*(\mathcal{H}) \le 3$.

- Our second claim is that Staller has a strategy that always results in $|\mathcal{G}| \ge 3$. As each vertex of $\mathcal{H}$ belongs to at most two 3-element edges, after Edge-hitter's first move the sum of the loads of the 3-element edges is at most 2. Thus, Staller can play a legal move that does not assign weights to v_1, v_2, v_3, v_4. After this move, the sum of the loads of the 2-element edges remains at most 2, and Edge-hitter has to spend a weight of at least 1 to finish the game. This shows $\tau_g^*(\mathcal{H}) \ge 3$. We therefore conclude $\tau_g^*(\mathcal{H}) = 3 > \tau_g^*(C_4)$.

(3) The removal of the edges which are *subsets* of other edges in a hypergraph $\mathcal{F}$ may also change the values of the parameters. For instance, let $\mathcal{F}$ be the hypergraph obtained from C_4 by adding the 4-element edge $E = \{v_1, v_2, v_3, v_4\}$. As the load of E equals 1 after the first move in the game, it is easy to see that $\tau_g^*(\mathcal{F}) = \tau_g^*(C_4) = 5/2$, while the removal of all 2-element edges results in a one-edge hypergraph $\mathcal{F}'$ with $\tau_g^*(\mathcal{F}') = 1$.

3 Some basic facts and the Continuation Principle

In this section we first observe some simple inequalities which are analogous to the ones in other games concerning graph domination and hypergraph transversal, most notably to the fractional domination game [15].

Proposition 2.

(*i*) *For every hypergraph $\mathcal{H}$, it holds that*

$$\tau^*(\mathcal{H}) \le \tau_g^*(\mathcal{H}) < 2\tau^*(\mathcal{H}) \qquad \text{and} \qquad \tau^*(\mathcal{H}) \le \tau_g^{*\prime}(\mathcal{H}) < 2\tau^*(\mathcal{H}) + 1.$$

(*ii*) *There is no universal constant C with $\tau_g(\mathcal{H}) \le C \cdot \tau_g^*(\mathcal{H})$, and not even with $\tau(\mathcal{H}) \le C \cdot \tau_g^*(\mathcal{H})$. The same holds true for $\tau_g^{*\prime}(\mathcal{H})$, too.*

Proof. No matter which player starts the game, at the end the cover function t_q is a fractional transversal. This implies the lower bounds $\tau_g^*(\mathcal{H}) \ge \tau^*(\mathcal{H})$ and $\tau_g^{*\prime}(\mathcal{H}) \ge \tau^*(\mathcal{H})$.

Concerning a fractional transversal game $\mathcal{G}$ on $\mathcal{H}$ and the upper bounds in (i), we can write the value of the game in the form $|\mathcal{G}| = W + W'$, where W and W' denote the total sum of weights assigned by Edge-hitter and Staller, respectively. To keep the claimed bounds, first Edge-hitter can fix an optimal fractional transversal f, i.e. one with $|f| = \tau^*(\mathcal{H})$. After that, in his moves he can apply the strategy to play submoves (v_{i_j}, w_j) with the largest possible weights w_j which are not only allowed by

$(*)$ but also respect the inequalities $t_{i-1}(v_{i_j}) + w_j \leq f(v_{i_j})$. If such a legal submove with a positive weight does not exist anymore, then $\mathcal{H}$ is fully covered and the game is finished.

This strategy yields $W \leq \tau^*(\mathcal{H})$, with strict inequality if the game is finished by Staller. We also have $W' \leq W$ or $W' \leq W + 1$, depending on whether the first move is made by Edge-hitter or Staller, both with strict inequalities if the game is finished by Edge-hitter. Since only one of the players can make the last move, the claimed strict upper bounds follow.

For the proof of (ii) we apply the following result of Alon [1]: For every $\epsilon > 0$ and for any sufficiently large k, there is a k-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that $\tau(\mathcal{H}) \geq (1-\epsilon)\frac{\ln k}{k}(|V| + |\mathcal{E}|)$. On the other hand, a very simple fractional transversal f with $|f| = |V|/k$ may be constructed by assigning $f(v) = 1/k$ to each vertex $v \in V$. Therefore, $\tau^*(\mathcal{H}) \leq \frac{|V|}{k}$ and we obtain

$$(1/2 - \epsilon)\ln k < \sup_{\mathcal{H}} \frac{\tau(\mathcal{H})}{2\tau^*(\mathcal{H})} < \sup_{\mathcal{H}} \frac{\tau(\mathcal{H})}{\tau_g^*(\mathcal{H})} \leq \sup_{\mathcal{H}} \frac{\tau_g(\mathcal{H})}{\tau_g^*(\mathcal{H})}$$

due to the obvious fact $\tau \leq \tau_g$ and the inequality $\tau_g^* < 2\tau^*$ from (i). For $\tau_g^{*\prime}(\mathcal{H})$ the proof is similar, by the second part of (i). □

Proposition 3. *The upper bounds in Proposition 2 (i) are tight apart from an additive constant at most 2.*

Proof. Consider the complete bipartite graph $G = K_{k,k^2}$ on $k + k^2$ vertices as a 2-uniform hypergraph. Clearly, $\tau^*(G) = k$. In any submove of a fractional transversal game, while G is not fully covered, Staller can always select a vertex from the bigger partite class. Following this strategy, during $k-1$ moves, Staller increases the sum of the loads by at most $(k-1)k$. As G has maximum degree k^2, $k-1$ moves of Edge-hitter increase the loads by at most $(k-1)k^2$. Hence, no matter whether Edge-hitter or Staller starts the game, after $2k-2$ moves we have

$$\sum_{E \in \mathcal{E}} \ell_{2k-2}(E) \leq (k-1)k + (k-1)k^2 = k^3 - k < |E|,$$

therefore the game is not over yet. This shows $\tau_g^*(G) > 2k - 2 = 2\tau^*(G) - 2$ and, similarly, $\tau_g^{*\prime}(\mathcal{H}) \geq 2\tau^*(G) - 1$ follows if Staller starts the game. □

A monotone property of the game fractional transversal number is expressed in the following idea, which provides a useful tool in simplifying several arguments. Let a hypergraph $\mathcal{H}$ with a pre-defined load function ℓ be given, which we consider as a non-zero starting configuration. We ask about the value $|\mathcal{G}|$ of the game started by Edge-hitter, where the game is finished when ℓ is completed to a load function under which $\mathcal{H}$ is fully covered. The rules are the same as they were in the case of $\ell_0 \equiv 0$, but here we have $\ell_0 = \ell$, while the value of the game is still computed by starting with the cover function $t_0 \equiv 0$. Under these conditions and assuming that Edge-hitter starts the fractional transversal game $\mathcal{G}$ on hypergraph $\mathcal{H}$ with the pre-defined ℓ, we consider the sets

$$\begin{aligned} U_{\mathcal{H}|\ell} &= \{a \in \mathbb{R} : \text{Edge-hitter has a strategy that ensures } |\mathcal{G}| \le a\}, \\ L_{\mathcal{H}|\ell} &= \{b \in \mathbb{R} : \text{Staller has a strategy that ensures } |\mathcal{G}| \ge b\}. \end{aligned}$$

Then the *game fractional transversal number with predefined load function* ℓ is defined as $\tau_g^*(\mathcal{H}|\ell) = \inf(U_{\mathcal{H}|\ell})$. It can be shown with a proof analogous to that of Proposition 1 that $\inf(U_{\mathcal{H}|\ell}) = \sup(L_{\mathcal{H}|\ell})$ always holds.

The corresponding value $\tau_g^{*\prime}(\mathcal{H}|\ell)$ is defined analogously for the Staller-start game on $\mathcal{H}|\ell$.

The imagination strategy is a useful technique applied in many proofs when domination and transversal games are considered (see e.g. [3, 6, 9, 15, 17, 21]). It was introduced and first used in [7]; for a detailed explanation and examples we refer the reader to [5, Chapter 2.2].

Theorem 4 (Continuation Principle)**.** *If ℓ and ℓ' are load functions on the hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that $\ell(E) \le \ell'(E)$ holds for every $E \in \mathcal{E}$, then $\tau_g^*(\mathcal{H}|\ell) \ge \tau_g^*(\mathcal{H}|\ell')$, and similarly $\tau_g^{*\prime}(\mathcal{H}|\ell) \ge \tau_g^{*\prime}(\mathcal{H}|\ell')$.*

Proof. Assume for a contradiction that $\tau_g^*(\mathcal{H}|\ell) < \tau_g^*(\mathcal{H}|\ell')$, and choose two reals t_1, t_2 with $\tau^*(\mathcal{H}|\ell) < t_1 < t_2 < \tau_g^*(\mathcal{H}|\ell')$. We use the imagination strategy between the following two games:

Game 1: Edge-hitter plays on $\mathcal{H}|\ell$ applying a strategy which ensures that the value of the game is at most t_1. Staller's moves in Game 1 are defined according to her moves in Game 2.

Game 2: Staller plays on $\mathcal{H}|\ell'$ applying a strategy which ensures that the value of the game is at least t_2. Edge-hitter's moves in Game 2 are defined according to his moves in Game 1.

Assume that Edge-hitter starts the game. He chooses his first move in Game 1 according to the prescribed strategy. We copy this move into Game 2 if it is a legal move there, or choose an appropriate replacement move for Edge-hitter in Game 2. In the next turn, Staller replies with a move according to her prescribed strategy in Game 2 and we copy the same move into Game 1. (We will see that it is always legal.) Then, Edge-hitter replies in Game 1 and we copy it or make the according move for Edge-hitter in Game 2. The two parallel games continue this way until at least one of them is finished.

The moves essentially are copied (or interpreted) between Game 1 and Game 2 such that $\ell(E) \le \ell'(E)$ remains true for all $E \in \mathcal{E}$ after every move. If this inequality is valid before Staller's move in Game 2, then her next move in $\mathcal{H}|\ell'$ is legal in $\mathcal{H}|\ell$ as well, so that we can simply copy it into Game 1. The condition $\ell(E) \le \ell'(E)$ for all $E \in \mathcal{E}$ clearly remains valid for the new load functions.

Suppose now that the inequality $\ell(E) \le \ell'(E)$ is true for all $E \in \mathcal{E}$ before Edge-hitter's move in Game 1. If it is legal, we simply copy it into Game 2 and the inequality remains valid. In the other case one or more submoves (v_{i_k}, w_k) made in Game 1 are not legal in Game 2. We then choose the maximum w'_k such that (v_{i_k}, w'_k) is a legal submove in Game 2. The remaining weight $w_k - w'_k$ can be distributed between arbitrary vertices such that the submoves are legal. Observe, however, that if this happens, all loads on the edges incident with v_{i_k} reach 1 after the submove (v_{i_k}, w'_k) in Game 2. We infer that $\ell(E) \le \ell'(E)$ remains true for all $E \in \mathcal{E}$ after Edge-hitter's move.

It follows that the loads will never become smaller in Game 2 than the corresponding ones in Game 1. Thus, the values g_1 and g_2 of Games 1 and 2 satisfy $g_1 \ge g_2$. By the strategies of the players, it is true that $t_1 \ge g_1$ and $g_2 \ge t_2$. We therefore obtain

$$t_1 \ge g_1 \ge g_2 \ge t_2 > t_1$$

and this contradiction proves $\tau_g^*(\mathcal{H}|\ell) \ge \tau_g^*(\mathcal{H}|\ell')$.

The analogous conclusion can be reached in the Staller-start game as well, literally by the same argument, deriving a contradiction from the assumption $\tau_g^{*\prime}(\mathcal{H}|\ell) < \tau_g^{*\prime}(\mathcal{H}|\ell')$. $\square$

We obtain the following immediate consequence.

Theorem 5. *The game fractional transversal numbers for the Staller-start and for the Edge-hitter-start games on $\mathcal{H}$ may differ by at most* 1.

Proof. Consider the Staller-start game. Whatever Staller moves first, she assigns total weight 1, and creates a situation which is at least as favorable for Edge-hitter as the all-zero load at the beginning of the original transversal game. Then, due to Theorem 4, Edge-hitter can ensure that the game ends using at most $\tau_g^*(\mathcal{H})$ further weight. This proves $\tau_g^{*\prime}(\mathcal{H}) \le \tau_g^*(\mathcal{H}) + 1$.

Similarly, if Edge-hitter starts, after his first move he is in at least as favorable position as with the all-zero load at the beginning of the Staller-start game. This proves the reverse inequality $\tau_g^*(\mathcal{H}) \le \tau_g^{*\prime}(\mathcal{H}) + 1$. $\square$

4 Infinite moves are not necessary

The definition of a legal move in the transversal game admits the option that a player splits the value 1 into an infinite number of pieces; e.g., $w_k = 2^{-k}$. It turns out, however, that each legal move on $\mathcal{H} = (V, \mathcal{E})$ is equivalent to a move which consists of at most $|V|$ submoves.

Theorem 6. *Every legal move in a fractional transversal game can be replaced with a legal move such that each vertex occurs in at most one submove of it and the two moves result in the same load function.*

Proof. First, consider a vertex v which occurs in two different submoves (v_{i_j}, w_j) and (v_{i_k}, w_k) of a move. That is, $v = v_{i_j} = v_{i_k}$ and we may assume $j < k$. By the condition $(*)$, there exists an edge $E \in \mathcal{E}$ such that $v \in E$ and the second submove (v_{i_k}, w_k) increases the load of E by exactly w_k. If the submove (v_{i_j}, w_j) is deleted from the sequence and the weight w_k is replaced by $w_j + w_k$ in the k^{th} submove, the submove and the whole move remain legal and result in the same load function as before. Performing this modification repeatedly we can achieve that every vertex occurs in either zero or exactly one or infinitely many submoves of the move in question. This already proves the statement if the move contains only a finite number of submoves.

Now, assume that the move is infinite. Then, the sequence of submoves can be split into two, such that the first part is finite, and in the second infinite part every vertex (which is present there) is repeated infinitely many times. Consider this infinite subsequence $S = (v_{i_s}, w_s), \dots$. By renaming the vertices of $\mathcal{H}$ if necessary, we may assume that $\{v_1, \dots, v_\ell\}$ is the set of the vertices which are present in S. We prove that the finite sequence $S' = (v_1, \sum_{j:\, i_j=1} w_j), \dots, (v_\ell, \sum_{j:\, i_j=\ell} w_j)$ is equivalent to S. Clearly, S and S' yield the same load function after the move. So, it is enough to prove that S' is legal. Assume for a contradiction that $(v_k, \sum_{j:\, i_j=k} w_j)$ is not a legal submove in S', and let k be the smallest such index. Then, after the $(k-1)^{\text{st}}$ submove of S', every edge E which contains v_k has a load $\ell(E) > 1 - \sum_{j:\, i_j=k} w_j$ and, moreover, there is a positive constant ϵ such that $\min_{v_k \in E} \ell(E) + \sum_{j:\, i_j=k} w_j = 1 + \epsilon$. Now, consider S again. There is an index $p = p(\epsilon)$ such that $\sum_{j \geq p} w_j < \epsilon$ and hence, before the p^{th} submove of S, each edge containing v_k is fully covered. As v_k occurs infinitely often in S, and also the occurrences after the p^{th} submove are legal, this is a contradiction. $\square$

5 Edge-hitter's moves are transposable

We say that a finite move $(v_{i_1}, w_1), \dots, (v_{i_k}, w_k)$ is *transposable* if for any permutation $\beta(1), \dots, \beta(k)$ of $1, \dots, k$, the move $(v_{i_{\beta(1)}}, w_{\beta(1)}), \dots, (v_{i_{\beta(k)}}, w_{\beta(k)})$ is legal. We will show that from the point of view of Edge-hitter, we can restrict our attention to transposable moves. Note that every transposable move is legal, but not conversely.

We first give a characterization of transposable moves:

Lemma 7. *A move $(v_{i_1}, w_1), \ldots, (v_{i_k}, w_k)$, where $w_j > 0$ for all $1 \le j \le k$ and no vertices are repeated, is transposable if and only if the total weight constraint $(**)$ is satisfied, and after performing the entire move, for every vertex v_{i_j},*

$$\min_{E \ni v_{i_j}} \sum_{v_i \in E} t(v_i) \le 1. \tag{1}$$

Proof. A legal move must satisfy the two constraints $(*)$ and $(**)$.

The total weight constraint $(**)$ for a legal move is explicitly required in the lemma, and it is insensitive to permuting the submoves.

Let us turn to $(*)$. If $\sum_{v_i \in E} t(v_i) \le 1$ for an edge E containing v_{i_j}, then omitting the submove (v_{i_j}, w_j) from the move we obtain $\sum_{v_i \in E \setminus \{v_{i_j}\}} t(v_i) \le 1 - w_j$. Hence (v_{i_j}, w_j) is a legal submove no matter when it is performed during the move. This means that the move is transposable whenever the condition (1) is satisfied for all j.

In the other direction, assume that for some v_{i_j}, the left-hand side of (1) is bigger than 1. Consider a permutation in which (v_{i_j}, w_j) is the last submove. Then (v_{i_j}, w_j) is not legal because $(*)$ is violated. Consequently the move is not transposable. $\square$

Theorem 8. *If a finite legal move $m = (v_{i_1}, w_1), \ldots, (v_{i_k}, w_k)$ is not transposable in the fractional transversal game, then it can be replaced by a transposable (and legal) move after which no edge gets smaller load than after m.*

Proof. First, consider the legal move $m = (v_{i_1}, w_1), \ldots, (v_{i_k}, w_k)$ and the move $m' = (v_{i_2}, w_2), \ldots, (v_{i_k}, w_k), (v_{i_1}, w_1)$ that is obtained by the cyclic permutation $\beta = 2, \ldots, k, 1$. It is clear that condition $(*)$ in the definition remains true for the first $k-1$ submoves of m' and consequently, these submoves are legal. For the last (and not necessarily legal) submove, determine w_1^* as the maximum weight which results in a legal submove with v_{i_1}. If $w_1^* \ge w_1$, then m' is legal and gives exactly the same load function as m. If $w_1^* < w_1$, then the same load function is obtained after the submove (v_{i_1}, w_1^*) as after m, because in both cases every edge incident with v_{i_1} is fully covered and the loads of the other edges are unchanged.

In the latter case, the sum of the weights is decreased by $w_1 - w_1^*$. After this change, the submove (v_{i_1}, w_1^*) will be legal in any permutation of $(v_{i_1}, w_1^*), (v_{i_2}, w_2), \ldots, (v_{i_k}, w_k)$. That is, if a permutation is not legal after this replacement, this is due to another vertex. The same is true if some weights w_s are replaced by smaller weights.

We repeat this step for the modified sequence with permutation $\beta = 3, \ldots, k, 1, 2$, then with $\beta = 4, \ldots, k, 1, 2, 3$, and so on, finally with $\beta = k, 1, 2, \ldots, k-1$, keeping all modifications incrementally. Decreasing the weight of the last submove in each step if necessary, at the end a legal transposable move m^* is obtained, which yields the same load function as m and satisfies $\sum_{j=1}^k w_j^* \le \sum_{j=1}^k w_j$. If $\Delta = \sum_{j=1}^k w_j - \sum_{j=1}^k w_j^*$ is positive, we use the weight Δ to increase the loads of some non-fully covered edges in an arbitrary way. The total absolute change of weights is 2Δ, or possibly less if the game is over. Finally, we normalize the move by eliminating any multiple occurrences of vertices, using Theorem 6.

The redistribution of Δ and the normalization may lead to a move that is not transposable. If so, we repeat the modification described above.

If the process terminates after a finite number of iterations, then the last version of the move is transposable by construction, and the proof is complete. Otherwise we obtain an infinite sequence $\Delta^{(1)}, \Delta^{(2)}, \ldots$ of re-distributions from the total unit weight of the move. The total load of edges increases by at least $\sum_{i\geq 1} \Delta^{(i)}$; hence this sum converges because altogether the total load is at most $|\mathcal{E}|$.

On the other hand, the weight of a vertex changes by at most $\Delta^{(i)}$ in the i^{th} iteration, hence the local changes in weight (at least one of which is negative in each iteration) in absolute value sum up to at most $2\sum_{i\geq 1} \Delta^{(i)}$, and therefore their sum also converges.

Let $m^{(i)}$ denote the move constructed in the i^{th} iteration, *before* the re-distribution of the weight $\Delta^{(i)}$. We know that this move is transposable. Let $w_v^{(i)}$ denote the weight used for vertex v in the corresponding submove of $m^{(i)}$, and set $w_v^{(i)} = 0$ if v does not appear in this move.

We have shown that the limit of these weights $w_v^* = \lim_{i\to\infty} w_v^{(i)} = 0$ exists. Let p denote the number of vertices with a positive weight in w^*. Relabeling these vertices in an arbitrary order, we define a move $m^* = (v_1, w_1^*), \ldots, (v_p, w_p^*)$ with p submoves. By continuity, the loads achieved by the moves $m^{(i)}$ converge to the corresponding loads after the move m^*.

We claim that m^* is transposable. This will complete the proof because the loads never decrease, hence under w^* no edge gets smaller load than by move m.

For showing that m^* is transposable, we use the characterization of Lemma 7. The inequality (1) follows from the fact that its left-hand side is a continuous function of the weights.

Let us finally check that the total weight constraint $(**)$ is satisfied: Since $\sum_v w_v^{(i)} \leq 1$ for all i, this inequality is satisfied in the limit. If in some iteration, the current move terminates the game (i.e., each edge gets load at least 1 while the total weight in the move is at most 1), this will hold in all successive moves, since the loads never decrease, and hence it holds also for m^*, by continuity. Therefore, we need to show $\sum_v w_v^* = 1$ only when none of the moves terminate the game. In this case, the difference $1 - \sum_v w_v^{(i)}$ is bounded by Δ_i, which goes to 0, and hence $\sum_v w_v^* = 1$ in the limit. □

Remark 9. *Based on Theorem 8, Edge-hitter may restrict his strategy to transposable moves. On the other hand, the result suggests that Staller is advised to perform moves, if possible, which are 'very non-transposable' in a sense.*

6 Algorithm for computing the value of the game

We consider an equivalent version of the game, the *structured game*, which is easier to analyze.

Each *move* consists of $n+1$ *rounds*. Each round consists of n *submoves*, which are dedicated to the vertices $v_1, \ldots, v_n$ in succession. In each submove, the player whose turn it is can decide the amount w, the weight spent in the submove, by which the cover value of $t(v_i)$ is increased, subject to the usual rules: The increase must be useful, i.e. each submove must satisfy the condition $(*)$, and it must be within the *budget constraint* of total weight 1 to be spent per move. It is possible to skip a submove by simply choosing w to be zero.

The first n rounds are identical, but the last round is special: In each submove, the weight is greedily chosen as the largest possible legal weight, hence not allowing any freedom in choosing w for the player in those submoves. This ensures that the whole move spends a total weight of 1 unless a cover is obtained.

There are n moves, alternating between the two players. This is enough to ensure that a cover is constructed when the game terminates. Every move consists of n^2+n submoves, and in total, the game consists of $N = n^3+n^2$ submoves. We illustrate this for a small example with $n = 4$ vertices, where the Edge-hitter starts. The sequence of submoves is

$$\begin{array}{lllll}
H_1, H_2, H_3, H_4; & H_1, H_2, H_3, H_4; & H_1, H_2, H_3, H_4; & H_1, H_2, H_3, H_4; & G_1, G_2, G_3, G_4; \\
S_1, S_2, S_3, S_4; & S_1, S_2, S_3, S_4; & S_1, S_2, S_3, S_4; & S_1, S_2, S_3, S_4; & G_1, G_2, G_3, G_4; \\
H_1, H_2, H_3, H_4; & H_1, H_2, H_3, H_4; & H_1, H_2, H_3, H_4; & H_1, H_2, H_3, H_4; & G_1, G_2, G_3, G_4; \\
S_1, S_2, S_3, S_4; & S_1, S_2, S_3, S_4; & S_1, S_2, S_3, S_4; & S_1, S_2, S_3, S_4; & G_1, G_2, G_3, G_4.
\end{array}$$

Here H_i denotes a move of Edge-hitter for vertex i, and S_i denotes a move of Staller for vertex i. The greedy moves are denoted by G_i.

We do not stipulate as part of the rules that the whole budget of 1 unit must be spent during a move. This capacity is only an upper bound. It is still true that the whole budget is spent in each move if the game is played from the beginning. However, this arises as a *consequence* of the new setup, due to the greedy moves.

As soon as a cover is found, the rules imply that no more weight can be spent, and thus the game is effectively over.

Lemma 10. *The structured game has the same value as the original game.*

Proof. According to Theorem 6, we can assume that every vertex occurs at most once in a move. We can realize this in the structured game by selecting one vertex per round and leaving the weight at 0 otherwise. Thus, the structured game does not restrict the players' strategies, when compared to the original game. On the other hand, the structured game does not give the players more power: The greedy moves ensure that the total weight of 1 is used as long as it is possible. □

Example for the structured game. Consider an Edge-hitter-start game on C_4 with strategies of the players as described in Section 2.1. A corresponding structured game, where we follow the idea in the proof of Lemma 10, can be given as follows.

$$
\begin{array}{lllll}
\text{Edge-hitter's first move:} & (v_1,\frac{1}{4}), & (v_2,0), & (v_3,0), & (v_4,0) \\
 & (v_1,0), & (v_2,\frac{1}{4}), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,\frac{1}{4}), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,\frac{1}{4}); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0). \\
 & & & & \\
\text{Staller's first move:} & (v_1,0), & (v_2,0), & (v_3,\frac{1}{2}), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,\frac{1}{2}); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0). \\
 & & & & \\
\text{Edge-hitter's second move:} & (v_1,0), & (v_2,\frac{1}{6}), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,0), & (v_2,0), & (v_3,0), & (v_4,0); \\
 & (v_1,\frac{1}{3}), & (v_2,0), & (v_3,0), & (v_4,0).
\end{array}
$$

For demonstration, the last move makes use of a greedy submove.

By Theorem 8, it is not a disadvantage for Edge-hitter restricting himself to transposable moves. Consequently, his submoves can always be performed in the order $(v_1,w_1),\dots,(v_k,w_k)$, and a single round $H_1,H_2,\dots,H_n$ followed by n greedy submoves would be a sufficient model for Edge-hitter's moves. For simplicity, we have however chosen to treat the two players uniformly. Note that the greedy submoves are necessary also in case of Edge-hitter. Otherwise, for example, he might pass on the first move and transform the game to the Staller-start version, which sometimes admits a smaller game value (as in the example of C_4).

Consider the situation after the j^{th} submove, $0 \le j \le N$. Let $\vec{x} \in [0,1]^{\mathcal{E}}$ be an arbitrary load vector, and let $r \in [0,1]$ be the budget for the current move that is still available. If j is written in the form $j = k(n^2+n)+i$, i.e. $k = \lfloor \frac{j}{n^2+n} \rfloor$ and $0 \le i < n^2+n$, then $1-r$ is the total weight spent in the last i submoves.

We define

$$T_j(\vec{x}, r)$$

as the sum of weights spent during the remaining part of the game, if both players play optimally, starting from the current situation. If j is large and the entries of $\vec{x}$

are small, it may happen that a complete fractional cover is not reached, because the game necessarily ends after the n^{th} round. Nevertheless, we have chosen our definition because it makes T_j well-defined for arbitrary $\vec{x}$ and r. (The definition of $T_j(\vec{x}, r)$ is related to the game fractional transversal number with *predefined load function* used in the proof of the Continuation Principle (Theorem 4)[6].)

The value of the original game is $T_0(\vec{0}, 1)$.

We will derive a backward recursion for the functions T_j, and thus show that they are piecewise linear and continuous.

Given j, we know the type of the j^{th} submove (H, S, or G) and the vertex v_i to which it applies. We denote the maximum permitted weight by

$$w_i^{\max}(\vec{x}, r) = \min\{r, \max\{\, 1 - x_E \mid E \ni v_i \,\}\}\,, \tag{2}$$

where x_E denotes the entry of $\vec{x}$ that corresponds to the edge $E \in \mathcal{E}$. For the result of increasing the cover value of v_i by w we write

$$\mathit{update}_i(\vec{x}, w) = \vec{x}\,' \quad \text{with} \quad x'_E = \begin{cases} x_E, & \text{if } v_i \notin E, \\ \min\{1, x_E + w\}, & \text{if } v_i \in E. \end{cases} \tag{3}$$

With these definitions, the recursion for a submove H_i for Edge-hitter can be written easily:

$$T_{j-1}(\vec{x}, r) = \min\{\, w + T_j(\mathit{update}_i(\vec{x}, w), r - w) \mid 0 \le w \le w_i^{\max}(\vec{x}, r) \,\} \tag{4}$$

If the submove is for the Staller (S_i), the recursion is the same as (4), except that min is replaced by max. In the greedy submoves G_i, we always choose $w = w_i^{\max}(\vec{x}, r)$:

$$T_{j-1}(\vec{x}, r) = w_i^{\max}(\vec{x}, r) + T_j(\mathit{update}_i(\vec{x}, w_i^{\max}(\vec{x}, r)), r - w_i^{\max}(\vec{x}, r)) \tag{5}$$

The last greedy submove G_n of each move is an exception: Since a different move is about to start, the budget r is reset to 1. Thus, when j is a multiple of $n^2 + n$, then

$$T_{j-1}(\vec{x}, r) = w_n^{\max}(\vec{x}, r) + T_j(\mathit{update}_n(\vec{x}, w_n^{\max}(\vec{x}, r)), 1). \tag{6}$$

As the recursion anchor, we use the value after the final move, which is simply

$$T_N(\vec{x}, r) = 0. \tag{7}$$

Theorem 11. *Each function $T_j(\vec{x}, r)$ for $0 \le j \le N$ is a piecewise linear continuous function with finitely many linear pieces defined on $[0,1]^{\mathcal{E}} \times [0,1]$. Moreover, all T_j are rational in the sense that each linear piece has rational coefficients and rational constant part. As a consequence, the boundaries between regions of the domain with different linear functions can be described by linear equations with rational coefficients.*

[6] In particular, if the j^{th} submove is the last submove in a move of Staller and ℓ is the load function corresponding to $\vec{x}$, then we would expect $T_j(\vec{x}, 1)$ to be $\tau_g^*(\mathcal{H}|\ell)$. However this does not hold in general because, as just discussed, the structured game may terminate too early.

Proof. We will call a function with all the desired properties — piecewise linear, continuous, and rational, with finitely many linear pieces — a PLCR function.

The proof proceeds by backward recursion from T_N down to T_0. The function T_N from (7) is obviously PLCR.

The sum, difference, maximum, or minimum of two PLCR functions is again PLCR, and the same holds true when substituting one PLCR function into another. It follows directly that the functions $w_i^{\max}$ and $update_i$ are PLCR functions on the domain $[0,1]^{\mathcal{E}} \times [0,1]$. This allows us to perform the induction step in the recursions (5)–(6) for G_i.

In the recursion (4) we additionally have a minimization (or, in the analogous recursion for Staller, a maximization) over some range of values w. It has the form

$$\min\{\, F(\vec{x}, r, w) \mid 0 \le w \le w_i^{\max}(\vec{x}, r) \,\}$$

with the PLCR function

$$F(\vec{x}, r, w) := w + T_j(update_i(\vec{x}, w), r - w)$$

To get rid of the varying upper bound on w, we rewrite the recursion in terms of another PLCR function

$$\hat{F}(\vec{x}, r, w) = F(\vec{x}, r, \min\{w, w_i^{\max}(\vec{x}, r)\}$$

as

$$T_{j-1}(\vec{x}, r) = \min\{\, \hat{F}(\vec{x}, r, w) \mid 0 \le w \le 1 \,\}.$$

Lemma 12 below establishes that T_{j-1} is a PLCR function.

The same argument applies to the recursion for the Staller (S_i), where min is replaced by max. $\square$

Lemma 12. *Suppose that $\hat{F}(y, w)\colon [0,1]^m \times [0,1] \to \mathbb{R}$ is a PLCR function. Then the function $T(y)\colon [0,1]^m \to \mathbb{R}$ defined by minimizing over w:*

$$T(y) := \min\{\, \hat{F}(y, w) \mid 0 \le w \le 1 \,\} \tag{8}$$

is also a PLCR function.

Proof. We first show that T is continuous. Since $\hat{F}$ is PLCR, it is Lipschitz-continuous. Let L be its Lipschitz constant with respect to the ∞-norm. (We can compute L as the maximum L_1-norm of all coefficient vectors of the linear pieces of $\hat{F}$.) It follows that the function T in (8) is also Lipschitz-continuous with Lipschitz-constant L. To see this, let $\|y_0 - y_1\| \le \varepsilon$, and let $T(y_0) = \hat{F}(y_0, w_0)$ for some w_0. Then $T(y_1) \le \hat{F}(y_1, w_0) \le \hat{F}(y_0, w_0) + L\varepsilon = T(y_0) + L\varepsilon$. The converse bound $T(y_0) \le T(y_1) + L\varepsilon$ follows in the same way.

We still need to show that T is piecewise linear. For an intuitive way to see this, one can interpret the minimization over w geometrically. The graph of $\hat{F}\colon [0,1]^m \times [0,1] \to \mathbb{R}$ is a subset of $\mathbb{R}^{m+2}$. Taking the minimum over all w amounts to projecting away the coordinate corresponding to w and taking the lower envelope (with respect to the last coordinate) in the projection in $\mathbb{R}^{m+1}$. Figure 2 shows a two-dimensional illustration. This picture can also be interpreted as a three-dimensional view of the graph of a bivariate function $\hat{F}(y,w)$ when the viewing direction is parallel to the w-axis. (In this hypothetical example, the resulting minimum is discontinuous; this cannot happen when $\hat{F}$ is continuous and its domain is the box $[0,1]^m \times [0,1]$.)

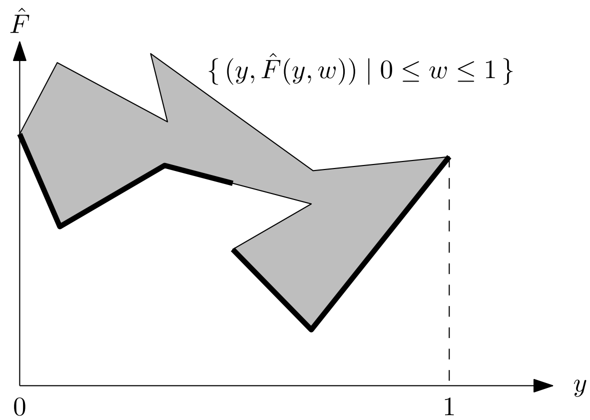

Figure 2: The lower envelope of a polyhedral set in 2 dimensions ($m = 1$).

Formally, we conduct the proof as follows. We know that the domain $[0,1]^{m+1}$ of $\hat{F}$ splits into finitely many rational convex $(m+1)$-dimensional polytopes P on which $\hat{F}$ is linear:

$$\hat{F}(y,w) = a_P y + b_P w + c_P, \qquad \text{for } (y,w) \in P$$

for some rational coefficient vector a_P and rational coefficients b_P and c_P. We can thus write $T(y)$ as the minimum of finitely many functions $T_P(y)$ of the form

$$T_P(y) := \min\{ a_P y + b_P w + c_P \mid 0 \le w \le 1, (y,w) \in P \}, \tag{9}$$

where the minimum of an empty set is taken as ∞.

For fixed y, the minimum in (9) depends on the sign of b_P. If $b_P > 0$, the minimum is achieved on a boundary point that lies on some facet P' of P whose outer normal has negative w-coordinate. On such a facet, w can be expressed as a linear function of y, and thus, T_P can be written as a linear function

$$T_{P'}(y) = a_{P'} y + c_{P'}, \qquad \text{for } y \in \bar{P}', \tag{10}$$

where $\bar{P}'$ is the projection of the facet P' to $[0,1]^m$. Thus, $T_P(y)$ is the minimum of finitely many functions $T_{P'}(y)$, with the understanding that $T_{P'}(y)$ is taken as ∞ when y is outside its domain $\bar{P}'$.

The situation is similar for $b_P < 0$. When $b_P = 0$, then $\hat{F}$ does not depend on w and we can simply write

$$T_P(y) = a_P y + c_P, \quad \text{for } y \in \bar{P}, \tag{11}$$

where $\bar{P}$ is the projection of P.

In summary, the function $T(y)$ can be written as the minimum of finitely many pieces $T_P(y)$, each of which can in turn be written as the minimum of finitely many *linear* pieces (10) or (11). All these pieces have rational coefficients and rational domain boundaries, and since continuity of T has already been established, the PLCR property of T follows. $\square$

The proof of Theorem 11 is constructive and, in principle, it provides an algorithm for computing the value $T_0(\vec{0}, 1)$ of the game. From this, we obtain the following important corollary.

Theorem 13. *For every finite hypergraph $\mathcal{H} = (V, \mathcal{E})$, the game fractional transversal number $\tau_g^*(\mathcal{H})$ and its Staller-start version $\tau_g^{*\prime}(\mathcal{H})$ are rational. Moreover, each player in every step has an optimal move with only rational weights, provided that the weights in all previous submoves were rational.*

Remark 14. *It is not true in general that every optimal strategy uses only rational weights. A simple counterexample is the graph C_4 (Example 1 from Section 2.1), where Staller can start by placing x and $1 - x$ on two vertices with any $x \in [0, 1]$, no matter if x is rational or irrational.*

7 Concluding remarks and open problems

Putting the fractional domination game [15] into a more general context, in this paper we introduced the fractional transversal game on hypergraphs. Among other results, we proved that the game value is rational, and both players have optimal strategies using rational weights and with a finite number of submoves. Since a dominating set is a transversal of the closed neighborhood hypergraph, and total a dominating set is a transversal of the open neighborhood hypergraph, the following consequence is immediate.

Theorem 15. *The fractional versions of both the domination game and the total domination game have rational game values on every hypergraph. Moreover, in either of these games, both players can achieve their common optimum using only rational weights and within finitely many submoves.*

We conclude this paper with some conjectures and open questions.

Conjecture 1. *If each of the first* $2k-1$ $(k \geq 1)$ *moves was an integer move in the fractional transversal game, i.e. of the form* $(v_{i_1}, 1)$*, then Staller has an integer move in the* $(2k)^{\text{th}}$ *turn, which is optimal in the fractional transversal game.*

This means that fractional moves would be advantageous for Edge-hitter only. If true then this conjecture implies the following weaker one.

Conjecture 2. *For every hypergraph* H*,* $\tau_g^*(H) \leq \tau_g(H)$*.*

Perhaps the following stronger version of Conjecture 1 is also true.

Conjecture 3. *Starting from any cover function, there is an optimal strategy for Staller where, in every submove, she always spends the largest legal weight.*

These conjectures could be approached by implementing the algorithm that is implicit in the proof of Theorem 11 by computer. We have not derived an estimate for the complexity (number of pieces) of the piecewise linear continuous functions $T_j(\vec{x}, r)$ that are involved in the construction. If the growth of the complexity is not too steep, there is hope to solve some examples of moderate size, beyond the range of small examples that we considered in Section 2.1, and this could shed some light on the conjectures.

One would naturally expect that $T_0(\vec{x}, r)$ is monotonically decreasing in $\vec{x}$, for fixed r, and moreover, that it is Lipschitz-continuous with Lipschitz constant 1. In other words, in every linear piece, the coefficient of each variable x_i is between 0 and -1. We have not explored these properties.

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