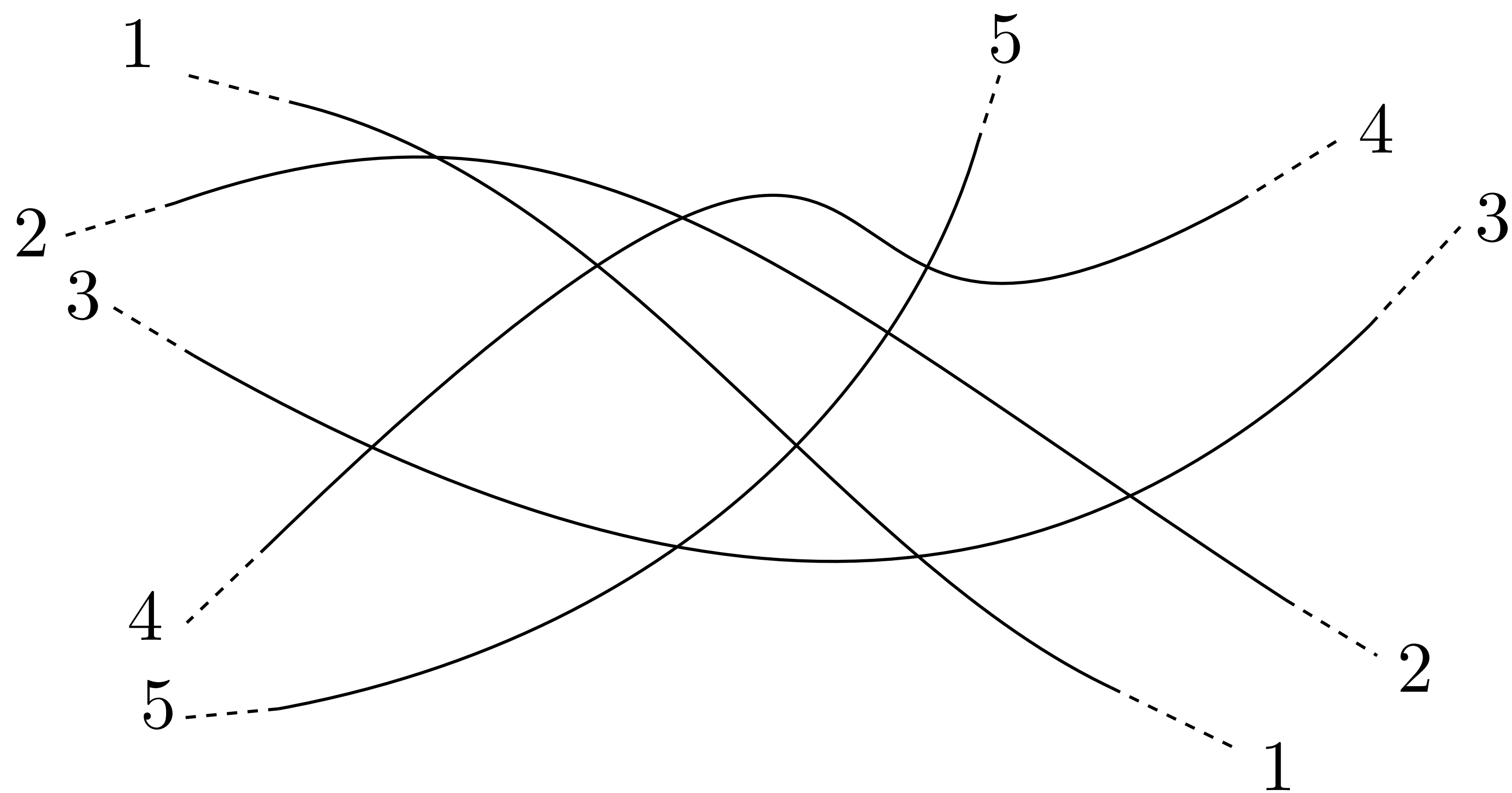
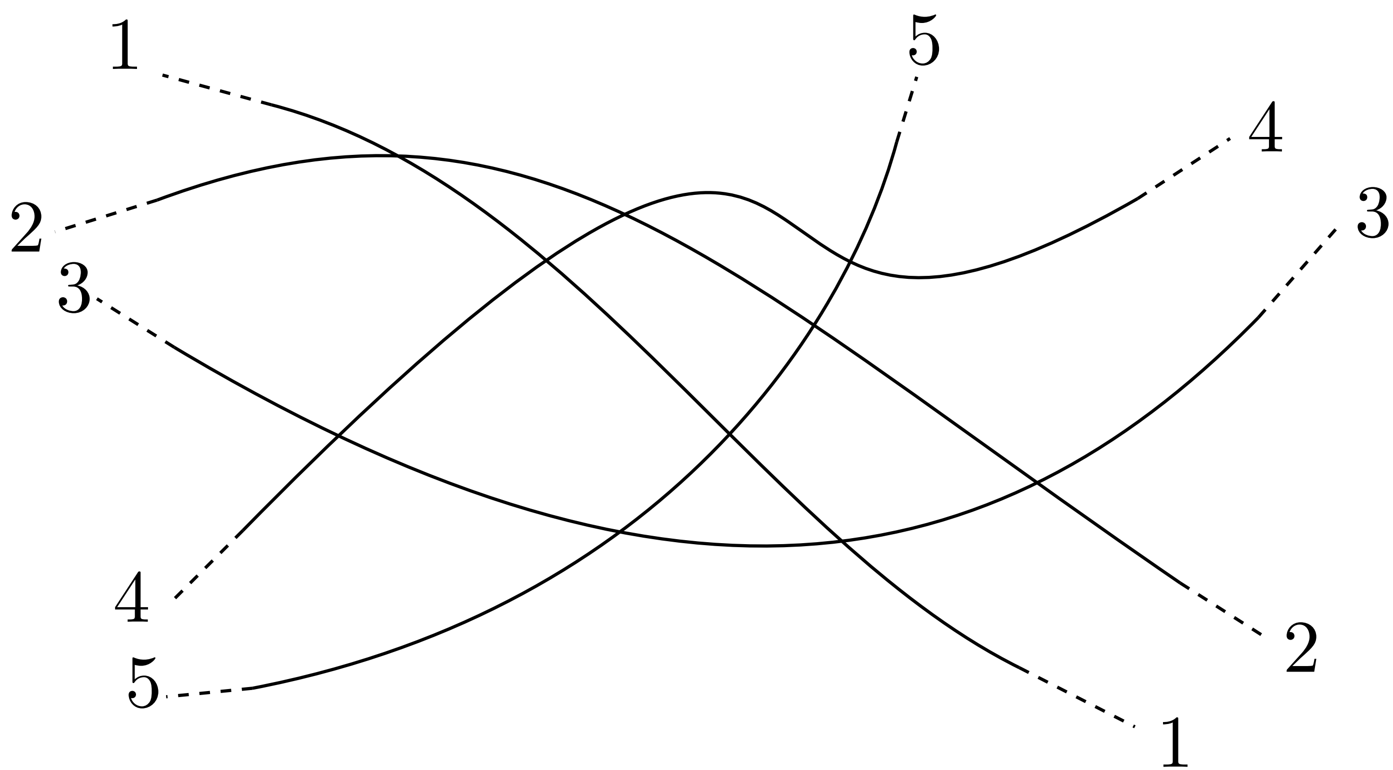
Enumeration and Counting of Pseudoline Arrangements

Günter Rote
Freie Universität Berlin



Pseudoline Arrangements

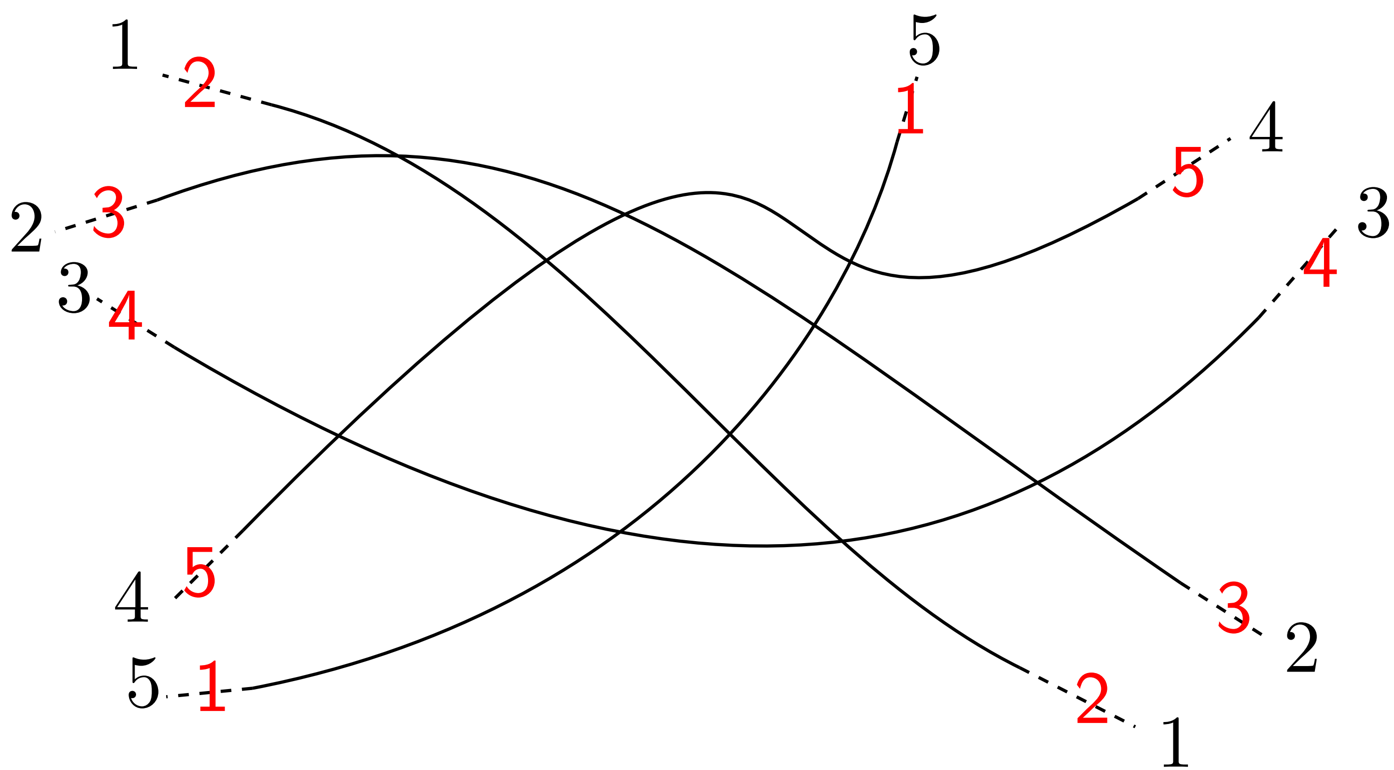


- n curves going to infinity
- Two curves intersect exactly once, and they cross.

- *simple* pseudoline arrangements: no multiple crossings
- x-monotone curves

Pseudoline Arrangements

a *different* arrangement



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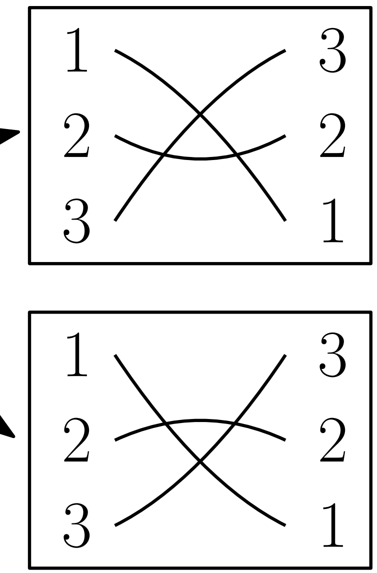
- x-monotone curves

How many pseudoline arrangements?

n	#PsA's with n pseudolines	
1	1	
2	1	
3	2	
4	8	
5	62	
6	908	
7	24698	
8	1232944	
9	112018190	
10	18410581880	
11	5449192389984	
12	2894710651370536	
13	2752596959306389652	
14	4675651520558571537540	[Yuma Tanaka, 2013]
15	14163808995580022218786390	[Yuma Tanaka, 2013]
16	76413073725772593230461936736	[G. Rote, 2021]



OEIS A006245

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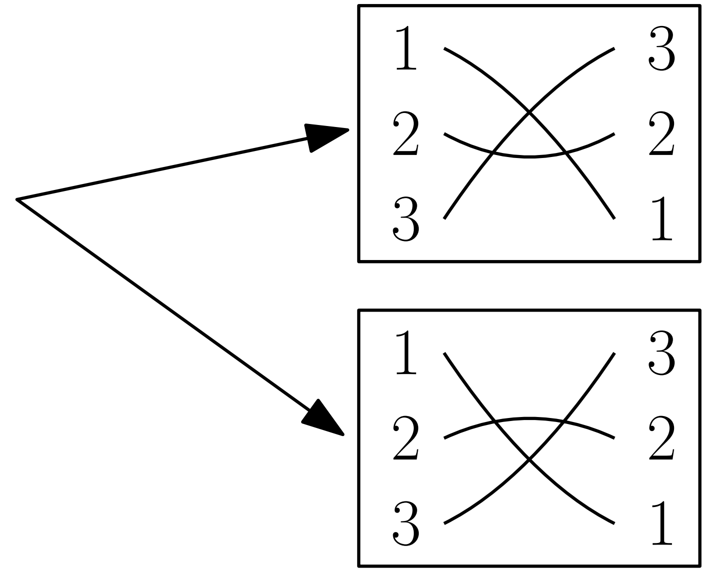
$\geq 2^{0.2083n^2}$
[Dumitrescu, Mandal 2018]

$\leq 2^{0.6571n^2}$
[Felsner, Valtr 2012]



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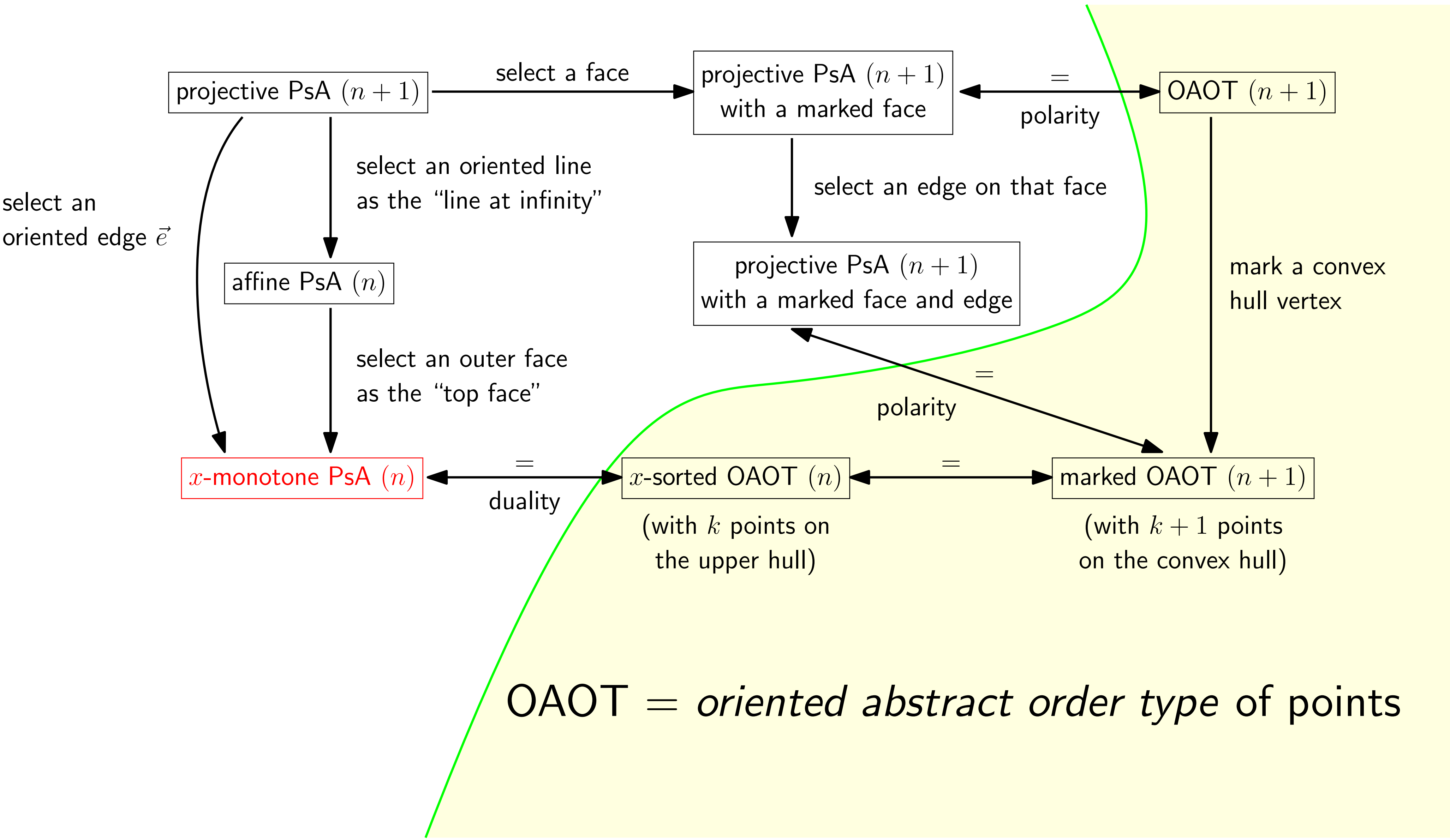
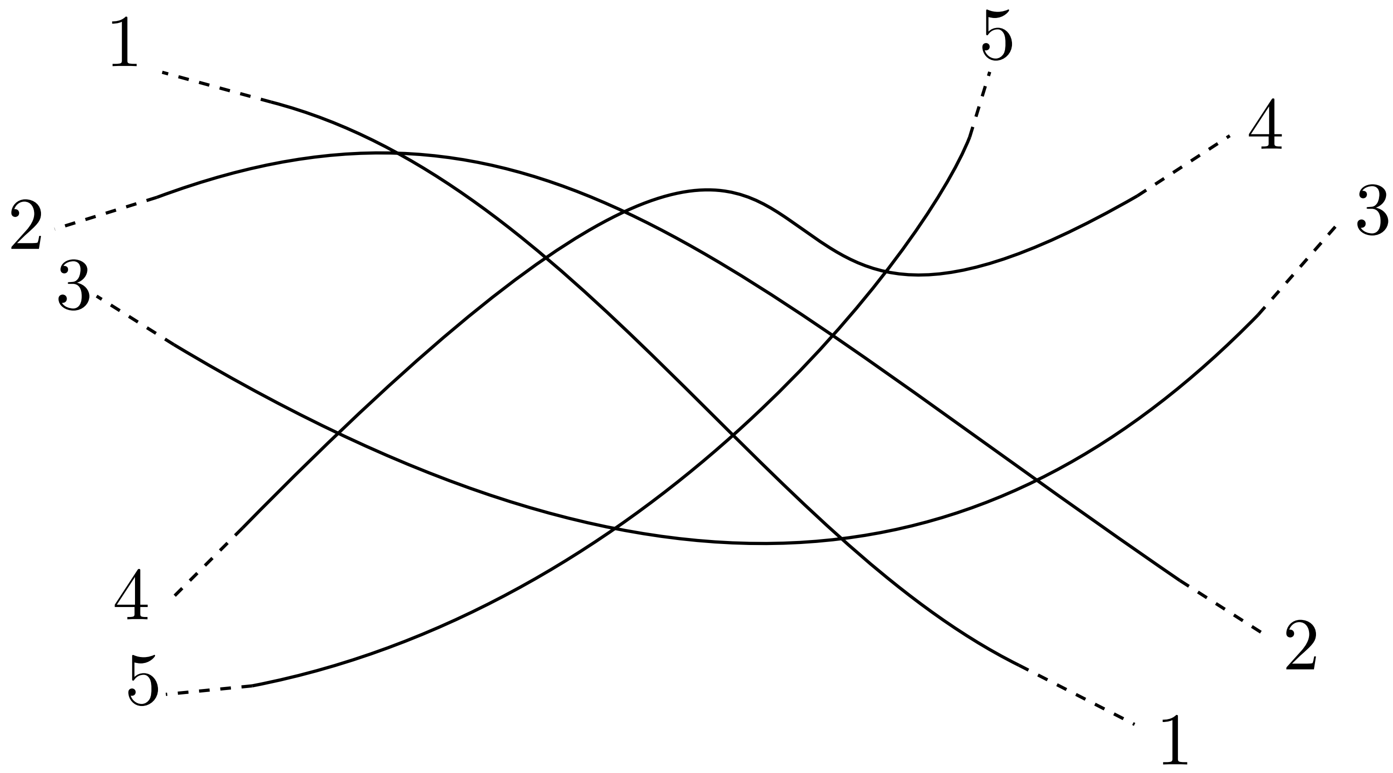
$n = 14, 15$: [Yuma Tanaka, 2013]

$n = 16$: [G. Rote, 2021]

Related concepts



Inductive Enumeration of PsA's



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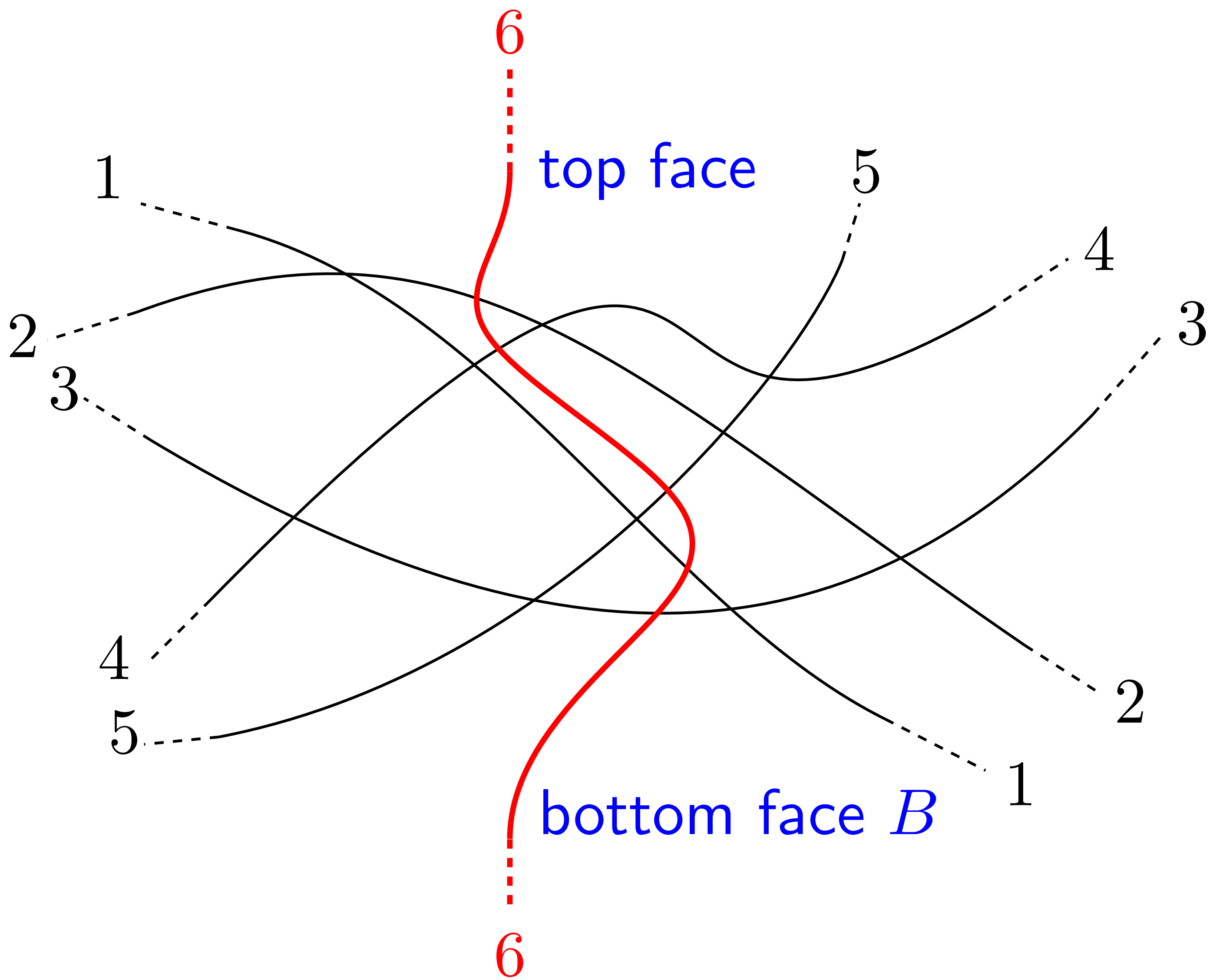

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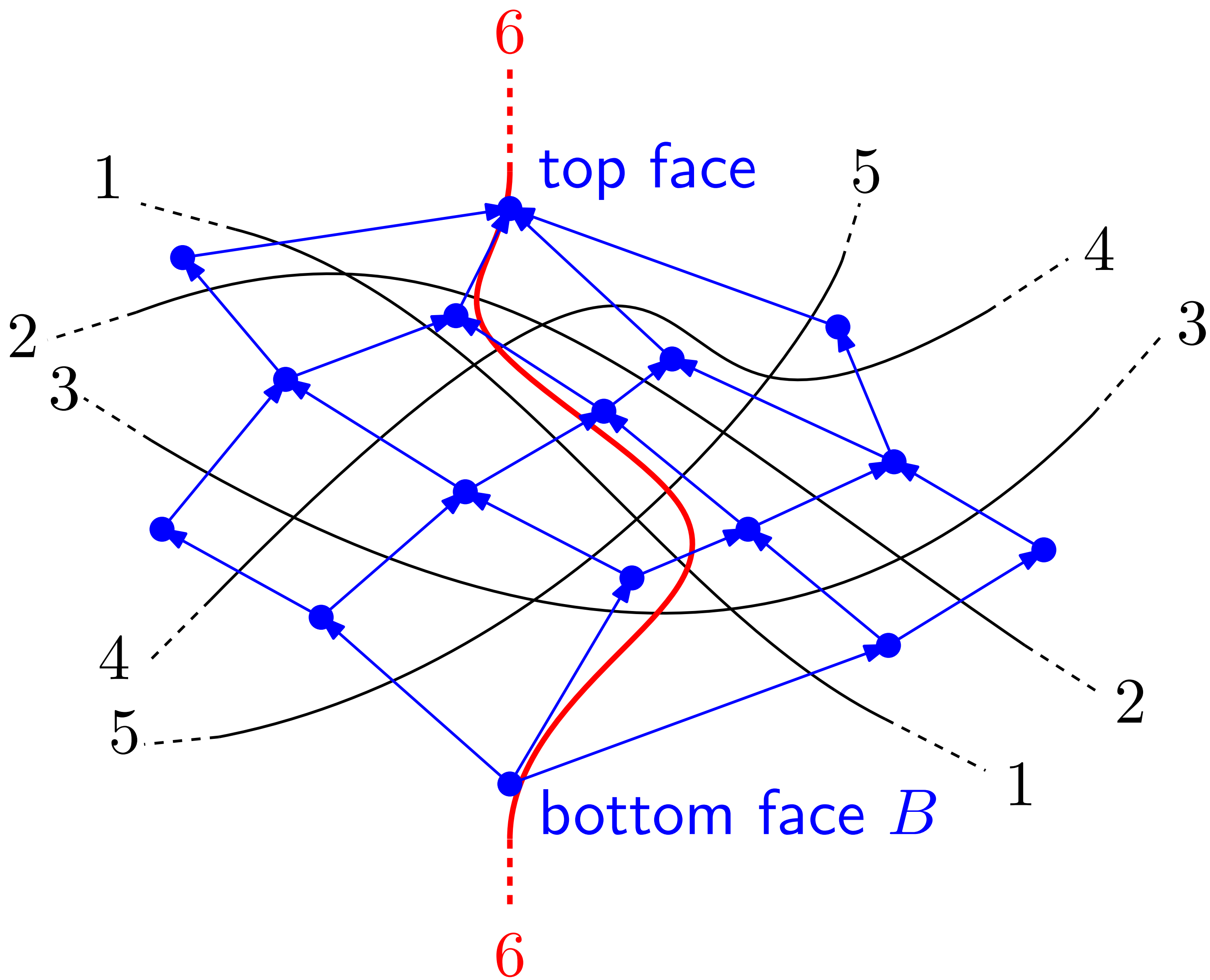
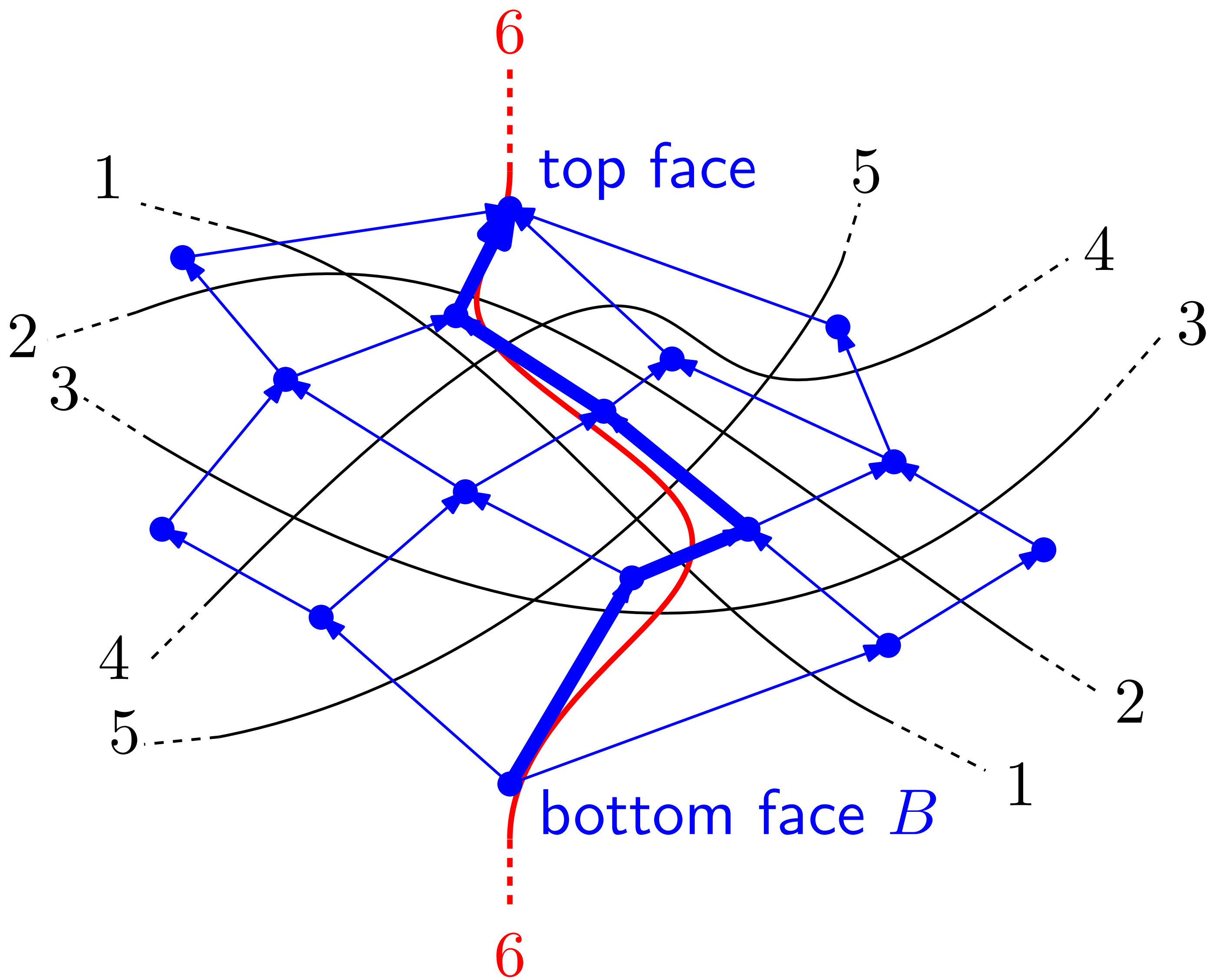


pseudoline $n + 1$ = path in the dual DAG

Inductive Enumeration of PsA's



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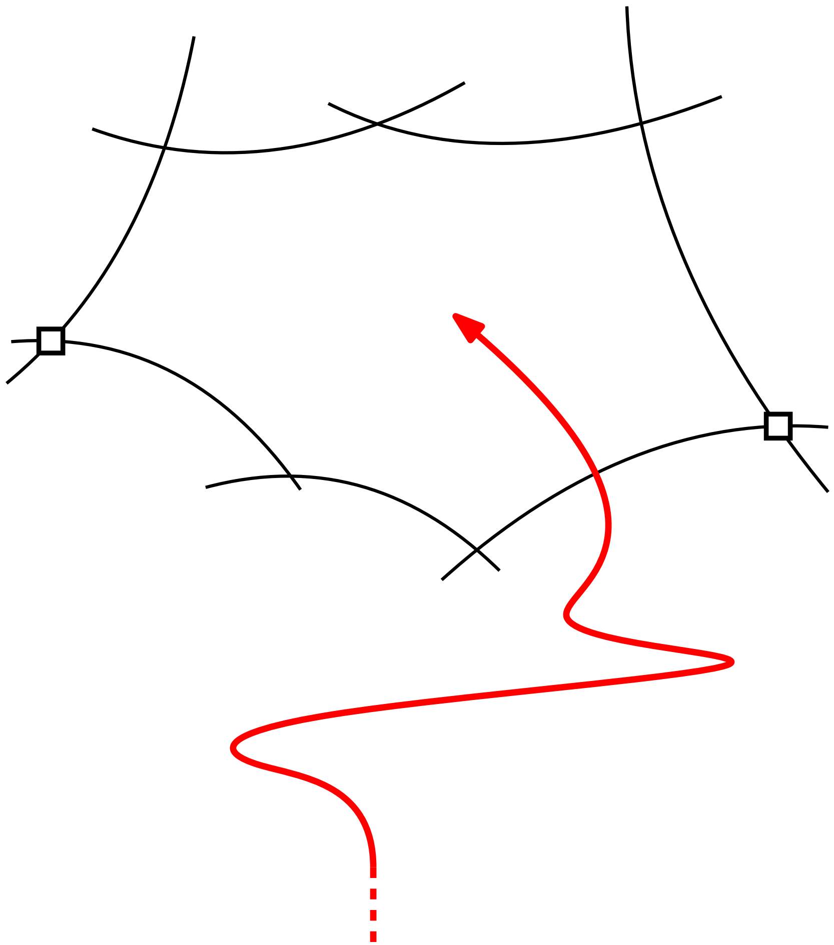
Inductive Enumeration of PsA's

Generation (enumeration) is straightforward. (No dead ends!)



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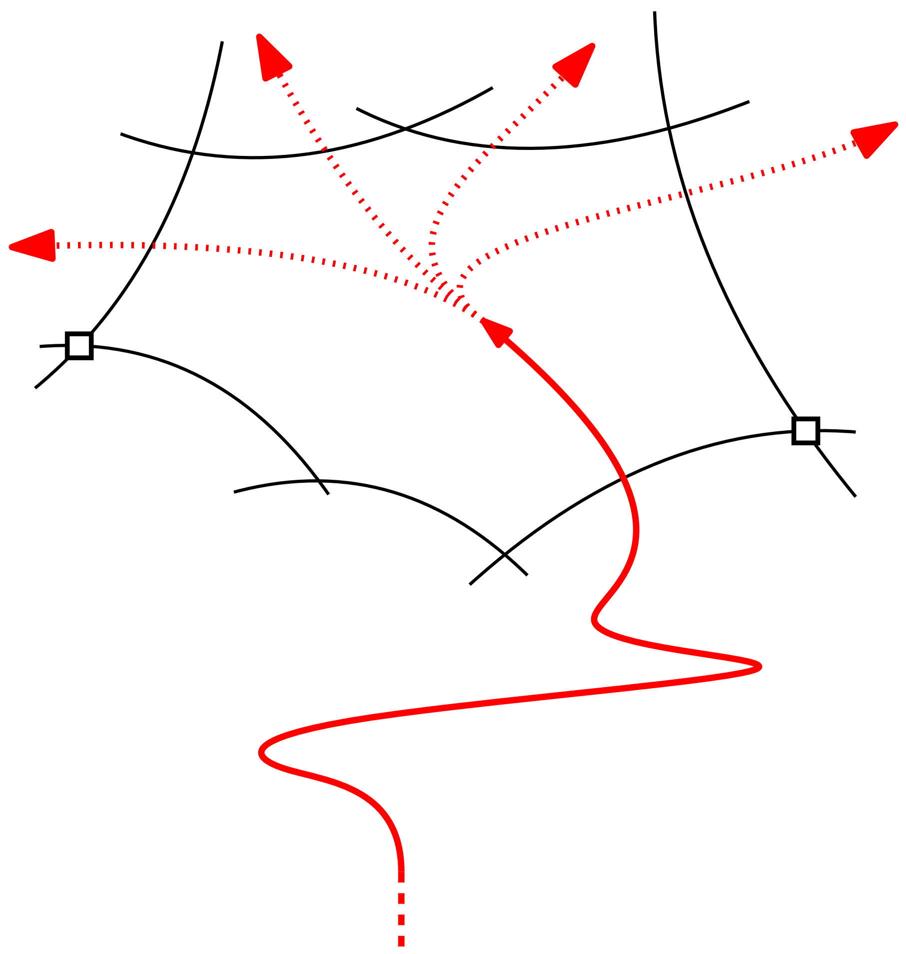
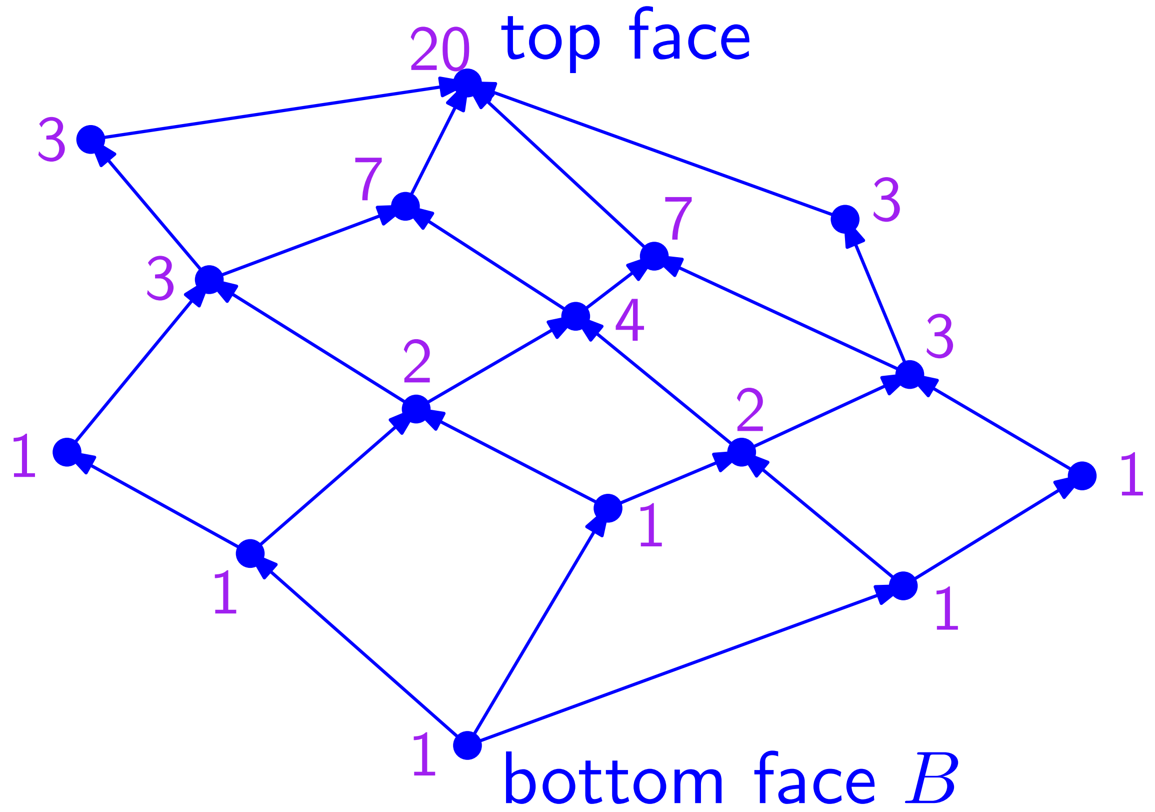


Inductive Enumeration of PsA's

Counting is straightforward. (#paths from B in a DAG)

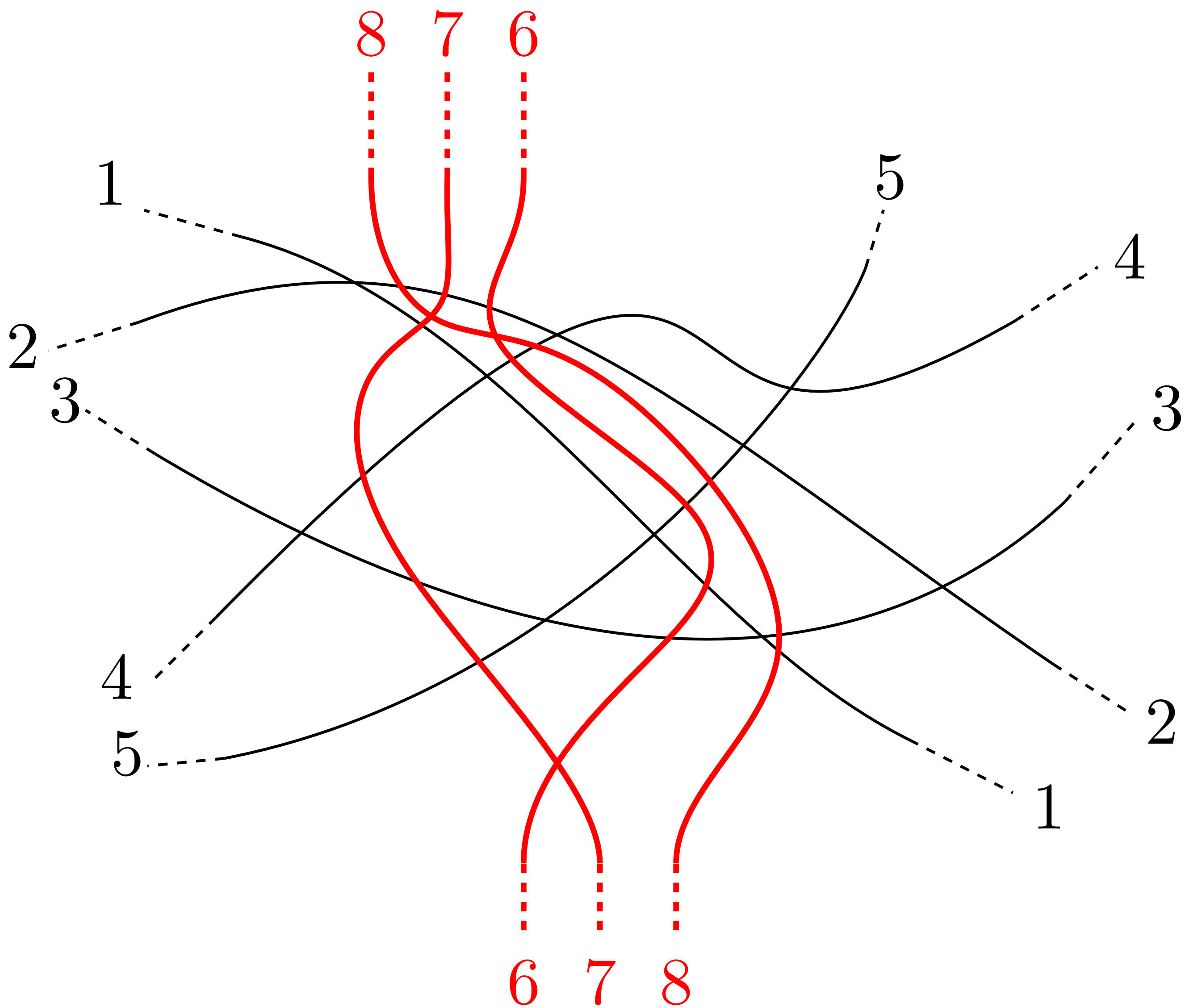


#paths $\leq 2.49^n$
[Felsner, Valtr 2012]

#paths can be as large as 2.076^n.
[O. Bílka 2010]

pseudoline $n+1$ = path in the dual DAG

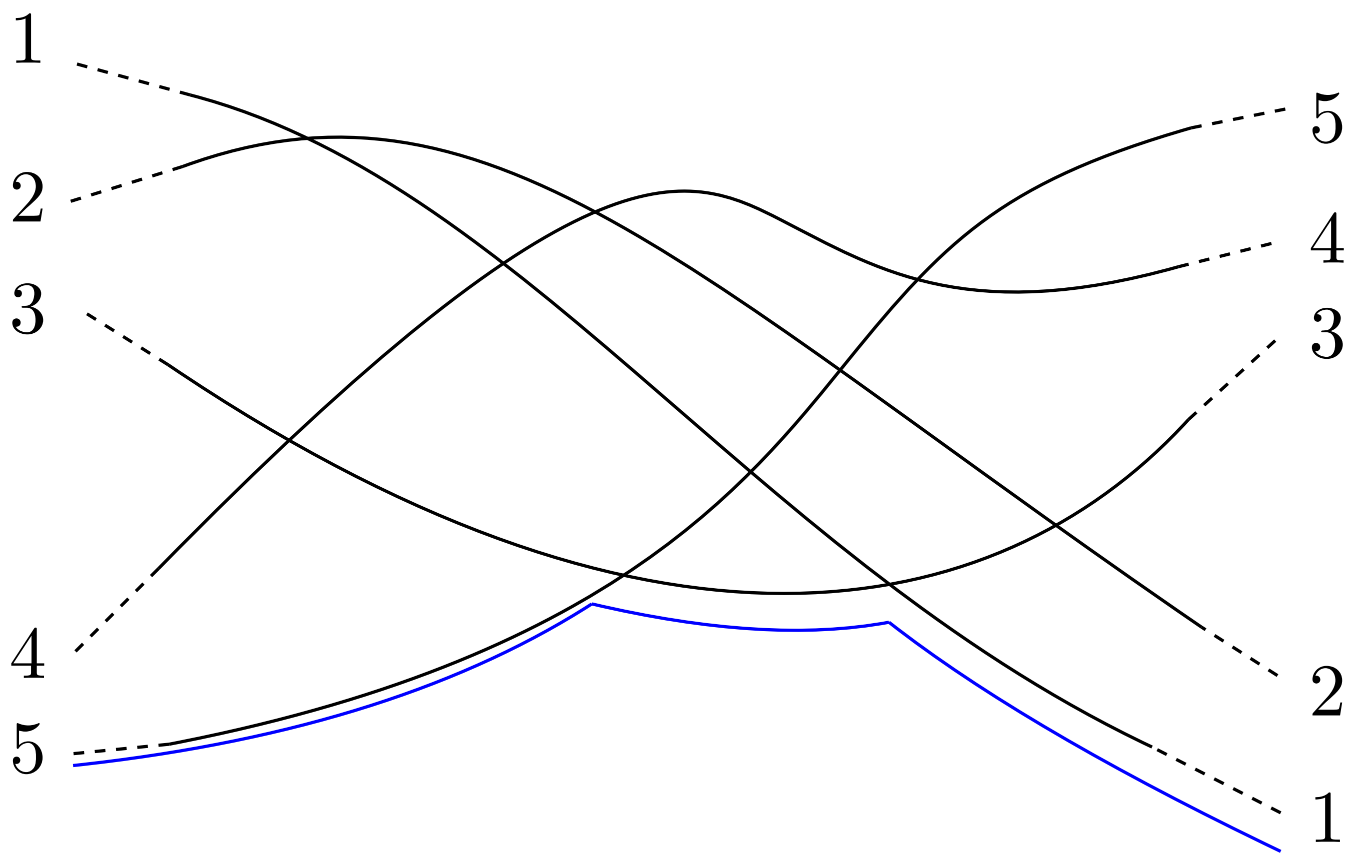
Threading several pseudolines at once



$k = 5$ *pseudolines*

$\ell = 3$ *strands*

A sequence of ropes



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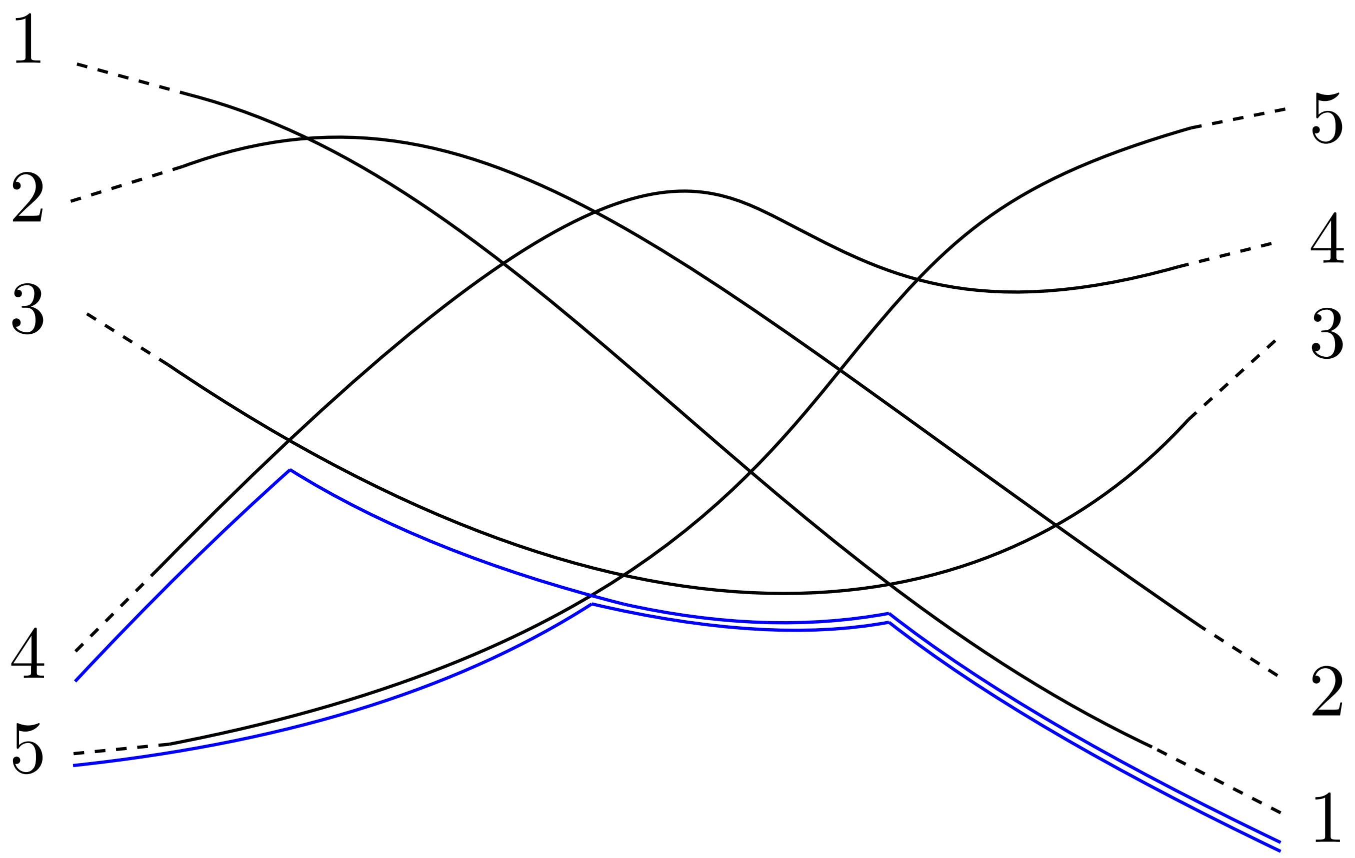

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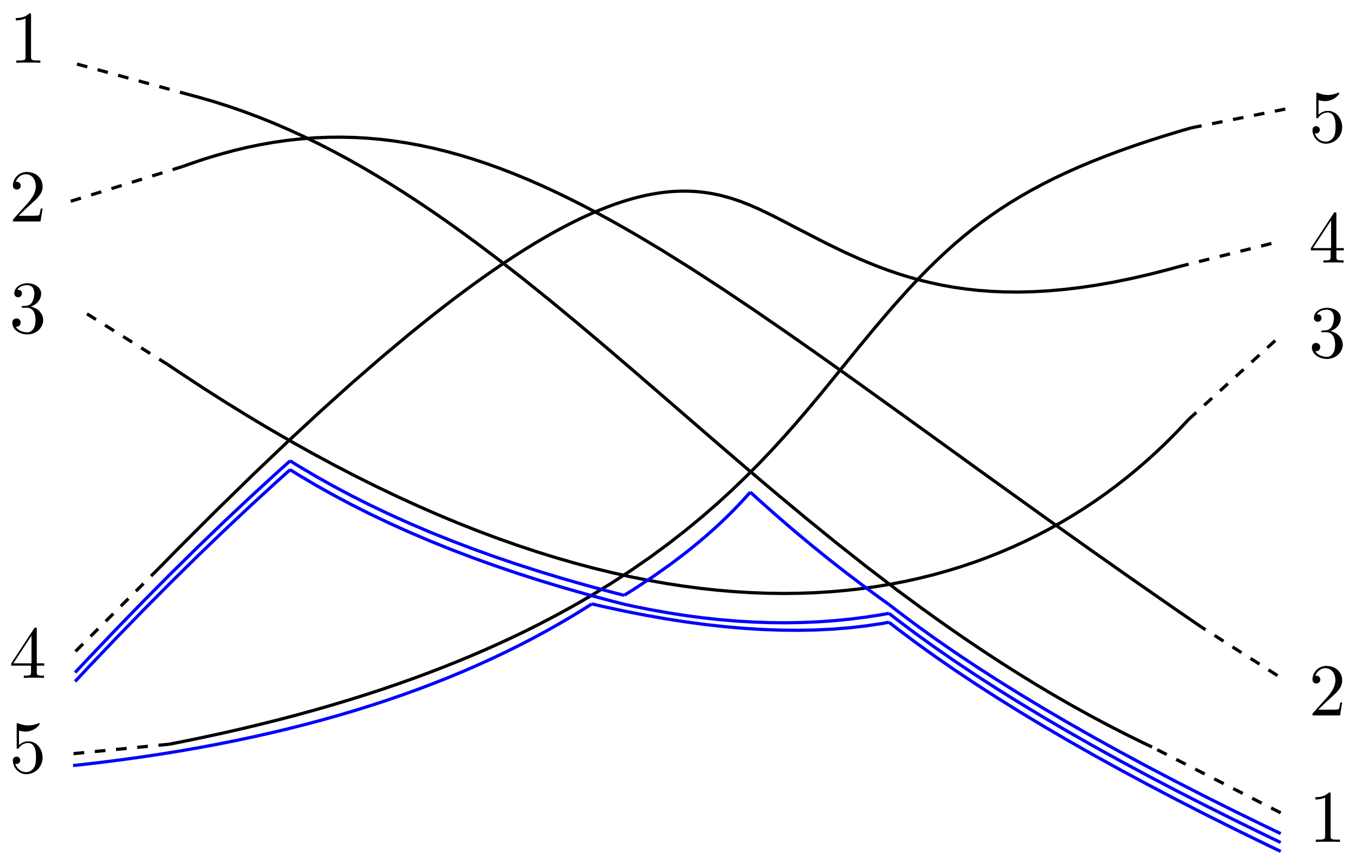

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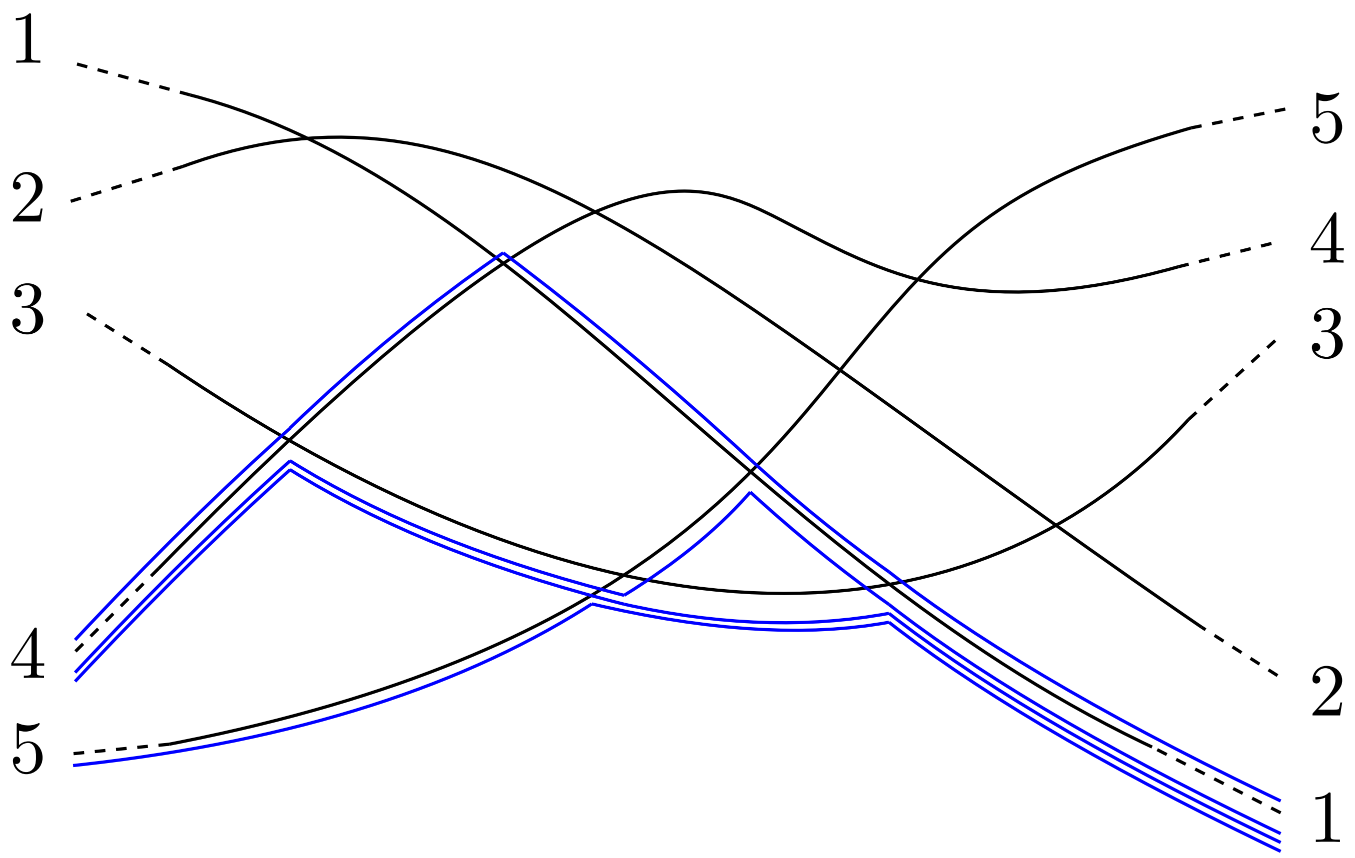

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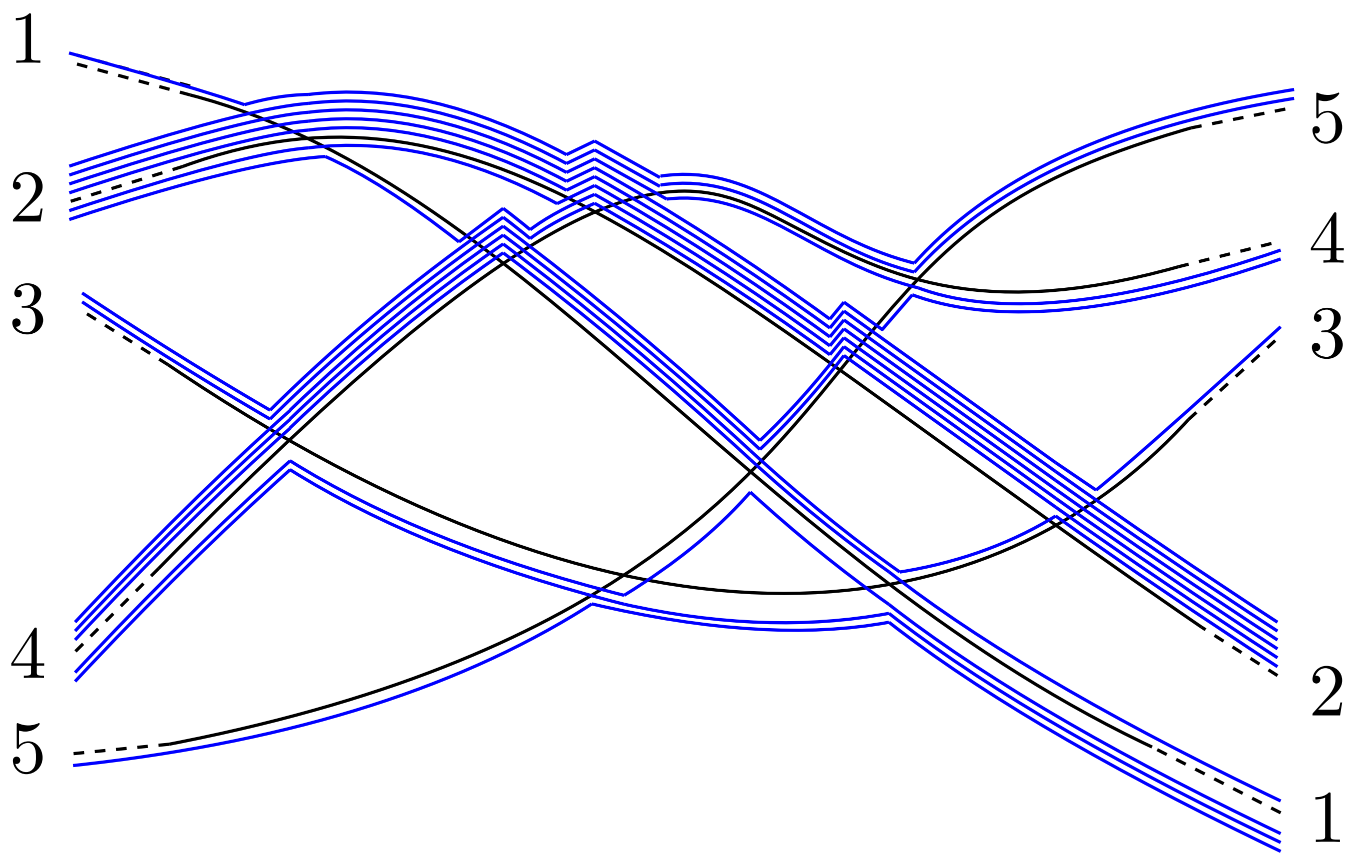
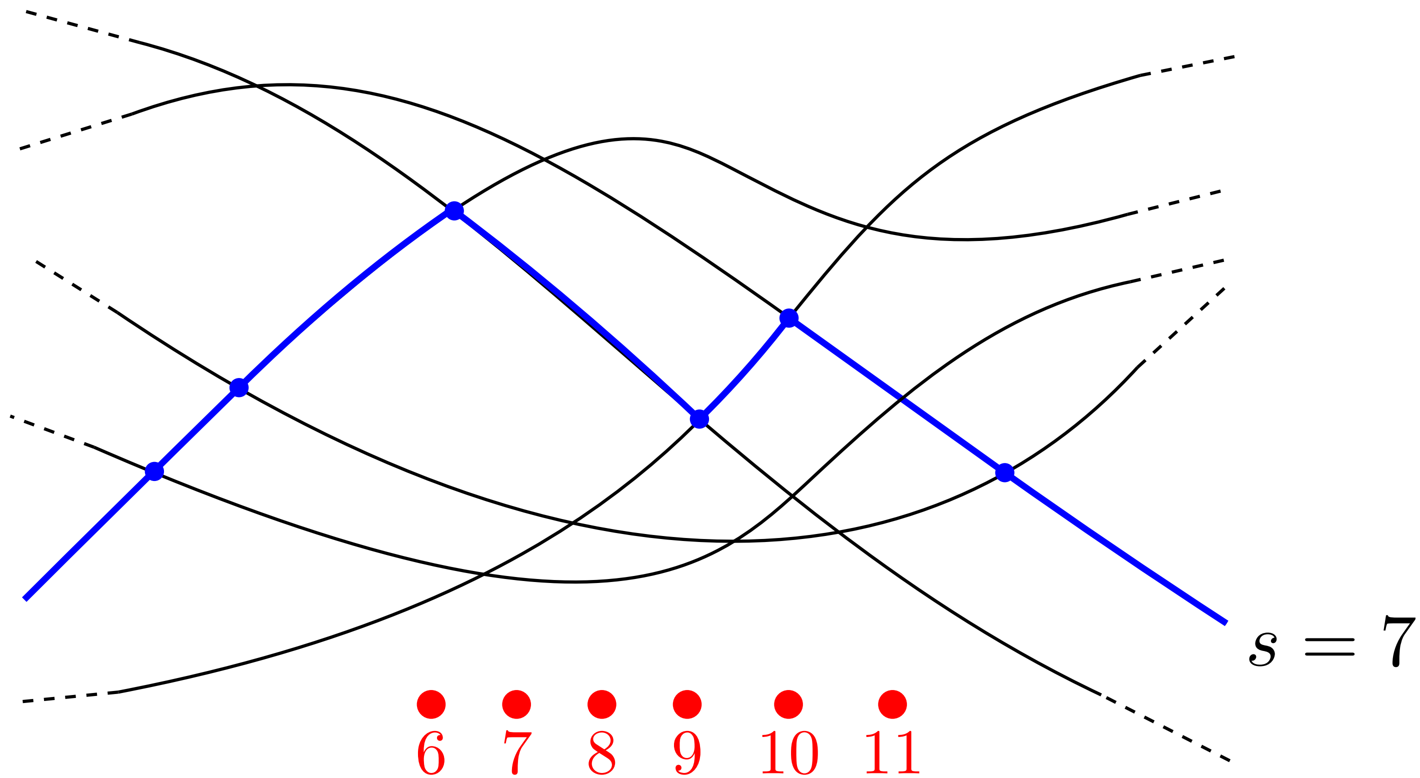


$\frac{k(k+1)}{2}$ ropes

Take a fixed sweep by a sequence of ropes.

Dynamic programming

For each rope:
(s pieces)



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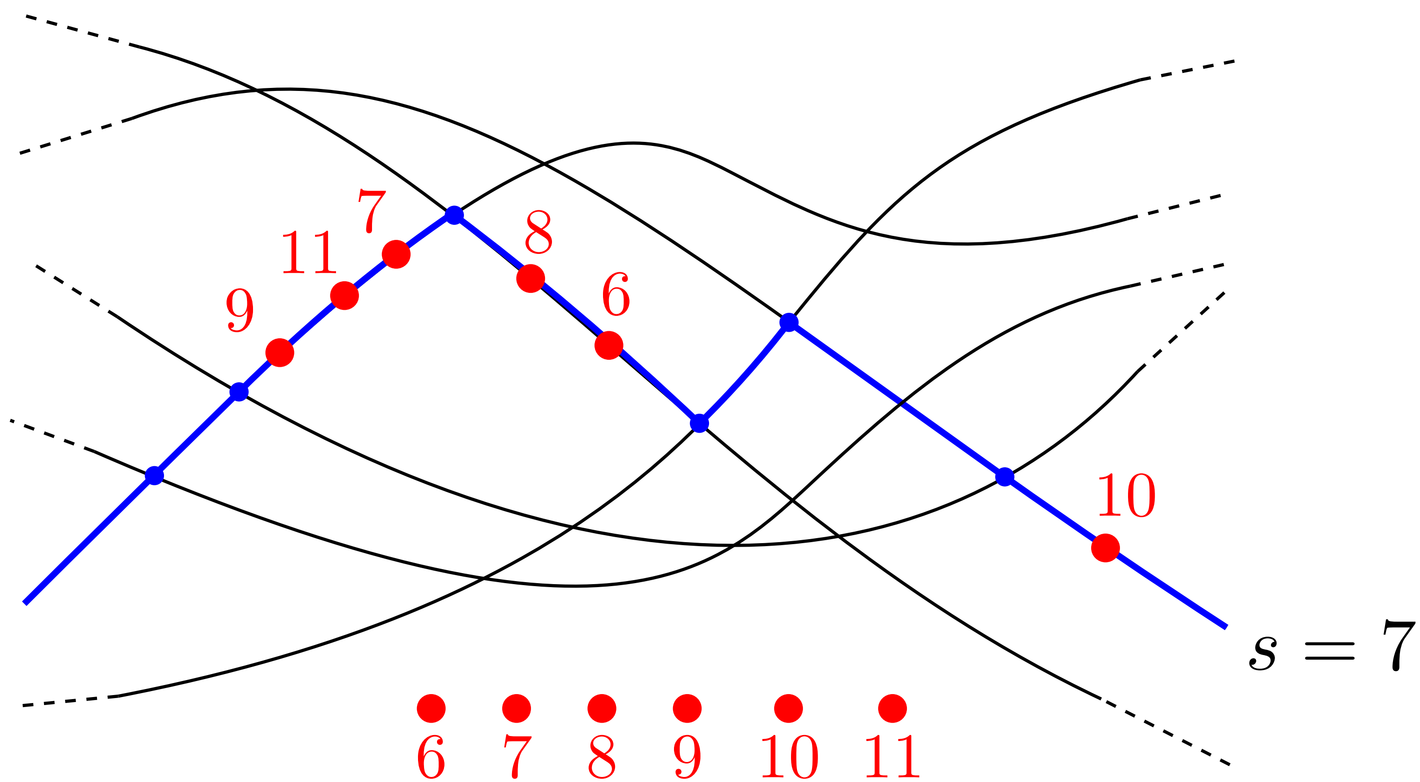

- For every distribution of the ℓ strands to the s pieces
- and for every permutation of the ℓ strands,

[$s(s+1)(s+2)\dots(s+\ell-1)$ entries]

store the number of possibilities to thread the ℓ strands from the bottom face to the rope.

Dynamic programming

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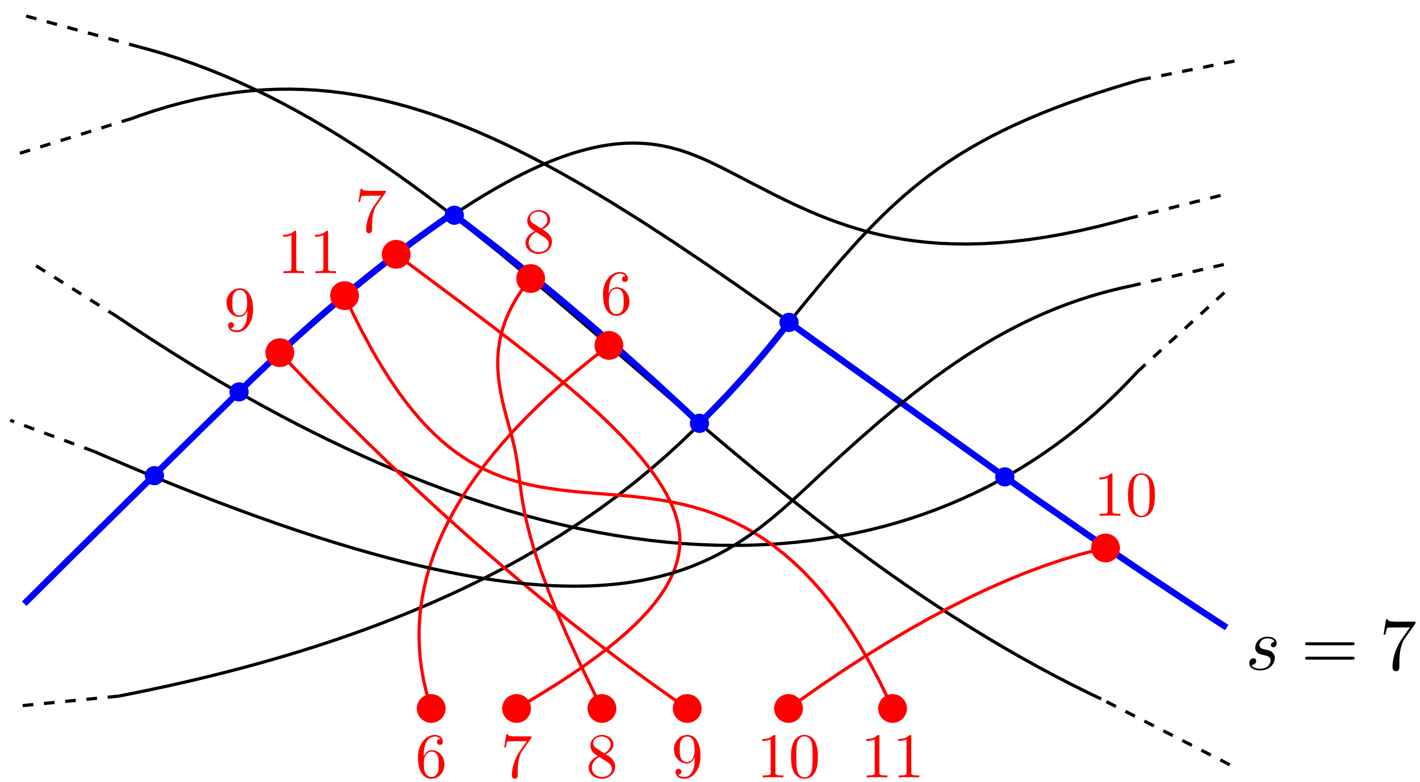

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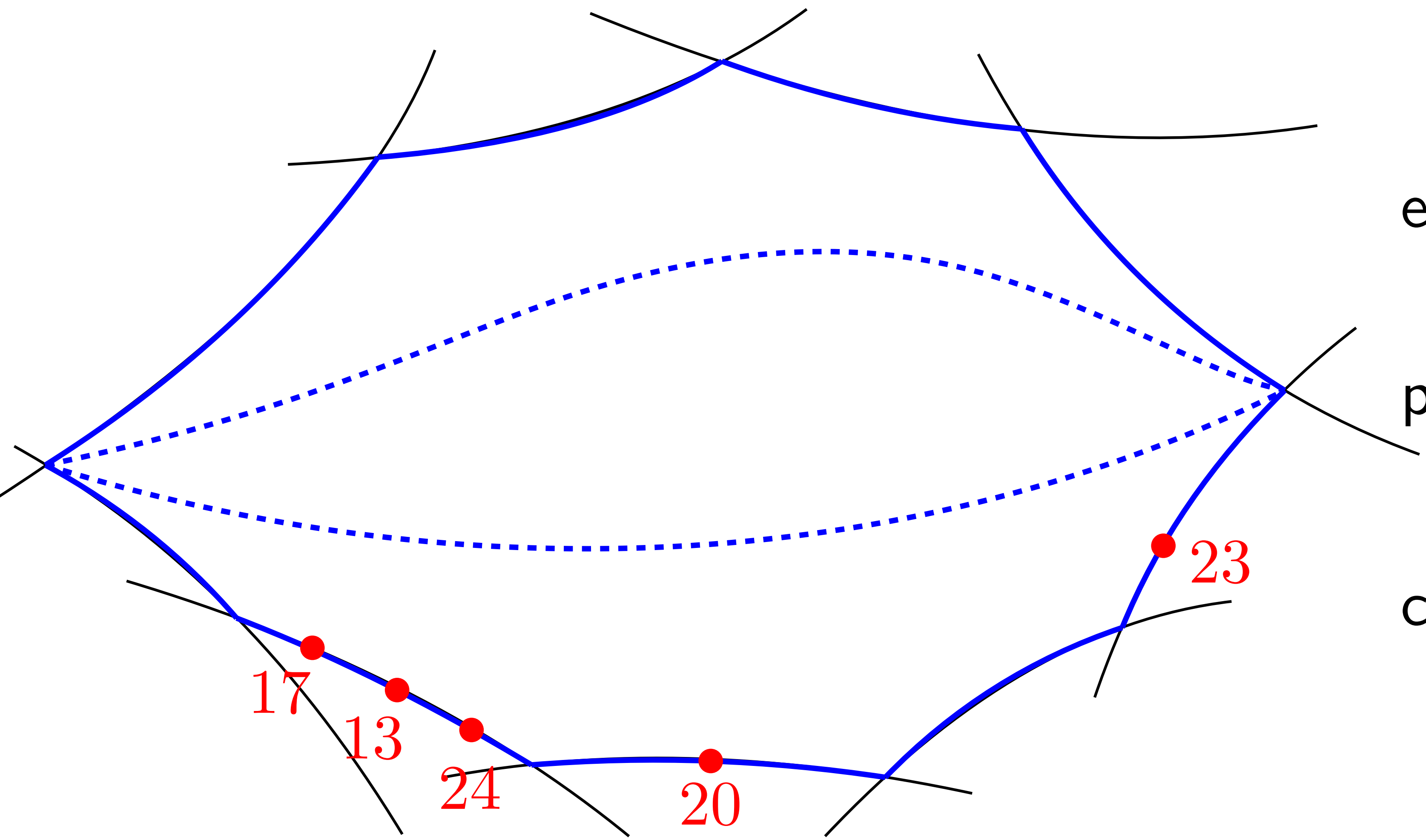
Advancing the rope across a face



expand

permute

contract

What is the contribution to the next rope?

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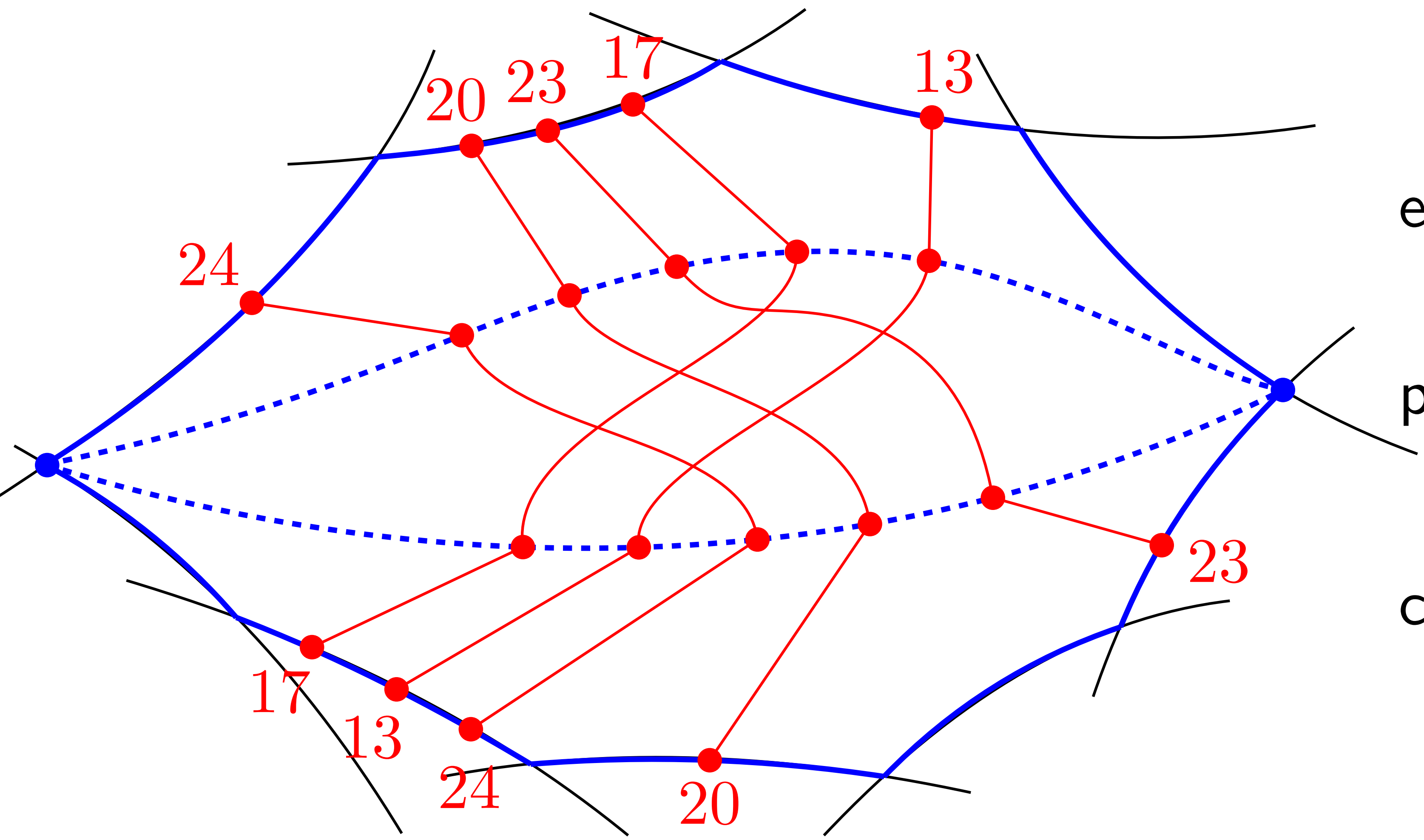

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PARTIAL pseudoline arrangements

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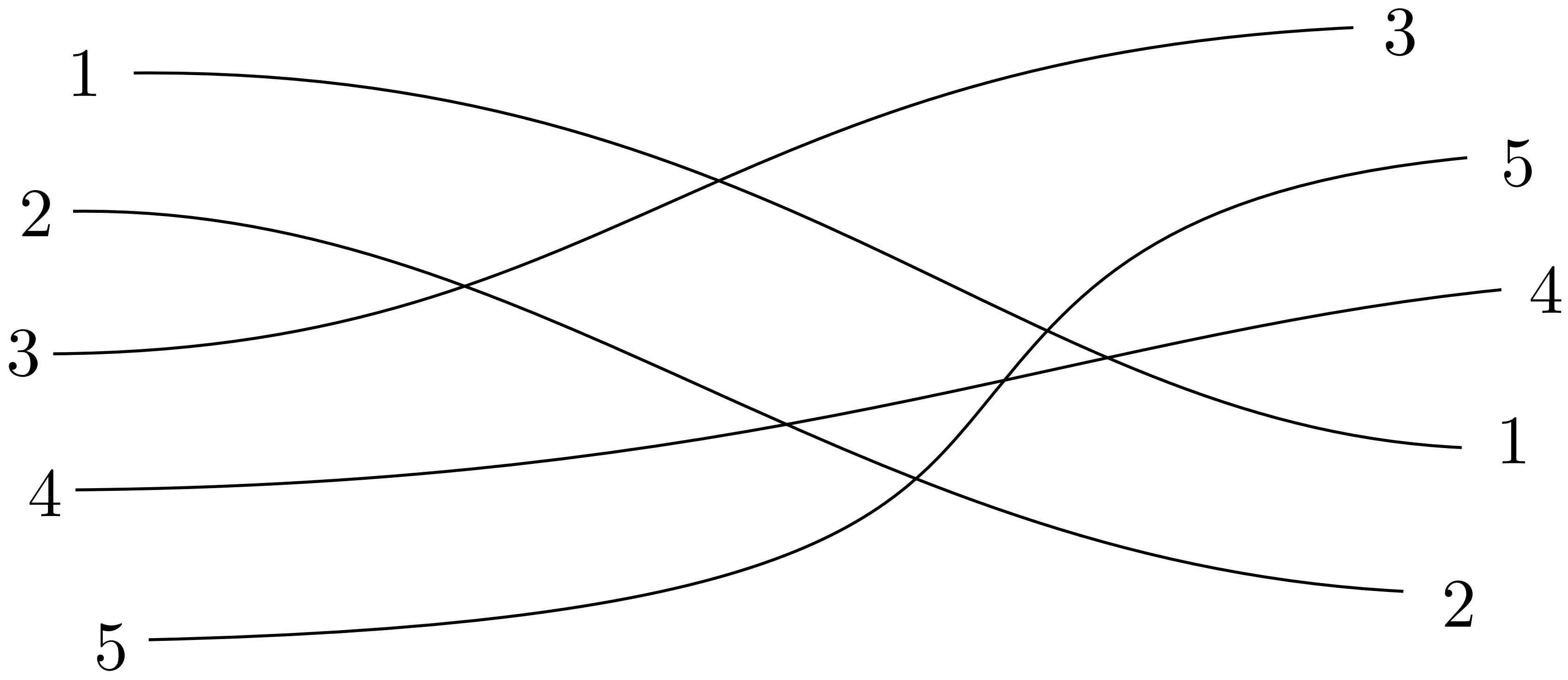
Pseudolines may not cross at all.



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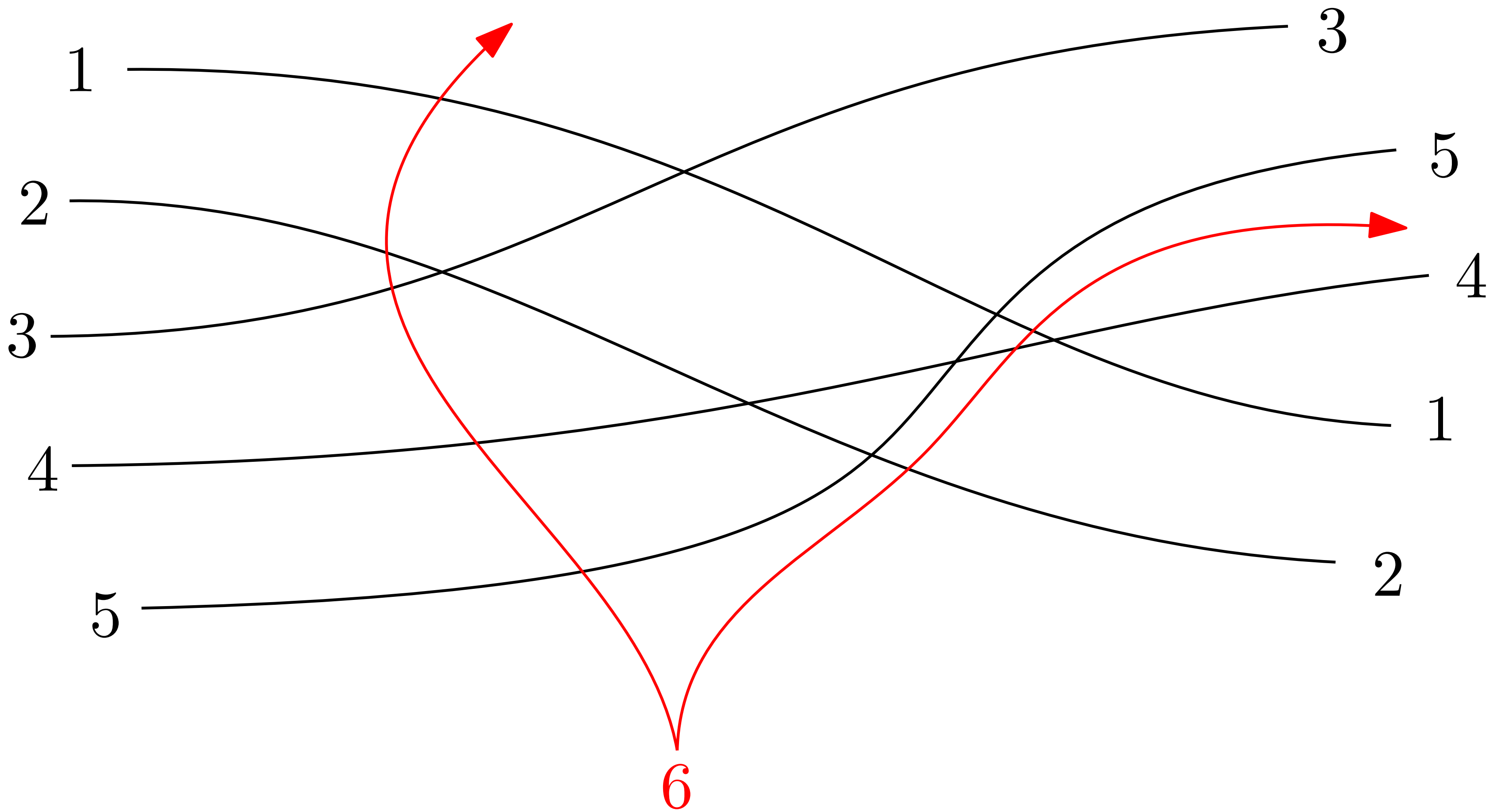

Enumeration is as easy as for full PsA's.

PARTIAL pseudoline arrangements

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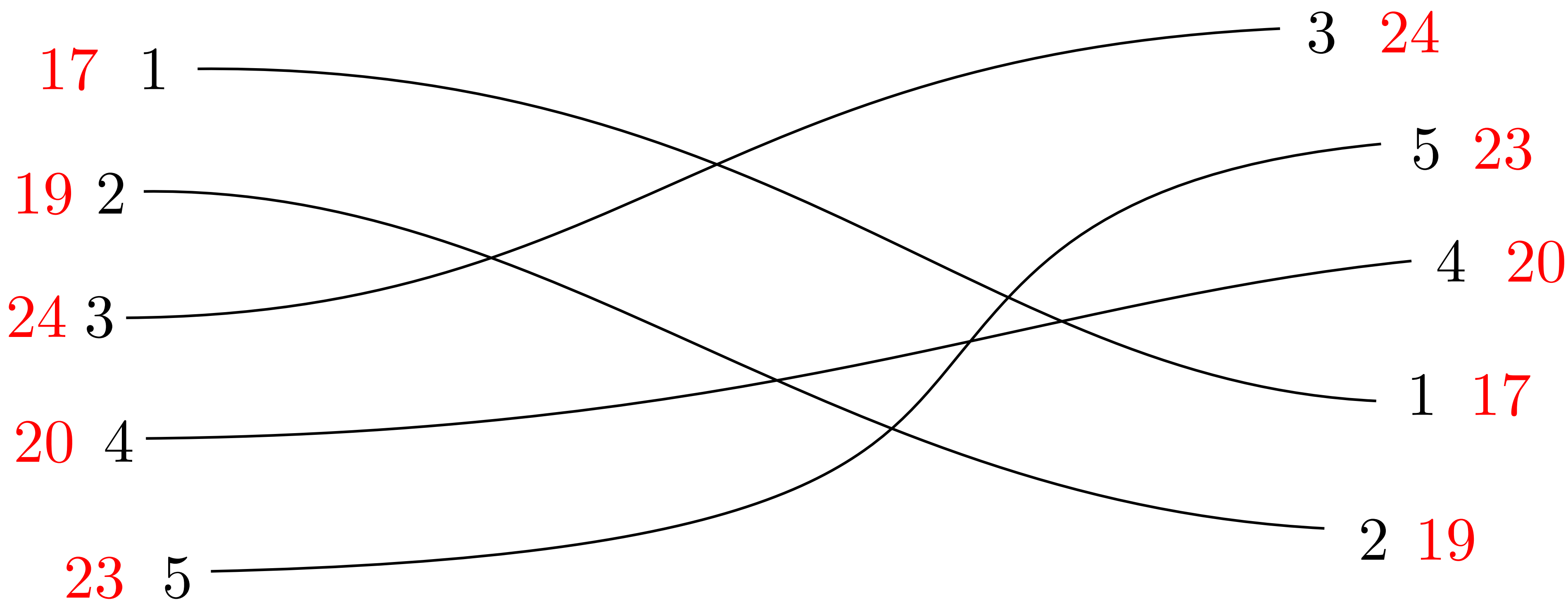

Preprocessing: $\rightarrow$ $\ell! \times \ell!$ table (sparse!)

Algorithm summary

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For each PsA of k pseudolines:

- Compute a sweep by ropes
- For each rope:
  - For each distribution and permutation of the ℓ strands:
    - Compute the contributions to the next rope, and accumulate them.

Some implementation details

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- PYTHON, with `scipy` for large arrays of 32/64-bit integers
- modular arithmetic, using 2^{64} plus two 30-bit moduli
- $n = 16 = k + \ell = 7 + 9$. Large memory!
  256 GBytes is enough; 128 GBytes sometimes failed.
- easy to parallelize:
  a large number (24,698) of independent tasks
- total CPU time: about 5.5 months, using various workstations of different speeds
- CPU time for $n = 15 = 6 + 9$ (exploiting symmetry): 6 h.
  By contrast*: PYTHON without `scipy` took 50 CPU days.
- There is also a version in C (using `CWEB`) for the task of *enumerating* PsA's.

The required rope length

- Every arrangement requires $\geq n+1$ pieces (for $n \geq 3$).
- can always do with $\leq 2n-2$ pieces. (greedy sweep)
- Some arrangements require $\lfloor \frac{7n}{4} \rfloor - 1$ pieces.

  (This is the true maximum for $n \leq 9$.)
- NP-hard? (homotopy height, cutwidth)

  [Biedl, Chambers, Kostitsyna, Rote, 2020, unpublished, + this week]

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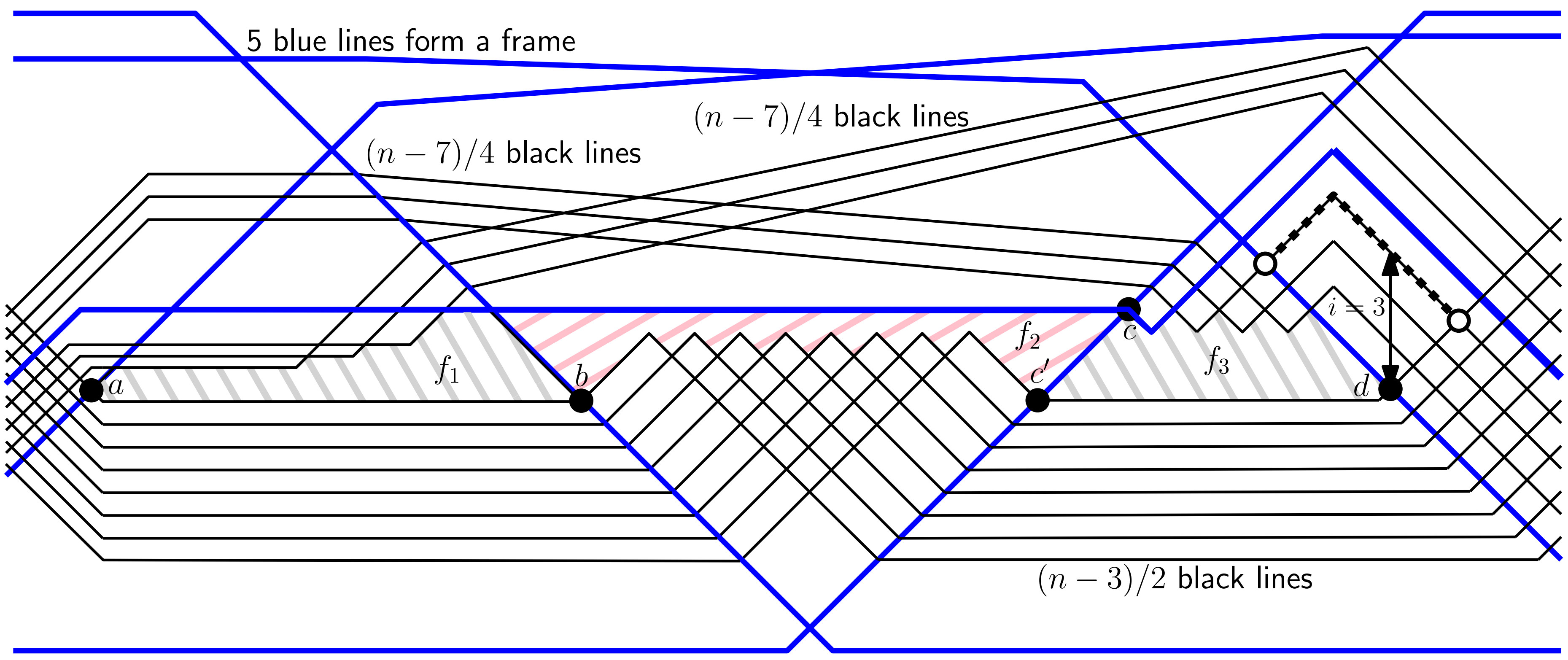
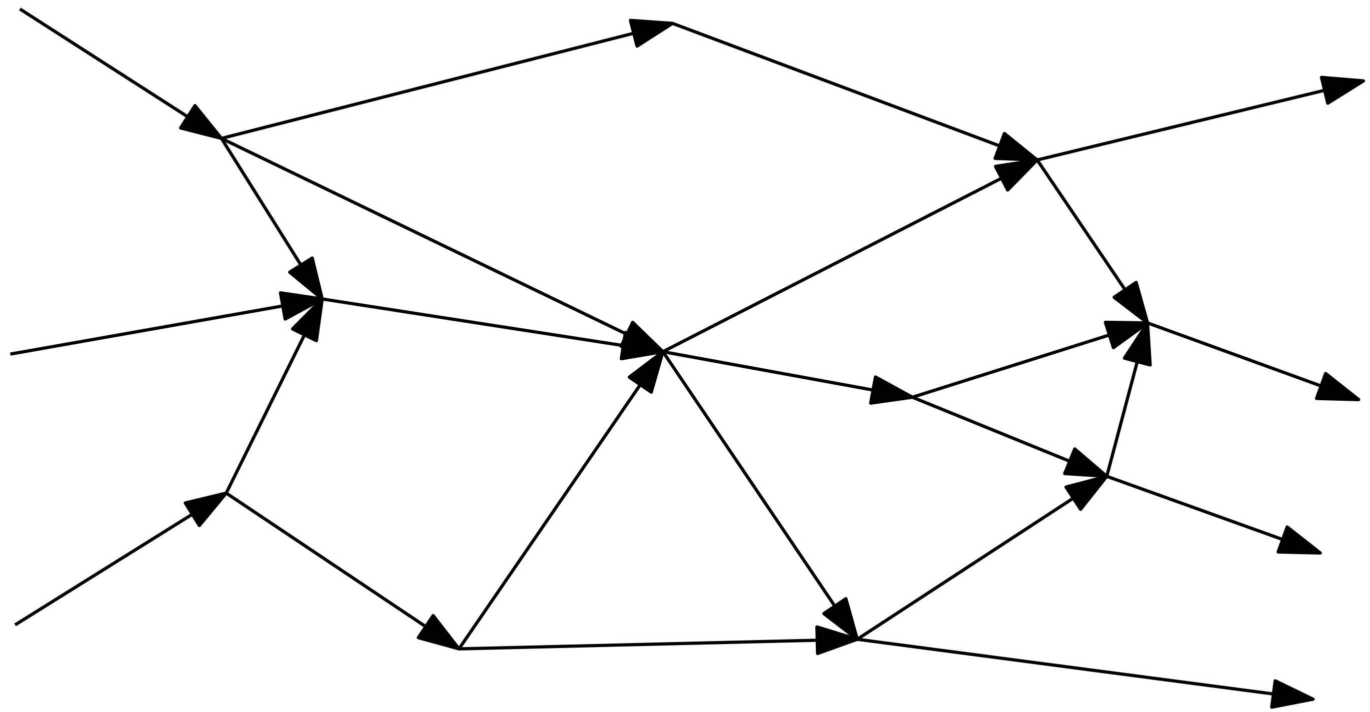


The required rope length

This is really about *bipolar orientations* (s-t-planar DAGs):



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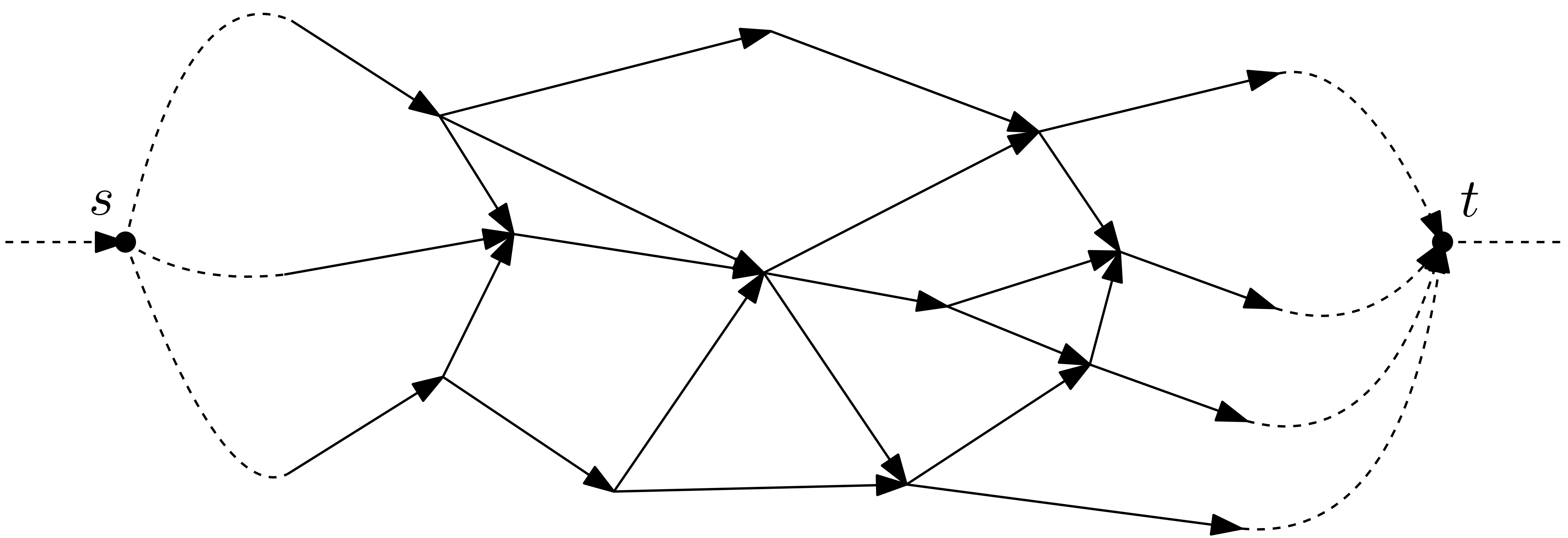

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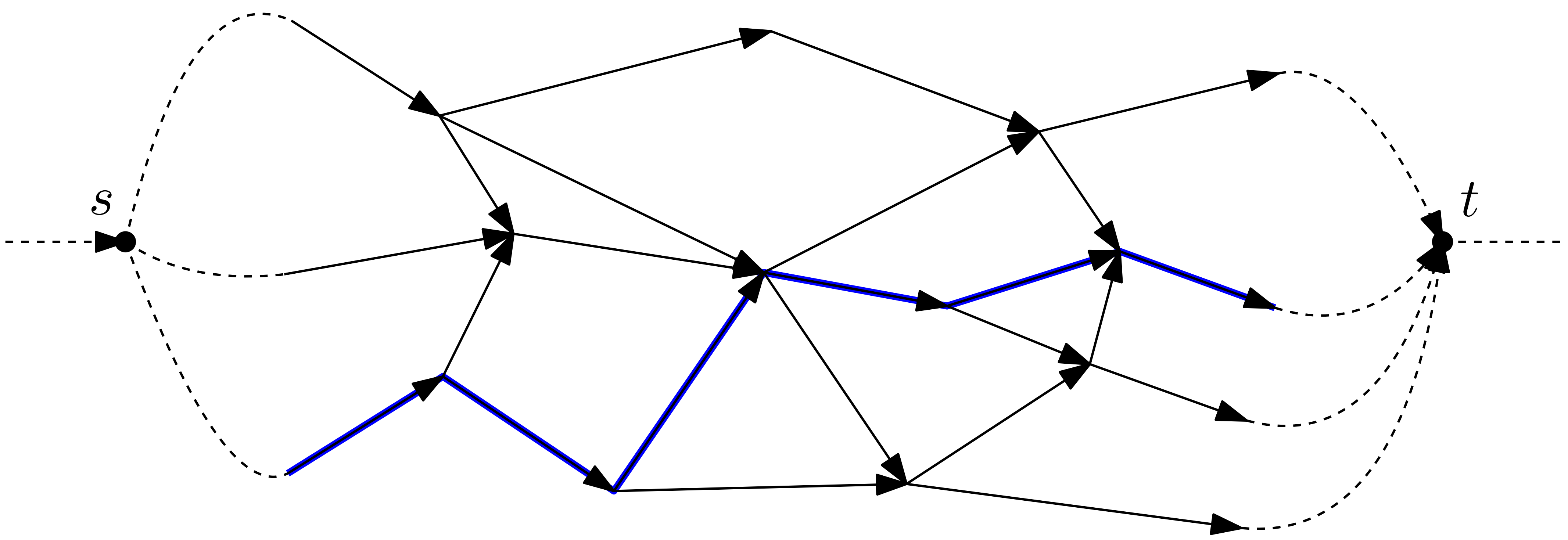

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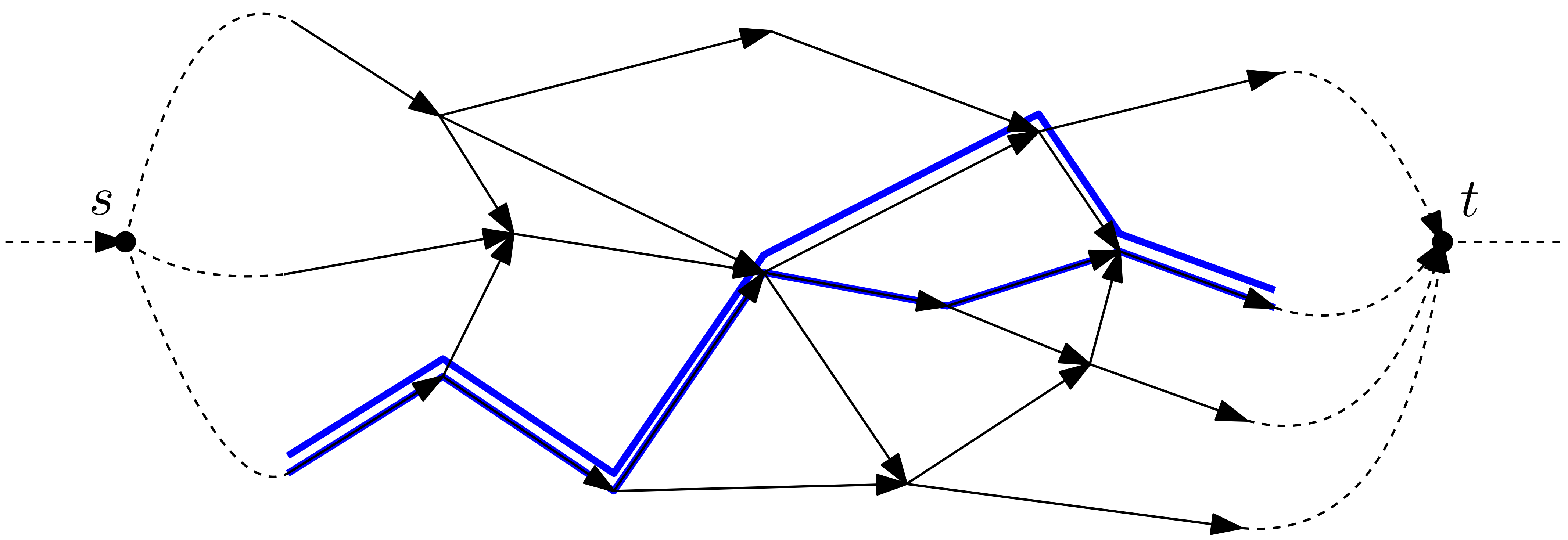

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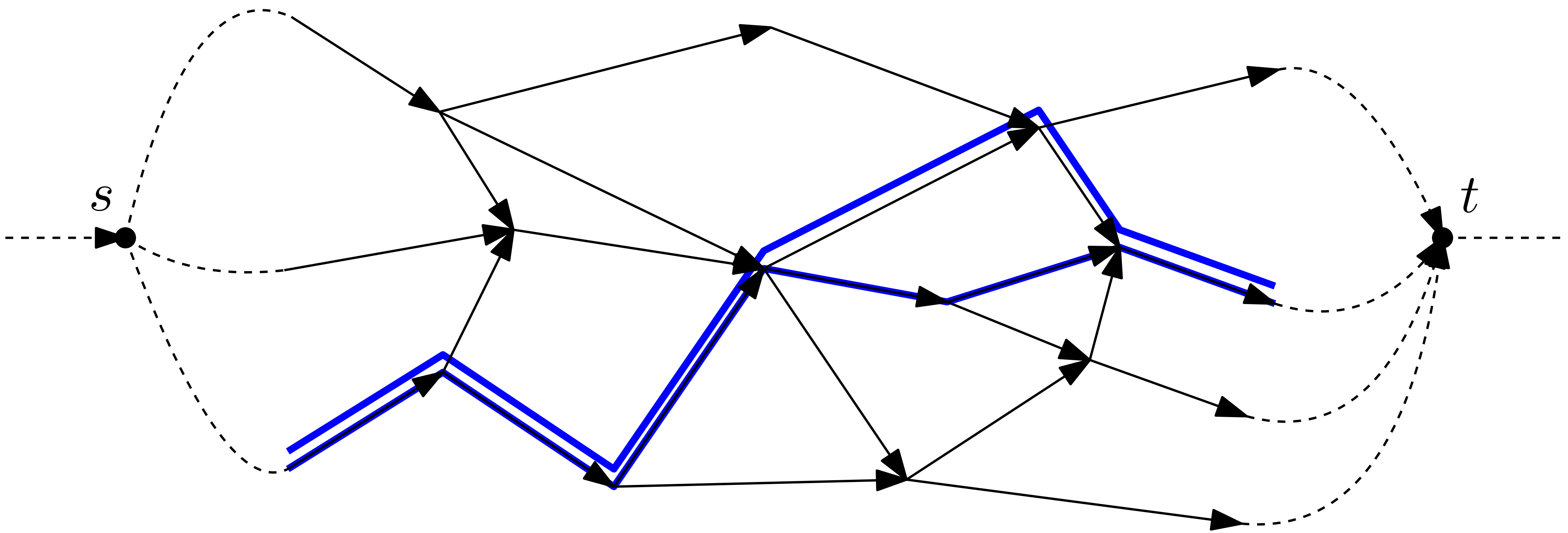
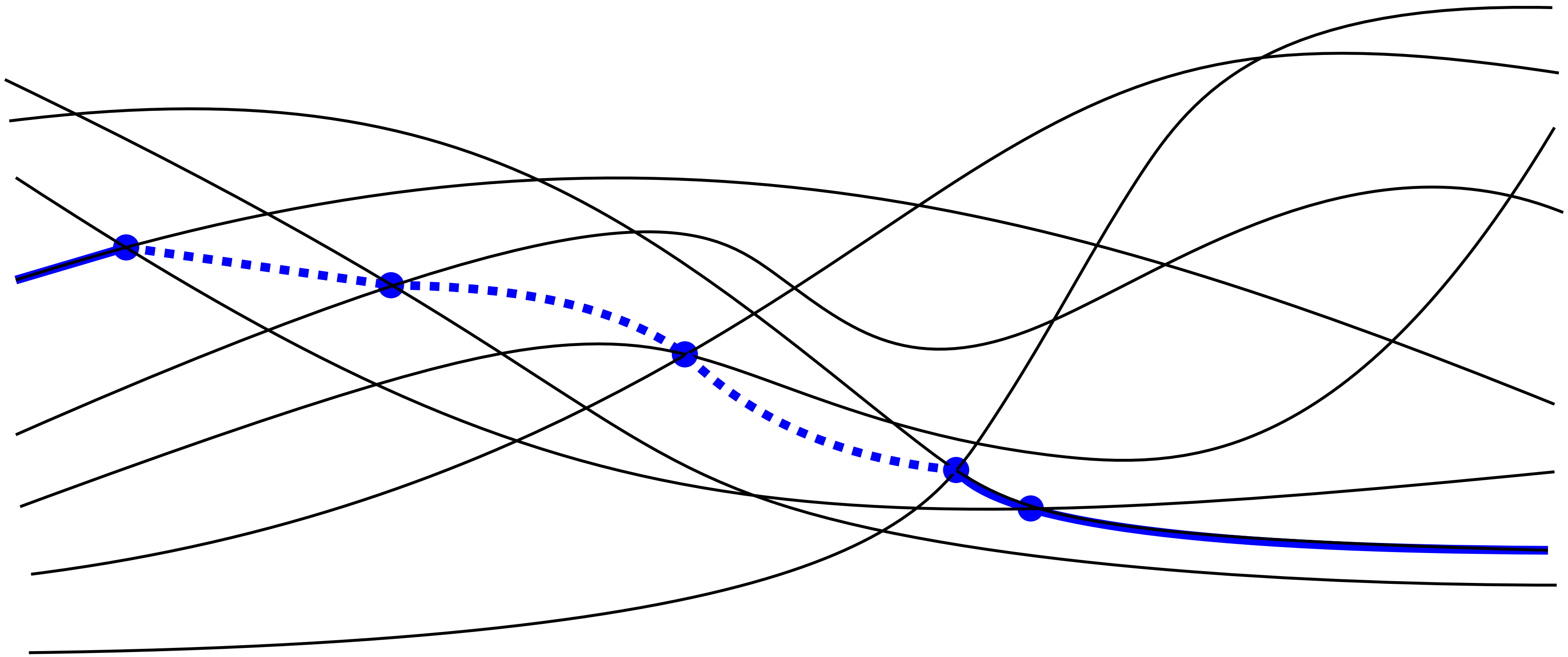


"leftmost-first" greedy sweep
$\rightarrow$ coordinated simultaneous primal-dual sweep

What really matters in practice



- several *expansions* simultaneously, followed by *combinations*
- *permute* steps separately

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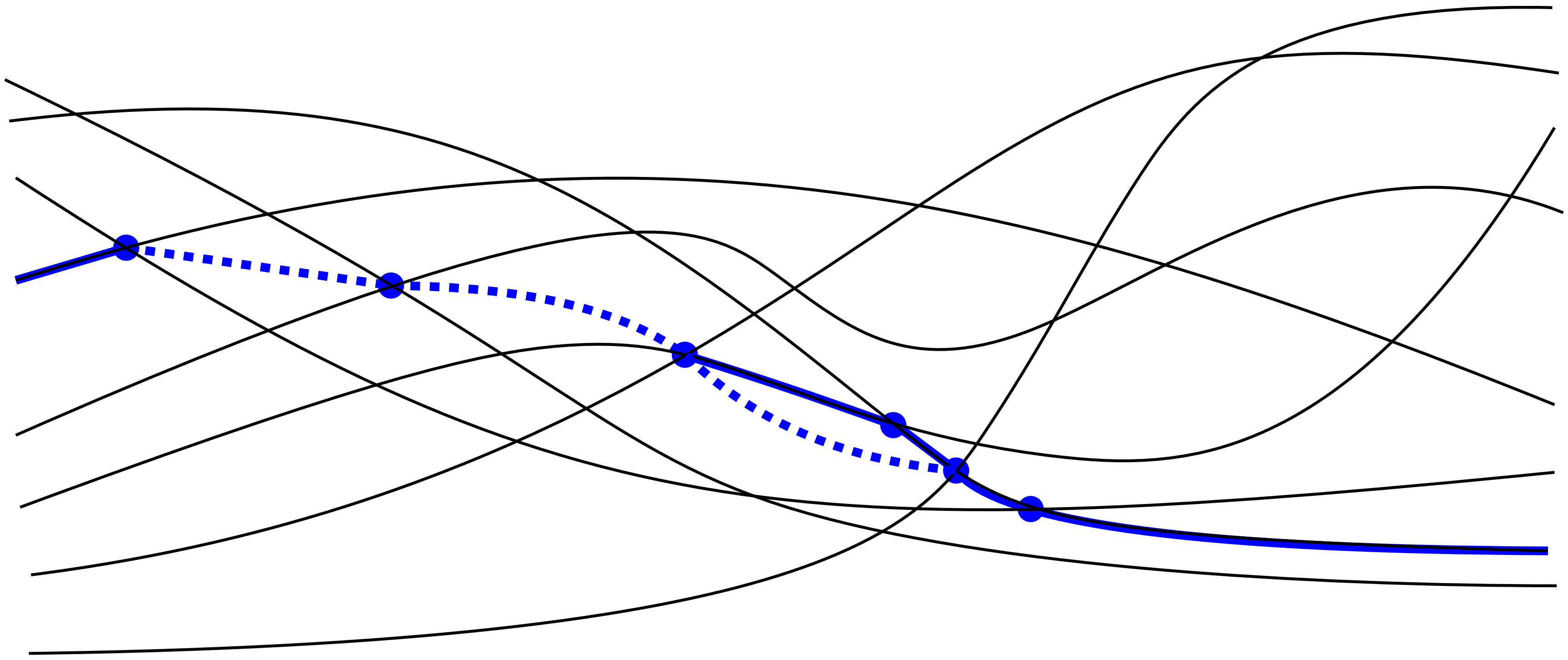

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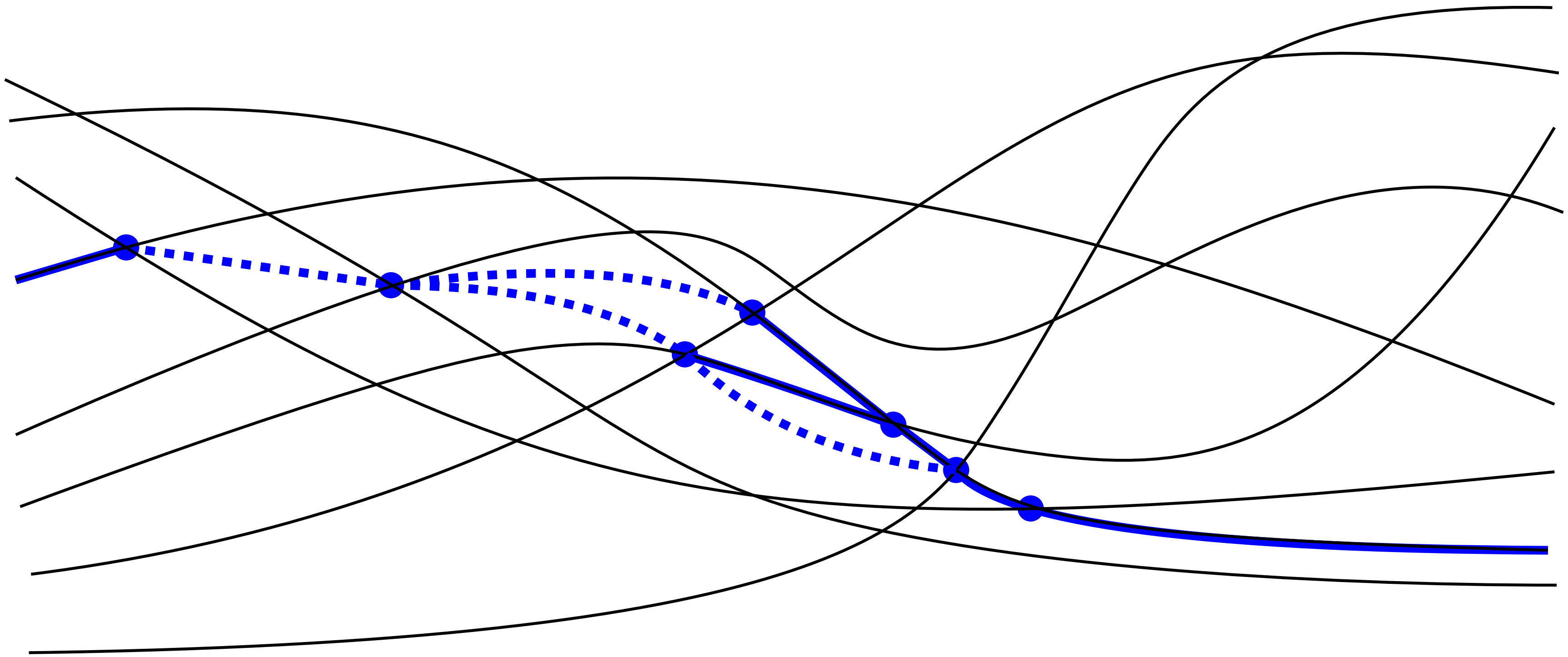

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