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Constant-Level Greedy Triangulations Approximate the MWT Well

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Abstract

The well-known greedy triangulation $GT(S)$ of a finite point set S is obtained by inserting compatible edges in increasing length order, where an edge is compatible if it does not cross previously inserted ones. Exploiting the concept of so-called light edges, we introduce a new way of defining $GT(S)$. The new definition does not rely on the length ordering of the edges. It provides a decomposition of $GT(S)$ into levels, and the number of levels allows us to bound the total edge length of $GT(S)$. In particular, we show $|GT(S)| \leq 3 \cdot 2^{k+1}|MWT(S)|$, where k is the number of levels and $MWT(S)$ is the minimum-weight triangulation of S. This constitutes the first non-trivial upper bound on $|GT(S)|$ for general points sets S.

1 Introduction

A *triangulation* of a given set S of n points in the plane is a maximal set of non-crossing line segments (called *edges*) which have both endpoints in S. Besides the Delaunay triangulation and the minimum-weight triangulation, the greedy triangulation (GT) is among the three most prominent ones. It is obtained by inserting compatible edges in increasing length order, where an edge is compatible if it does not cross previously inserted ones. Various algorithms for computing the GT are known, and the GT has been used in several applications. See, e.g., [DDMW] for a short history.

One use of the greedy triangulation is a length approximation to the minimum-weight triangulation (MWT). For a given point set S, the MWT minimizes the total edge length for all possible triangulations of S. Unfortunately, there are no known efficient algorithms for computing a MWT for general point sets. Therefore, efficiently computable approximations to the MWT are of importance.

Although the GT tends to be short in practical applications (and the GT for uniformly distributed points is expected to be within a constant factor of the MWT, see [LL]), its worst-case length behaviour is fairly unexplored. The only known result [L] in this respect is that the GT can be a factor of $\Omega(\sqrt{n})$ longer than the MWT. In particular, no non-trivial worst-case upper bounds have been known.

In this paper, we prove an upper bound of the form $|GT(S)| \leq c_k \cdot |MWT(S)|$, where c_k is a constant depending on the shape of S but not on its size. In Section 2, we introduce a decomposition of $GT(S)$ into k disjoint sets of edges, called *levels*, where the parameter $k \geq 1$ results from the shape of S. The level decomposition provides a new way of viewing, or defining, $GT(S)$ without having at hands the sorted list of edges spanned by S. It is based on the concept of *light* edges introduced in [AACRTX] and allows us to gain more insight into the structure of $GT(S)$. In particular, we show $c_k \leq 3 \cdot 2^{k+1}$ for $k \geq 2$ in Section 3. This generalizes the result $c_1 = 1$ in [AACRTX] and implies that a GT with constantly many levels is a constant approximation to the MWT. Section 4 studies the number of levels in a GT and offers a short discussion of the presented topic.

2 A Level Decomposition of GT

The usual procedural definition of $GT(S)$, as given in the introduction, resorts to the length properties of the edges spanned by S as well as to their crossing properties[1]. In particular, an edge which is not crossed by any shorter edge will surely belong to $GT(S)$. Let us call an edge *light* in this case. Below is a catalog of basic properties of light edges.

Lemma 1 *Let L denote the set of all light edges defined by S.*

(a) L is a non-crossing set of edges.

(b) L contains all edges bounding the convex hull of S.

(c) L is a subset of $GT(S)$.

(d) In general, L is no subset of $MWT(S)$.

The following result is less obvious and is proved in [AACRTX]. Let $|A|$ be the *weight* of a given set A of edges, that is, the sum of the lengths of all the edges in A.

Lemma 2 $|L| \leq |MWT(S)|$.

In conjunction with Lemma 1(c), Lemma 2 immediately implies: if L happens to form a triangulation of S, then $|L| = |GT(S)| = |MWT(S)|$. In any case, we learn that at least a subset of the edges in $GT(S)$ can be bounded in length by the weight of $MWT(S)$.

[1] By length of an edge we mean the Euclidean distance between its endpoints. Two distinct edges are said to cross if they intersect in their interiors. To ease the presentation, let us assume throughout that S is in general position and that no two edges have the same length.

In general, the following fact prevents particular edges in $GT(S)$ from being light. An edge e, though showing up in $GT(S)$, may still be crossed by some edge f shorter than e, as f might be non-compatible, that is, f is crossed by shorter edges which have already been inserted into $GT(S)$. This fact suggests to generalize the definition of light edges.

The edges in L are called *light of level 1*. Let E be the total set of edges defined by S, and let C_1 collect all edges of E that are crossed by some edge in L. Notice that each edge in L, and therefore no edge in C_1, appears in $GT(S)$. Define $E_2 = E \setminus (L \cup C_1)$. An edge $e \in E_2$ is called *light of level 2* if e is not crossed by a shorter edge in E_2. Let L_2 be the set of all edges which are light of level 2, and let C_2 collect all edges of E_2 that are crossed by some edge in L_2. Again, each edge in L_2, and therefore no edge in C_2, appears in $GT(S)$. By setting $E_3 = E_2 \setminus (L_2 \cup C_2)$ we now can define, in the obvious way, the set L_3 of edges which are light of level 3. Repeating this process until $E_{k+1} = \emptyset$ yields a hierarchy of levels $L_1, L_2, \ldots, L_k$ with $L_1 = L$.

It is evident that levels are pairwise disjoint, and that no edge of level i can cross an edge of level j, for $1 \leq i, j \leq k$. More specifically, we have:

Lemma 3 $GT(S) = L_1 \cup L_2 \cup \ldots \cup L_k$.

Lemma 3 gives an alternate, though still procedural, definition of $GT(S)$ which is of interest in its own right. It provides a more structured view of $GT(S)$ than does the original definition and does not require the sorted order of the edges spanned by S. Clearly, a level decomposition of $GT(S)$ can be computed in polynomial time. We leave open the question whether the level decomposition proves useful in the design of new and efficient greedy triangulation algorithms.

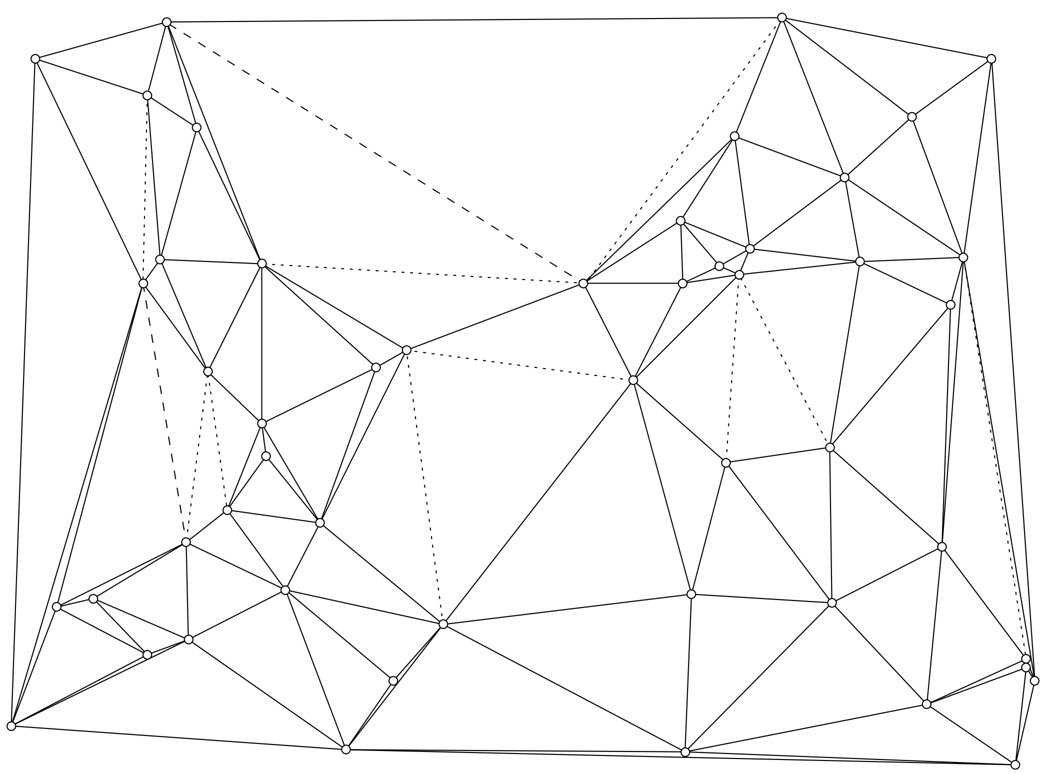

Figure 1: GT with three levels shown full, dotted, and dashed.

From Lemma 2 we learn that a GT with a single level is in fact a MWT. This suggests the conjecture that, if $GT(S)$ has only a few levels, it should not decline in weight too much from $MWT(S)$. We affirmatively settle this conjecture in the next section.

3 Bounding the Weight of a k-Level GT

The goal of this section is to establish an upper bound on the weight of $GT(S)$ that depends on the number of levels of $GT(S)$.

Theorem 1 *Let S be a finite set of points in the plane, and let k be the number of levels attained by $GT(S)$. Then $|GT(S)| \leq c_k \cdot |MWT(S)|$, where $c_1 = 1$ and $c_k = 3 \cdot 2^{k+1}$ for $k \geq 2$.*

The special case $k = 1$ follows from Lemma 2. To our knowledge, Theorem 1 constitutes the first non-trivial upper bound on the weight of a GT for general point sets. Still, the exponential dependence on the number of levels calls for improvement. Notice, however, that the bound is independent of the cardinality of the underlying set of points.

The reminder of this section contains a proof of Theorem 1. The proof is mainly based on an appropriate weighting scheme for the points in S. It proceeds in three stages. First, each point is associated with an initial weight, such that the sum of these weights can be related to the weight of $MWT(S)$ (Lemma 4). Next, point weights are updated stepwise, where each step corresponds to a level of $GT(S)$, and the increase of weight per point is controlled (Lemma 5). At last, the sum of the final point weights is used to bound the weight of $GT(S)$ (Lemma 7).

The weight for each point $p \in S$ is obtained by assigning to p a certain star of incident edges. A star can also contain edges which are not in $GT(S)$. The initial star of p, $\sigma_0(p)$, consists of three edges such that the angles between consecutive edges are less than π, and the sum of the edge lengths is minimum. (An exception are points lying on the convex hull of S. Their initial star consists of the two respective convex hull edges instead.) Here, and during the subsequent extensions and modifications of stars, the weight of a point p is defined to be the length of the longest edge in its star. Let $w_0(p)$ be the initial weight of p.

Lemma 4 $\sum_{p \in S} w_0(p) \leq 2 \cdot |MWT(S)|$.

Proof. For any point $p \in S$, any triangulation of S has to contain edges incident to p whose total length is at least $w_0(p)$. In particular, this is true for $MWT(S)$. The factor 2 is obtained because each edge of $MWT(S)$ is counted twice in this way, once for each endpoint. □

Let $GT(S)$ consist of k levels. For each point $p \in S$, its star is now updated during k steps. Let $\sigma_i(p)$ denote the star of p after step i, for $i = 1, \ldots, k$. The following three invariants for $\sigma_i(p)$ are maintained.

(1) $\sigma_i(p)$ contains – among possible other edges – all incident edges in $GT(S)$ which are of level at most i.

(2) All angles between consecutive edges in $\sigma_i(p)$ are less than π.

(3) No edge of $\sigma_i(p)$ is crossed by any edge in $GT(S)$ of level $\leq i$.

To maintain invariant (1), all incident level-i edges are added to $\sigma_{i-1}(p)$. Adding such an edge is called an *expansion* of a star. Clearly, an expansion does not destroy invariant (2).

After having expanded the stars for all points in S, invariant (3) may be violated. Assume this is the case for point p. For each edge g in p's star that is crossed by edges of level $\leq i$ we do the following; see Figure 2. Let ℓ be the edge of level $\leq i$ that crosses g closest to p. Consider the (topologically closed) triangle spanned by ℓ and p, and let x and y be the points in this triangle that are hit first when g is rotated about p. (Note that x or y can be an endpoint of ℓ.) We remove g, and add the edges px and py if they have not been part of the star yet. This action is called a *modification* of a star.

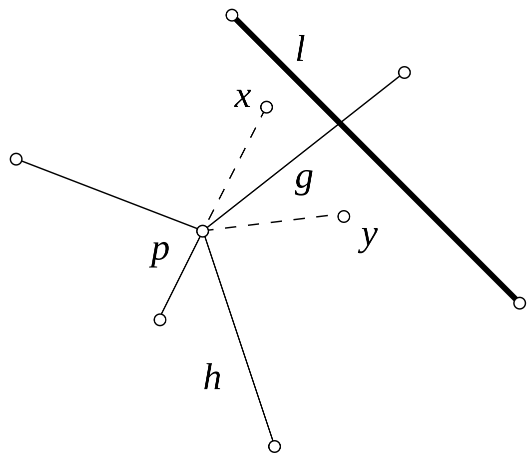

Figure 2: Modification of a star.

Indeed, after the modification, no edge of level $\leq i$ crosses px or py. By construction, any such edge e would have to cross g, too. As being no edge of $GT(S)$, g was already present in $\sigma_{i-1}(p)$. Hence, by invariant (3), e cannot be of level $\leq i-1$. It also cannot be of level i, as this would contradict the definition of ℓ, x, and y. Notice finally that invariant (2) is maintained during a modification.

Recall that the weight of a point p after step i, $w_i(p)$, was defined to be the length of the longest edge in $\sigma_i(p)$.

Lemma 5 *$w_i(p) \leq 2 \cdot w_{i-1}(p)$, for $i = 1, \ldots, k$.*

Proof. We argue that an expansion, as well as a modification, can produce edges of length at most twice the length of the longest edge in $\sigma_{i-1}(p)$.

We first consider an expansion of $\sigma_{i-1}(p)$. Let ℓ be an edge of level i that is added, and let g and h be the edges neighbored to ℓ in $\sigma_{i-1}(p)$; see Figure 3. We have $|g|, |h| \leq w_{i-1}(p)$. Assume $|\ell| > w_{i-1}(p)$ as nothing is to prove, otherwise. By invariant (2), ℓ then crosses the edge qr, where q and r is the second endpoint of g and h, respectively. Consider the convex hull of all points of S that lie in the triangle pqr, including q and r but excluding p. Let e be the edge of the convex hull which is crossed first by ℓ when coming from p. (Note that $e = qr$ is possible.) By construction, e cannot be crossed by any edge of level $\leq i-1$ as such an edge would have to cross g or h, too, contradicting invariant (3). Hence $|\ell| \leq |e|$, as e instead of ℓ would be of level i, otherwise. Moreover, by the triangle inequality, $|e| \leq |g| + |h|$. We conclude $|\ell| \leq 2 \cdot w_{i-1}(p)$.

Similar arguments can be used to bound the length of edges stemming from a modification of the expanded star. Let g, ℓ, x, and y be defined as in the modification step; cf. Figure 2. Recall that g already appears in $\sigma_{i-1}(p)$. So $|g| \leq w_{i-1}(p)$. Furthermore, g cannot cross edges of level $\leq i-1$ by invariant (3), but by definition crosses the level-i edge ℓ added in step i. This implies $|\ell| \leq |g|$. By construction of x and y, $\max\{|px|, |py|\} \leq |g| + |\ell|$. We conclude $|px|, |py| \leq 2 \cdot |g| \leq 2 \cdot w_{i-1}(p)$. □

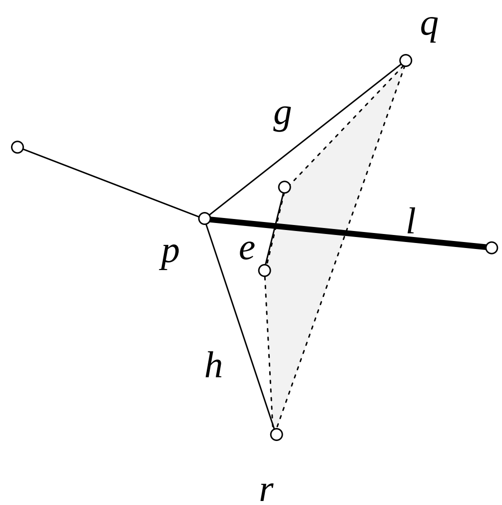

Figure 3: Expansion of a star.

Finally, in order to bound the weight of $GT(S)$ by means of the weights of the points in S after step k, we utilize the following result proved in [AAR].

Lemma 6 *Let T be an arbitrary triangulation of S. Then the edges of T can be oriented such that each point $p \in S$ has an in-degree of at most 3.*

The assertion below is now easy to prove.

Lemma 7 $|GT(S)| \leq 3 \cdot \sum_{p \in S} w_k(p)$.

Proof. According to Lemma 6, we orient the edges of $GT(S)$ such that each point $p \in S$ has at most three incident edges pointing at it. By invariant (1), these edges also have to be present in the final star $\sigma_k(p)$. Therefore, their total length is bounded by $3 \cdot w_k(p)$. Summing over all points in S gives the stated result. □

Theorem 1, the main result of this paper, now follows from combining Lemmas 4, 5, and 7.

4 On the Number and Length of Levels

The quality of the bound on GT expressed in Theorem 1 depends on how a GT is structured into levels. Below we summarize several observations on the number of levels as well as on lower bounds on the weight of particular levels.

Let us first report on some experimental results. We have run a level decomposition algorithm for the GT of $n \leq 200$ points uniformly distributed in the unit square. Figure 4 displays the number

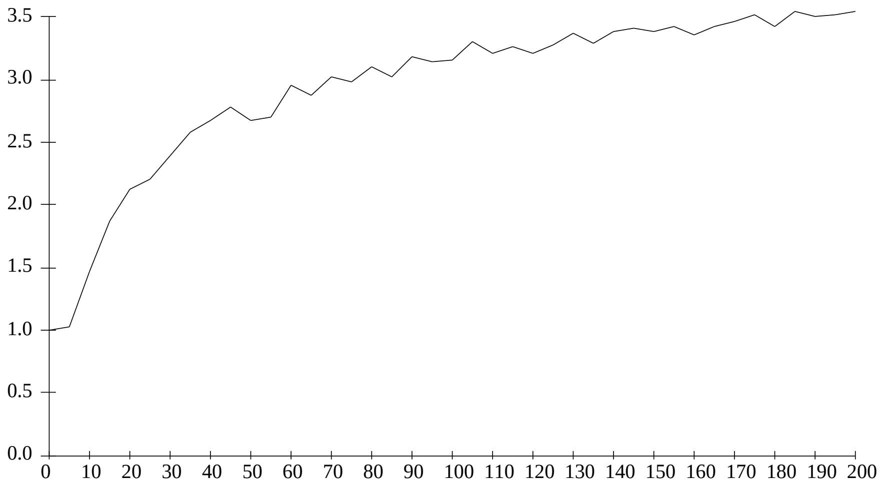

Figure 4: Expected number of levels.

k of levels in dependence of n. For each n, this number has been averaged over 100 sets of cardinality n. As can be observed, k shows an almost constant behaviour already for moderately large n.

For constant k, Theorem 1 implies that the GT is a constant approximation to the MWT. Hence, for uniformly distributed points, our observed behaviour of GT is in accordance with the theoretical result in [LL].

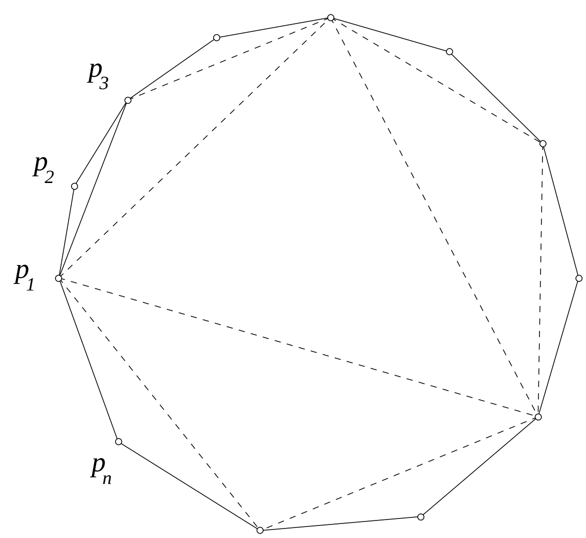

Figure 5: GT with many levels.

For specially constructed point sets, however, k can be up to linear in n. The points $p_1, p_2, \ldots, p_n$ in Figure 5 are placed on a circle where, for $i = 2, \ldots, n$, $|p_{i-1}p_i|$ increases by a fixed amount such that $|p_n p_1| = 2 \cdot |p_1 p_2|$. Aside from convex hull edges, only $p_1 p_3$ belongs to level L_1. This is because, for $2 \leq i \leq n$, $|p_i p_j| > |p_{i-1} p_{i+1}|$ if $j \geq i + 2$. For similar reasons, only $p_3 p_5$ belongs to level L_2. By repeating this type of argument, we see that in fact each inner edge of $GT(\{p_1, \ldots, p_n\})$ constitutes a separate level. This gives a lower bound of $k \geq n - 3$.

As a curious fact, notice that $GT(\{p_1, \ldots, p_n\})$ – though far from being a light triangulation –

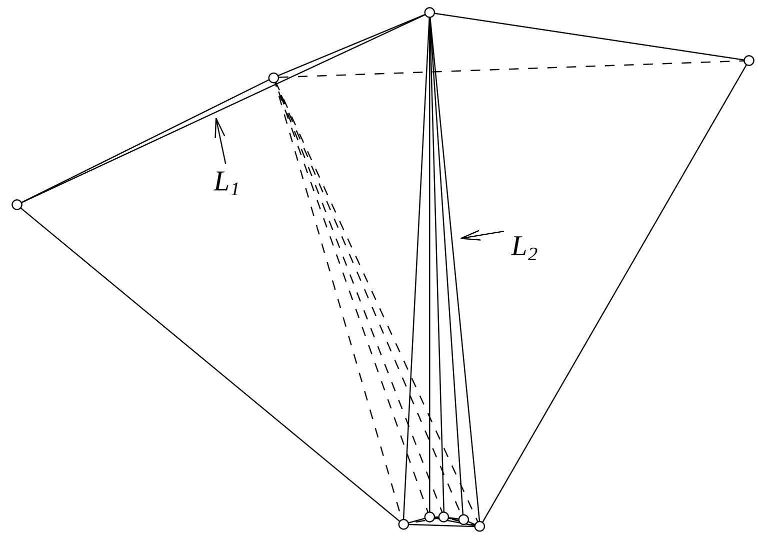

Figure 6: Level 2 (full) exceeds the MWT (dashed).

coincides with $MWT(\{p_1, \ldots, p_n\})$ in this case.

From Lemma 2 the question arises whether each particular level of a GT can be bounded in weight by the corresponding MWT. (The proof of Theorem 1 shows that levels can at most double in weight in the worst case.) A result of this kind would give a bound on GT that depends linearly on k rather than exponentially. Figure 6 exhibits a point set S where $|L_2| > |MWT(S)|$. The inner edge shown in full at the left top of the picture belongs to level L_1. This edge is shorter than all dashed edges it crosses and excludes them from $GT(S)$. Therefore, the bundle of full edges emanating from the points on the bottom of the picture belongs to level L_2. The dashed edges emanating from these points, however, are shorter than the full ones and belong to $MWT(S)$. Thus, if the number of bottom points is sufficiently large, L_2 will dominate $MWT(S)$ in weight. A similar construction can be done for a higher level L_i, yielding $|L_i| > |MWT(S)|$ for a given value $i \leq k$.

Despite of these negative results, it still remains open whether $|GT(S)| \leq c_k \cdot |MWT(S)|$, where c_k is polynomial in k. We have taken a look at the number of levels in the GT example in [L] – where the GT is a factor of $\Omega(\sqrt{n})$ longer than the MWT – and have found this number to be $\Theta(\sqrt{n})$. From this we conjecture $c_k = O(k)$. A proof of this conjecture, however, seems to require methods different from those used in the present paper.

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