Fixed-parameter tractability and lower bounds for stabbing problems

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Abstract

In this paper we study the parameterized complexity of several "stabbing" problems.

For a general class of objects in $\mathbb{R}^2$, we show that the (decision) problem of stabbing a set of translates of such an object with k axis-parallel lines is W[1]-hard with respect to k, and thus, not fixed-parameter tractable, unless W[1]=FPT. When the lines can have arbitrary directions, it is even W[1]-hard for disjoint objects. Nevertheless, for some special cases such as stabbing disjoint unit squares with axis-parallel lines, we show that the problem is fixed-parameter tractable by giving an algorithm that runs in $\mathcal{O}(n \log n)$ time for every fixed k.

We also show that deciding whether n given unit balls in $\mathbb{R}^d$ can be stabbed by one line (the decision version of the minimum enclosing cylinder problem) is W[1]-hard with respect to the dimension d.

1 Introduction

We study several instances of the following general stabbing problem: Given a set S of n translates of an object in $\mathbb{R}^d$ (e.g., squares or circles in the plane), find a set of k lines with the property that every object in S is "stabbed" (intersected) by at least one line. All these problems are known to be NP-hard and for most of them only polynomial time constant-factor approximation algorithms are known. We study these problems from a *parameterized complexity* point of view: We examine whether algorithms that run in $\mathcal{O}(f(k,d) \cdot n^c)$ time on inputs of size n (where c is a constant independent of k, d, n) do exist; see [2] for an introduction to parameterized complexity theory.

Results. For a general class of objects in $\mathbb{R}^2$, we show that the problem of stabbing with (even axis-parallel) lines is W[1]-hard with respect to k, and thus, not fixed-parameter tractable, unless W[1]=FPT.

Theorem 1 *Let O be a connected object in the plane (which is not a point). (i) The problem of stabbing a set of translates of O is W[1]–hard with respect to the parameter k if the stabbing lines are to be parallel to two different directions u, v, unless O is contained in a line parallel to u or v. (ii) Stabbing disjoint translates of O with k lines in arbitrary directions is W[1]–hard with respect to the parameter k.*

Corollary 2 *Stabbing a set of disjoint rectangles in the plane by k axis-parallel lines is W[1]-hard with respect to k.*

Let $\mathcal{D}$ be a set of directions. A line with a direction from $\mathcal{D}$ is called a $\mathcal{D}$-line. A set of objects with the property that the maximum number of objects that can be simultaneously intersected by two $\mathcal{D}$-lines with different directions is bounded by $c \in \mathbb{N}$ is called *c–shallow for $\mathcal{D}$*.

Theorem 3 *Stabbing sets that are $\mathcal{O}(1)$-shallow for $\mathcal{D}$ with k $\mathcal{D}$-lines can be decided in $\mathcal{O}(n \log n)$ time for every fixed k.*

Finally, we prove that the minimum enclosing cylinder of n points in $\mathbb{R}^d$ can probably not be computed by an algorithm with a running time of the form $\mathcal{O}(f(d) \cdot n^c)$, by showing the corresponding decision problem to be W[1]-hard:

Theorem 4 *Stabbing unit balls in $\mathbb{R}^d$ with a line is W[1]-hard with respect to the parameter d.*

Our reduction also implies that for this problem and, consequently, for the problem of computing the minimum enclosing cylinder of n points in $\mathbb{R}^d$, no $n^{o(d)}$-time algorithm exists, unless the Exponential Time Hypothesis (which asserts that n-variable 3SAT cannot be solved in $2^{o(n)}$-time) fails to hold [4].

Related Results. Langerman and Morin [5] showed that an abstract NP-hard covering problem that models a number of concrete geometric (as well as purely combinatorial) covering problems is in FPT.

Stabbing unit balls in unbounded dimension was shown to be NP-hard by Megiddo [6]. Hasin and Megiddo [3] developed constant factor approximations for special instances of the stabbing problem.

In a recent work, and independently of our results, Dom et al. [1] also show that the problem of stabbing (overlapping) rectangles with axis-parallel lines

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is W[1]-hard (by a different and more complicated reduction from the multi-colored k-clique problem) but in FPT for disjoint unit squares.

2 Stabbing with k lines

2.1 Hardness Results

We first prove that stabbing axis-parallel unit squares with k axis-parallel lines is W[1]-hard with respect to k by an fpt-reduction from the W[1]-complete k-clique problem in directed graphs (c.f. [2]). For proving Theorem 1, we modify our construction in order to handle general objects and lines of arbitrary direction. In this section we consider all objects to be open.

Let $[n] = \{1, \dots, n\}$ and $G = ([n], E)$ be a simple directed graph with no self edges. From G we construct a set $\mathcal{S}$ of axis-parallel squares in $\mathbb{R}^2$ such that $\mathcal{S}$ can be stabbed by $6k$ axis-parallel lines if and only if G has a clique of size k.

We first create instances $\mathcal{S}^*(G, k)$ with squares of *two different* sizes. There will be k horizontal and k vertical "double strips" $S_h^1, \dots, S_h^k$ and $S_v^1, \dots, S_v^k$, respectively, to choose lines from (c.f. Fig. 1). Out of each of those strips, two "consistent" lines will have to be chosen in order to get a solution of the specified size. Around every intersection of a vertical and a horizontal double strip, we place a gadget consisting of a set of squares within a region of suitable size. The gadgets represent the adjacency relation of the graph. We will ensure that (**P0**): Any selection of $2k$ such line pairs corresponds to a set C of vertices of G. (**P1**): Two orthogonal line pairs in strips S_h^i, S_v^j with $i \neq j$ will intersect all the squares in these strips iff these line pairs correspond to vertices that are connected in G. (**P2**): Two orthogonal line pairs in strips S_h^i, S_v^i will intersect all the squares in these strips iff these line pairs correspond to the same vertex. The $2k$ line pairs intersect all the squares iff C forms a k–clique. We will need $2k$ more lines to guarantee the consistency of such a selection.

Let $\square_l(x, y)$ denote the axis-parallel square with side length l and lower left corner (x, y). A gadget T will consist of a collection of axis-parallel squares. $T(x, y)$ denotes a copy of T whose squares are placed relative to (x, y).

The A–gadget (adjacency). A-gadgets represent the adjacency relation of the graph G. Each consists of the set $A := \{\square_{n-1}(i, j) \mid (i, j) \notin E\}$. There are $2n$ *v(ertex)–strips* strips in each direction to chose (combinatorially) different axis-parallel lines through an A–gadget from. If a line l goes through the i-th v–strip of the gadget we say that it *represents* the vertex $\mathrm{rep}(l) := i \mod n$ of G. If $0 < i < n$ the line l is called *negative*, otherwise it is called *positive*. Two lines are called *antipodal* if one is positive and the other is negative.

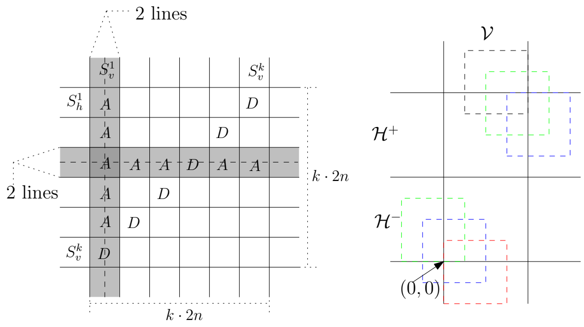

Figure 1: Double strips and a C_h–gadget ($n = 4$).

The construction ensures that if we have two antipodal horizontal lines that represent vertex i and two antipodal vertical lines that represent vertex j, then they intersect all the squares inside A iff i and j are adjacent in G, which will ensure (P1).

The D–gadget (diagonal). This type of gadget is an A–gadget for the graph with adjacency defined by the identity,i.e., $D := \{\square_{n-1}(i, j) \mid 1 \leq i \neq j \leq n\}$. The only line pairs allowed are those that correspond to the same vertex, which will ensure property (P2).

The F–gadget (forcing). This type of gadget will be used to force the existence of horizontal (or vertical) lines in a specified strip in any solution of size $6k$. We define them as $F_h := \{\square_n(-i \cdot n, 0) \mid 1 \leq i \leq 6k + 1\}$ and $F_v := \{\square_n(0, -i \cdot n) \mid 1 \leq i \leq 6k + 1\}$.

To intersect all squares of an F_h–gadget with only vertical lines requires at least $6k + 1$ such lines, thus in any solution with $\leq 6k$ lines there must be a horizontal line intersecting the gadget.

The C–gadget (consistency). This type of gadget is used to guarantee a certain distance between two lines *of the same direction* corresponding to the same double–strip of an A–(or D)–gadget. With this it will be possible to identify such line pairs with the vertices (P0). It will consist of three F–gadgets and $2(n - 1)$ additional squares. We describe the C_h–gadget for horizontal lines only (C_v–gadgets are defined analogously). C_h consists of the union of the two sets $\{\square_{n-1}(1 - i, i - n + 1) \mid 1 \leq i \leq n - 1\}$ and $\{\square_{n-1}(n + 1 - i, n + i - 1) \mid 2 \leq i \leq n\}$ together with three F–gadgets that force the existence of lines in the regions $\mathcal{H}^- = \mathbb{R} \times (0, n)$, $\mathcal{H}^+ = \mathbb{R} \times (n, 2n)$ and $\mathcal{V} = (0, n) \times \mathbb{R}$. See Figure 1 for an example.

The next lemma states the main property of the C_h–gadgets (C_v–gadgets have a symmetric property):

Lemma 5 *Given two antipodal horizontal lines h^-, h^+, in $\mathcal{H}^-, \mathcal{H}^+$, respectively, then there exists a vertical line that together with the others intersects all of the squares belonging to a C_h–gadget iff $\mathrm{rep}(h^+) \geq \mathrm{rep}(h^-)$. In fact, we can always assume that $\mathrm{rep}(h^+) = \mathrm{rep}(h^-)$.*

Putting it all together. We now describe the placement of the gadgets defined above. The main part, expressing the adjacency–relation of the graph, will be a $k \times k$ grid of A–gadgets and D–gadgets: $\mathcal{A} := \{A(i \cdot 3n, j \cdot 3n) \mid 1 \leq i \neq j \leq k\}$ and $\mathcal{D} := \{D(i \cdot 3n, i \cdot 3n) \mid 1 \leq i \leq k\}$. Around this grid, we add several C–gadgets to allow only specific solutions: $\mathcal{C}_h := \{C_h(-i \cdot 3n, i \cdot 3n) \mid 1 \leq i \leq k\}$ and $\mathcal{C}_v := \{C_v(i \cdot 3n, -i \cdot 3n) \mid 1 \leq i \leq k\}$. The size of the set $\mathcal{S}^*(G,k) := \mathcal{A} \cup \mathcal{D} \cup \mathcal{C}_h \cup \mathcal{C}_v$ is $\mathcal{O}\left(k^2 n^2\right)$.

Lemma 6 *$\mathcal{S}^*(G,k)$ can be stabbed by $6k$ axis-parallel lines iff G has a k–clique.*

To yield set $\mathcal{S}(G,k)$ of unit squares, we shrink the squares in the F–gadgets by letting $F_h := \{\Box_{n-1}(-in, 1/2) \mid 1 \leq i \leq 6k+1\}$ (similarly for F_v).

Corollary 2 follows by shifting all the squares by a small amount and replacing all (parallel) diagonals by very thin pairwise disjoint rectangles.

Arbitrary directions and general objects. First, the forcing gadgets are modified to contain n^2 squares; this will ensure that the chosen lines have to be "almost" parallel. Then, the squares are shrinked by some small value ϵ; this ensures that for each almost parallel line there is a parallel line that intersects the same squares. Thus, in any solution the lines can actually be assumed to be axis-parallel, i.e., Lemma 6 holds also for this new set. Again, shifting the squares a little and replacing the diagonals by rectangles as above yields a set of disjoint rectangles. This set is then linearly transformed to yield a combinatorially equivalent set of disjoint unit squares.

To prove the analogous results for arbitrary, connected objects we simply linearly transform them so that their axis-parallel bounding box is a unit square.

2.2 Fixed Parameter Tractable Cases

We prove that stabbing *disjoint* axis-parallel unit squares in the plane with k axis-parallel lines is fpt (with respect to k) by combining data reduction with branching and kernelization techniques. The algorithm can be easily modified to handle the case of general objects and line directions for Theorem 3. In this section we consider all sets to be closed.

Let $\mathcal{S}$ be a set of n disjoint axis-parallel unit squares. We want to determine if $\mathcal{S}$ can be stabbed by k axis-parallel lines. It suffices to consider only lines that support the boundary of a square in $\mathcal{S}$; there are at most $4n$ of them. For a line l let $I(l)$ denote the set of squares that are stabbed by l. We first apply the following data reduction rule to $\mathcal{S}$:

DR: *For all $\kappa > k+1$ squares with the same x–coordinates, delete all but $k+1$ of them, and also for $\kappa > k+1$ squares with the same y–coordinates.*

A set on which DR has been applied is called a *DR–set*. The algorithm is based on the following lemma.

Lemma 7 *Let l be a horizontal line that intersects $\kappa > k$ unit squares $I(l) = \{\Box(x_i, y_i) \mid 1 \leq i \leq \kappa\} \subseteq \mathcal{S}$. Then in order to stab the set $\mathcal{S}$ with k lines, there has to be a horizontal line l^* that intersects at least two squares from $I(l)$. l^* can be chosen from $B(I(l)) := \{y = y_i \mid 1 \leq i \leq \kappa\} \cup \{y = y_i + 1 \mid 1 \leq i \leq \kappa\}$.*

An analogous lemma holds for vertical lines as well. Observe that for a DR–set $\mathcal{S}$, if there is an axis-parallel line l with $|I(l)| > 2k+1$, then there is also a line l^* parallel to l with $k+1 \leq |I(l^*)| \leq 2k+1$. For such a line we use Lemma 7 by branching on all the $|B(I(l))|$ possible lines. If no such line exists, we end up with a problem kernel (in each branch): All the lines now intersect at most k squares, so the SOLVE function simply rejects if the number of squares left is more than k^2 and otherwise uses brute force.

STAB(S, k)
  if $S = \emptyset$ "SOLUTION FOUND"
  else if $k = 0$ **return**
  apply DR
  if there is a line l with $k+1 \leq |I(l)| \leq 2k+1$
    for all lines l' from the set $B(I(l))$
      STAB($S - I(l')$, $k-1$)
  else SOLVE(S, k)

Lemma 8 *If there is a solution of size k, the above algorithm finds a solution of size $k' \leq k$.*

The algorithm branches on at most $2(2k+1)$ possibilities at most k times and each call of the STAB procedure needs $\mathcal{O}(n \log n)$ steps. In each branch it ends up with a problem kernel, which can be solved in time $\left(4k^2\right)^k$ by exhaustive search, so the total running time is $\mathcal{O}(n \log n)$ for every fixed k.

3 Stabbing balls with one line

We show that the problem of stabbing unit balls in $\mathbb{R}^d$ with a line is W[1]-hard with respect to d by an fpt-reduction from the W[1]-complete k-independent set problem is general graphs [2]. Given an undirected graph $G([n], E)$ we construct a set $\mathcal{B}$ of balls in $\mathbb{R}^{2k}$ such that $\mathcal{B}$ can be stabbed by a line if and only if G has an independent set of size k.

For every ball $B \in \mathcal{B}$ we will also have $-B \in \mathcal{B}$. This allows us to restrict our attention to lines through the origin. For a line l, let $\vec{l}$ be its unit direction vector. We view $\mathbb{R}^{2k}$ as the product of k orthogonal planes $E_1, \ldots, E_k$, where each E_i has coordinate axes X_i, Y_i. The component on X_i, Y_i of a point p is denoted by $x_i(p)$, $y_i(p)$ respectively.

For each plane E_i, we define $2n$ $2k$-dimensional balls, with centers regularly spaced on the unit circle

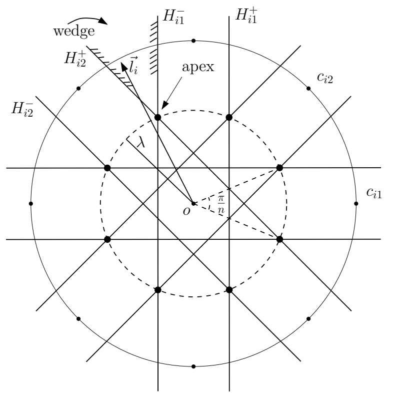

Figure 2: Centers of the balls and their respective half-planes and wedges on a plane E_i, for $n = 4$.

on E_i centered at the origin o. Let $c_{iu} \in E_i$ be the center of the ball B_{iu}, $u \in [2n]$, with $x_i(c_{iu}) = \cos(u-1)\frac{\pi}{n}$ and $y_i(c_{iu}) = \sin(u-1)\frac{\pi}{n}$. We define a scaffolding ball set $\mathcal{B}^0 = \{B_{iu}, i = 1, \dots, k \text{ and } u = 1, \dots, 2n\}$. We have $|\mathcal{B}^0| = 2nk$. All balls in $\mathcal{B}^0$ will have the same radius $r < 1$, to be defined later.

Two antipodal balls B, $-B$ are stabbed by the same set of lines. A line l stabs $\mathcal{B}^0$ if and only if it satisfies the system of nk inequalities: $(c_{iu} \cdot \vec{l})^2 \geq ||c_{iu}||^2 - r^2$, for $i = 1, \dots, k$ and $u = 1, \dots, n$.

Geometrically, the inequality asserting that l stabs B_{iu} says that the projection $\vec{l_i}$ of $\vec{l}$ on the plane E_i lies in one of the half-planes $H_{iu}^+ = \{p \in E_i | c_{iu} \cdot p \geq \sqrt{||c_{iu}||^2 - r^2}\}$ or $H_{iu}^- = \{p \in E_i | c_{iu} \cdot p \leq -\sqrt{||c_{iu}||^2 - r^2}\}$. Consider the situation on a plane E_i. Line l stabs all balls B_{iu} (centered on E_i) if and and only if $\vec{l_i}$ lies in one of the $2n$ wedges $\pm(H_{i1}^- \cap H_{i2}^+), \dots, \pm(H_{i(n-1)}^- \cap H_{in}^+)$, and $\pm(H_{i1}^- \cap H_{in}^-)$; see Fig 2. The apexes of the wedges are regularly spaced on a circle of some radius λ. By choosing $r = \sqrt{1 - (1 - \cos\frac{\pi}{n})/(2k)}$ we have that $\lambda = 1/(\sqrt{k})$.

Since $\vec{l} \in \mathbb{R}^{2k}$ is unit, we have that $||\vec{l_i}|| = 1/(\sqrt{k})$, for every $i \in \{1, \dots, k\}$. Hence, for line l to stab all balls in $\mathcal{B}^0$, every projection $\vec{l_i}$ must be one of the apexes in A_i. There are $2n$ choices for each $\vec{l_i}$, and the total number of lines that stab $\mathcal{B}^0$ is $n^k 2^{k-1}$.

For a tuple $(u_1, \dots, u_k) \in [2n]^k$, we denote by $l(u_1, \dots, u_k)$ the line with vector $\frac{1}{\sqrt{k}}(\cos(2u_1 - 1)\frac{\pi}{2n}, \sin(2u_1 - 1)\frac{\pi}{2n}, \dots, \cos(2u_k - 1)\frac{\pi}{2n}, \sin(2u_k - 1)\frac{\pi}{2n})$. Two lines $l(u_1, u_2, ..., u_k)$ and $l(v_1, v_2, ..., v_k)$ are said to be equivalent if $u_i \equiv v_i (\mod n)$, for all i. We have n^k equivalence classes $L(u_1, \dots, u_k)$, with $(u_1, \dots, u_k) \in [n]^k$, where each class consists of 2^{k-1} lines. There is a bijection between the possible equivalence classes of lines that stab $\mathcal{B}^0$ and $[n]^k$.

Constraint balls. For each pair of distinct indices $i \neq j$ ($1 \leq i, j \leq k$) and for each pair of (possibly equal) vertices $u, v \in [n]$, we define a constraint set of balls $\mathcal{B}_{ij}^{uv}$ with the property that (all lines in) all classes $L(u_1, \dots, u_k)$ stab $\mathcal{B}_{ij}^{uv}$ except those with $u_i = u$ and $u_j = v$. The centers of the balls in $\mathcal{B}_{ij}^{uv}$ lie in the 4-space $E_i \times E_j$. All lines in a class $L(u_1, \dots, u_k)$ project into only two lines on $E_i \times E_j$. We use a ball B_{ij}^{uv} (to be defined shortly) that is stabbed by *all* lines $l(u_1, \dots, u_k)$ except those with $u_i = u$ and $u_j = v$. Similarly, we use a ball $B_{ij}^{u\bar{v}}$ that is stabbed by *all* lines $l(u_1, \dots, u_k)$ except those with $u_i = u$ and $u_j = \bar{v}$, where $\bar{v} = v + n$. We have, $\mathcal{B}_{ij}^{uv} = \{\pm B_{ij}^{uv}, \pm B_{ij}^{u\bar{v}}\}$.

Consider a line $l = l(u_1, \dots, u_k)$ with $u_i = u$ and $u_j = v$. The center c_{ij}^{uv} of B_{ij}^{uv} will lie on a direction $\vec{z} \in E_i \times E_j$ orthogonal to $\vec{l}$, but for which $\vec{l'} \cdot \vec{z} \neq 0$ for any line $l' = l(u_1, \dots, u_k)$ with $u_i \neq u$ or $u_j \neq v$. Let ω be the angle between $\vec{l'}$ and $\vec{z}$. We choose $\vec{z}$ such that $|\cos\omega| > \frac{\lambda}{\sqrt{k}}$, where $\lambda < 1$ is an appropriate function of n. This helps us place B_{ij}^{uv} sufficiently close to the origin so that it is still intersected by l'. We can choose any point c_{ij}^{uv} on z with $r < ||c_{ij}^{uv}|| < r\sqrt{k/(k - \lambda^2)}$.

We add to $\mathcal{B}^0$ the $4n\binom{k}{2}$ balls in $\mathcal{B}_V = \bigcup \mathcal{B}_{ij}^{uu}$, $1 \leq u \leq n$, $1 \leq i < j \leq k$, to ensure that all components u_i in a solution (class of lines $L(u_1, \dots, u_k)$) are distinct. For each edge $uv \in E$ we also add the balls in $k(k-1)$ sets $\mathcal{B}_{ij}^{uv}$, with $i \neq j$. This ensures that the remaining classes of lines $L(u_1, \dots, u_k)$ represent independent sets of size k. In total, the edges are represented by $\mathcal{B}_E = \bigcup \mathcal{B}_{ij}^{uv}$, $uv \in E$, $1 \leq i, j \leq k$, $i \neq j$. The final set $\mathcal{B} = \mathcal{B}^0 \cup \mathcal{B}_V \cup \mathcal{B}_E$ has $2nk + 4\binom{k}{2}(n + 2|E|)$ balls. The constraint sets of balls exclude tuples with two equal indices $u_i = u_j$ or with indices u_i, u_j when $u_i u_j \in E$, thus, the classes of lines that stab B represent exactly the independent sets of G.

Lemma 9 *Set $\mathcal{B}$ can be stabbed by a line if an only if G has an independent set of size k.*

References

[1] M. Dom, M. R. Fellows, and F. A. Rosamond. Parameterized complexity of stabbing rectangles and squares in the plane. In *Proc. of the 3rd WALCOM*, LNCS, 2009. To appear.

[2] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006.

[3] R. Hassin and N. Megiddo. Approximation algorithms for hitting objects with straight lines. *Discrete Applied Mathematics*, 30:29–42, 1991.

[4] R. Impagliazzo and R. Paturi. On the complexity of k-sat. *J. Comput. Syst. Sci.*, 62(2):367–375, 2001.

[5] S. Langerman and P. Morin. Covering things with things. *Discrete & Computational Geometry*, 33(4):717–729, 2005.

[6] N. Megiddo. On the complexity of some geometric problems in unbounded dimension. *J. Symb. Comput*, 10:327–334, 1990.