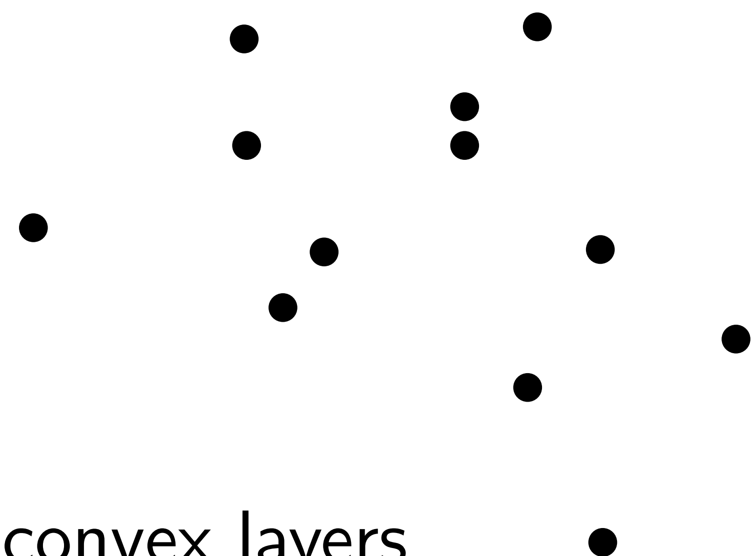
Grid Peeling and the Affine Curvature-Shortening Flow (ACSF)

Günter Rote and Moritz Rüber
Freie Universität Berlin



convex layers
onion layers

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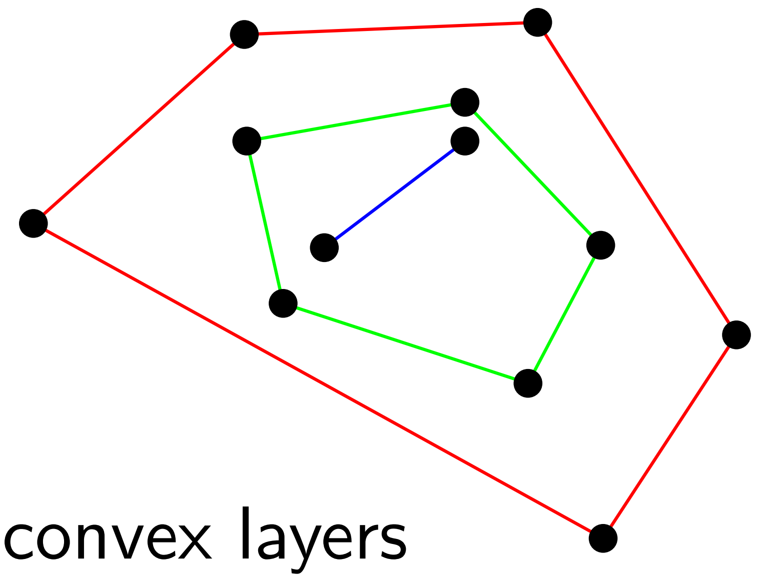

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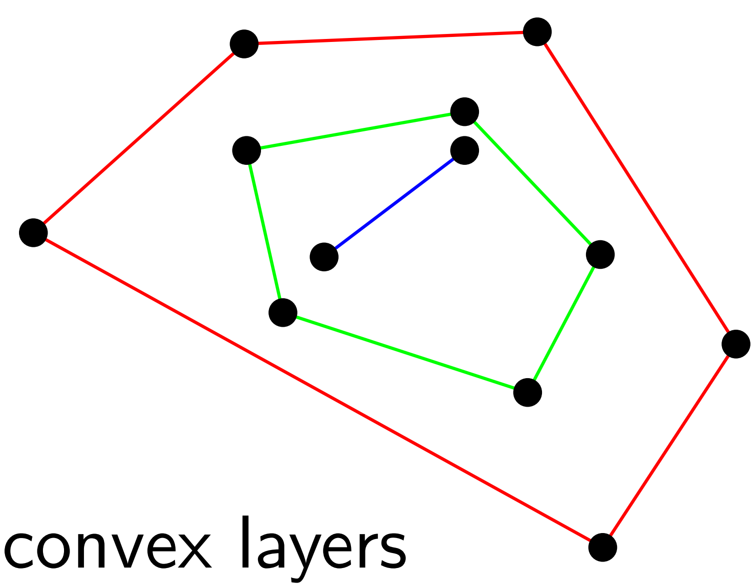

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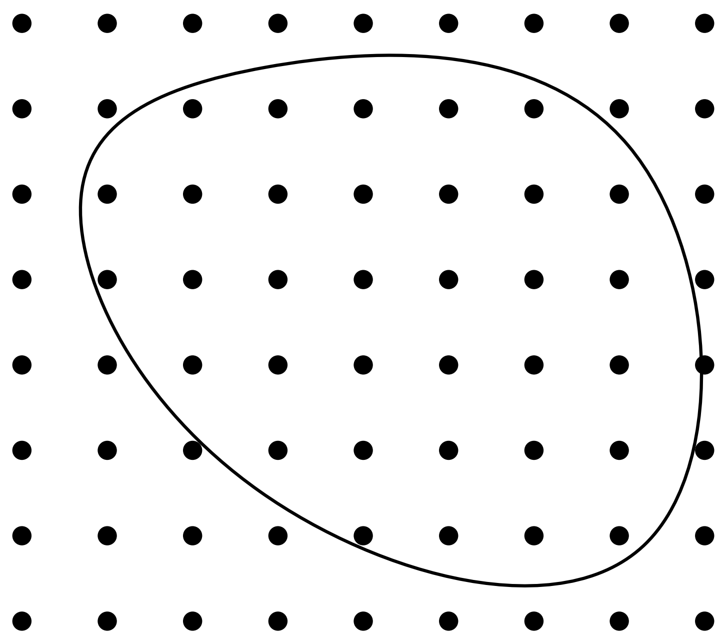

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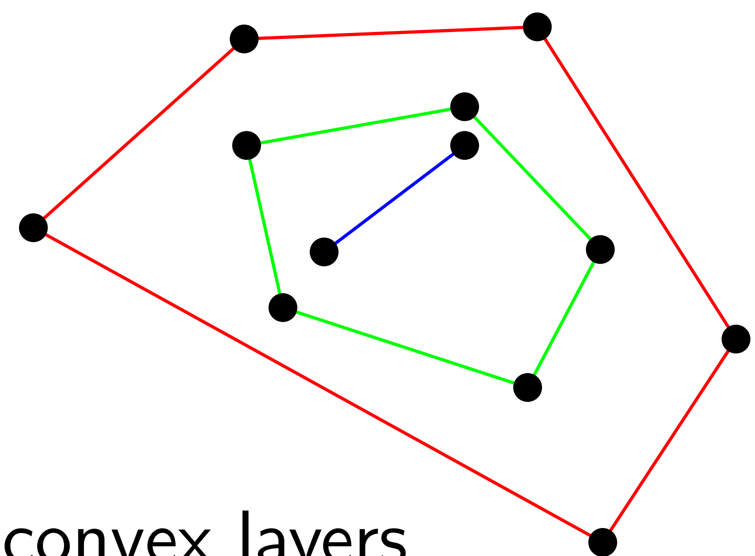

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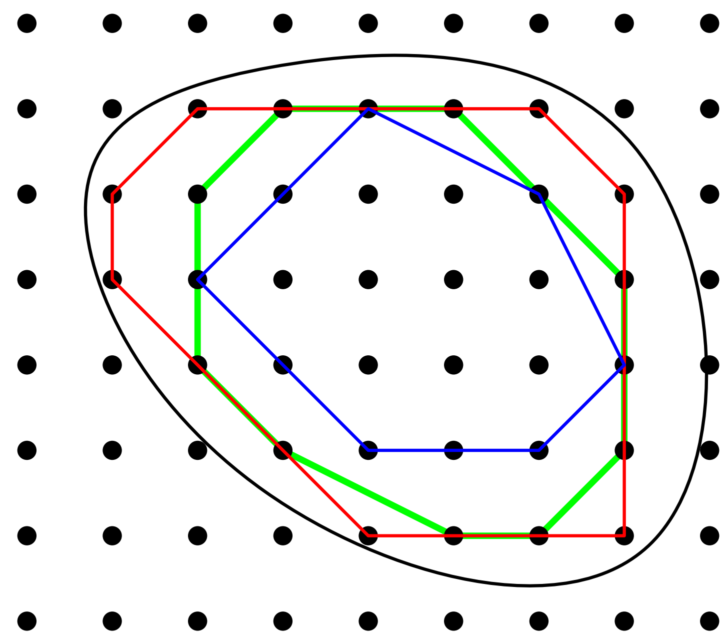
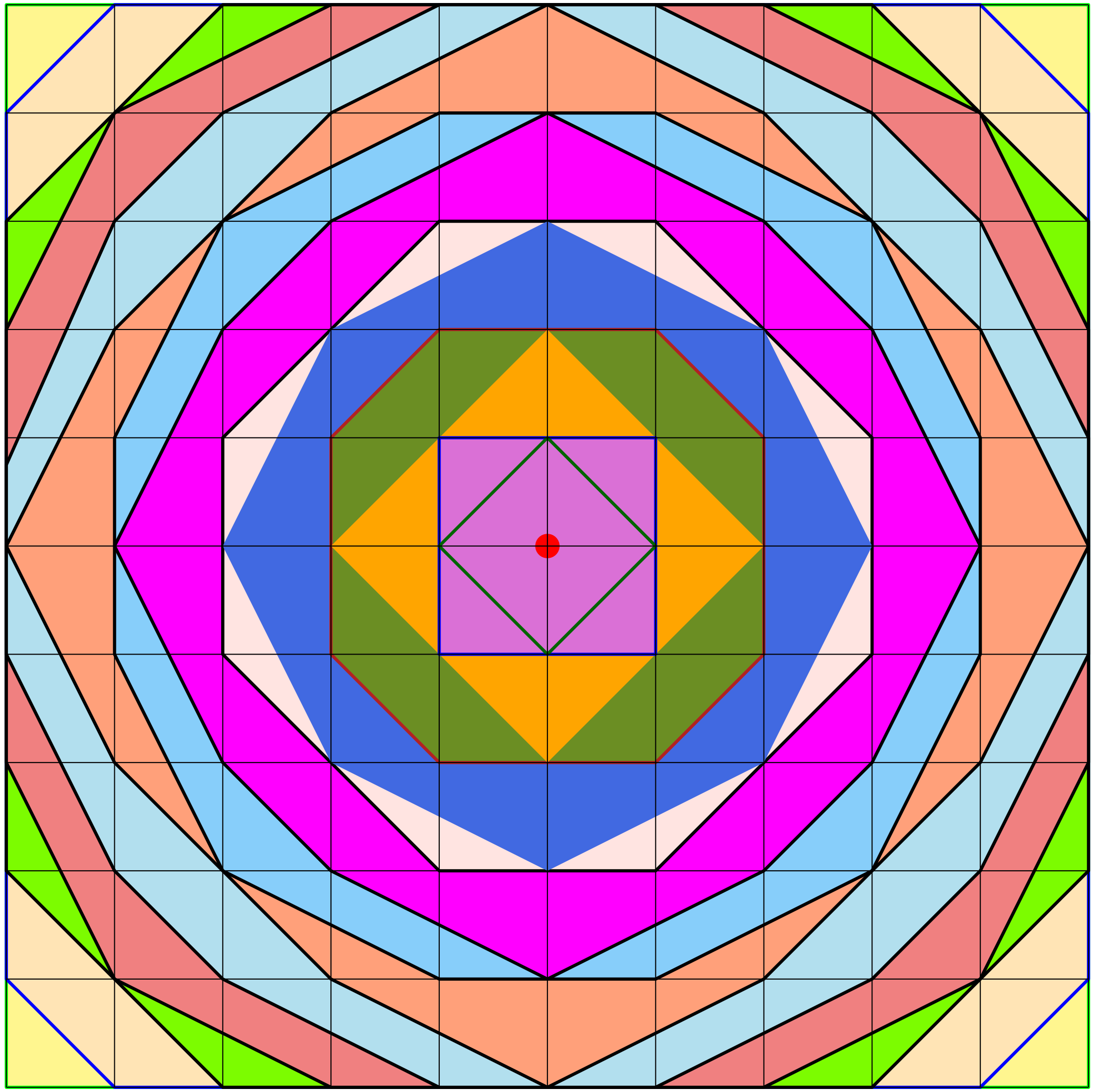


grid peeling

Grid Peeling of the Square

[Sariel Har-Peled and Bernard Lidický 2013]

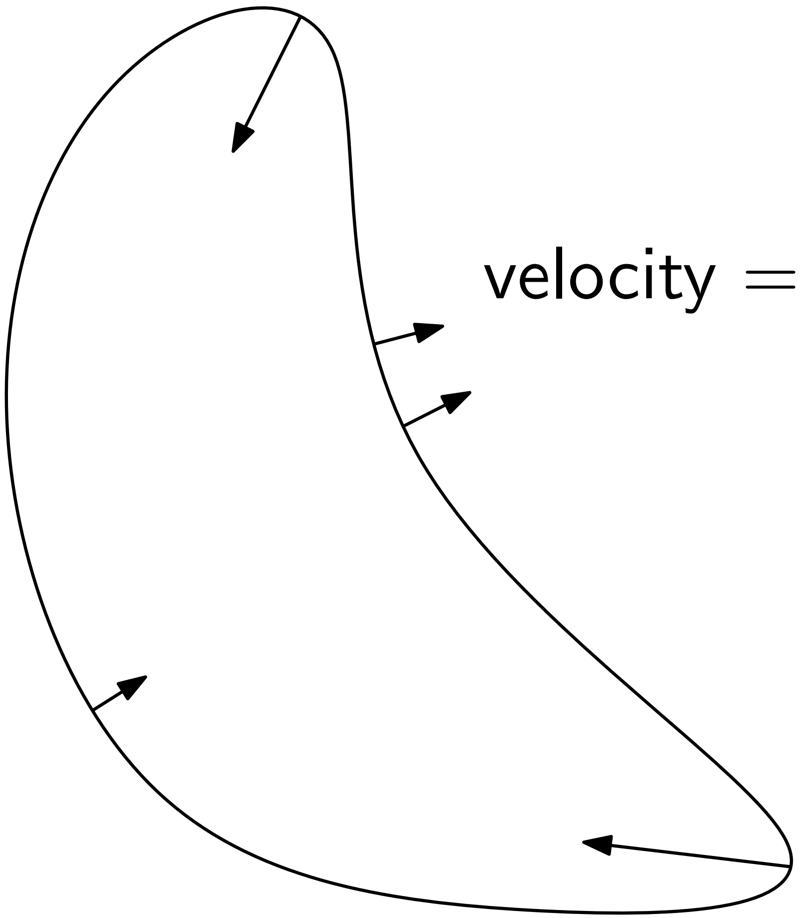


The $n \times n$ grid has $\Theta(n^{4/3})$ convex layers.

Affine Curvature-Shortening Flow (ACSF)

[L. Alvarez, F. Guichard, P.-L. Lions, J.-M. Morel:
"Axioms and fundamental equations of image processing" 1993]

[G. Sapiro and A. Tannenbaum:
"Affine invariant scale-space." Int. J. Computer Vision 1993]



velocity $= \kappa^{1/3}$ ($\kappa =$ curvature)

invariant under area-preserving
affine transformations!

Peeling and the ACSF

Freie Universität Berlin

Conjecture:

David Eppstein, Sariel Har-Peled, and Gabriel Nivasch. Grid peeling and the affine curve shortening flow. Experimental Mathematics **29** (2020), 306–316

As the grid is more and more refined, grid peeling approaches the ACSF.

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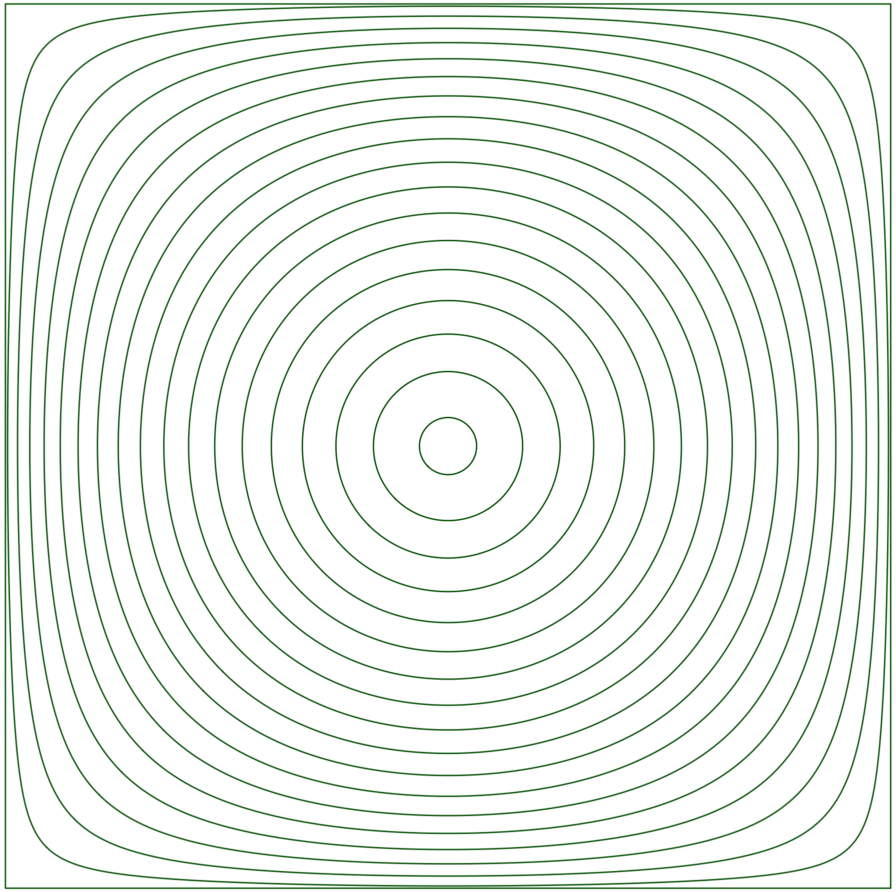

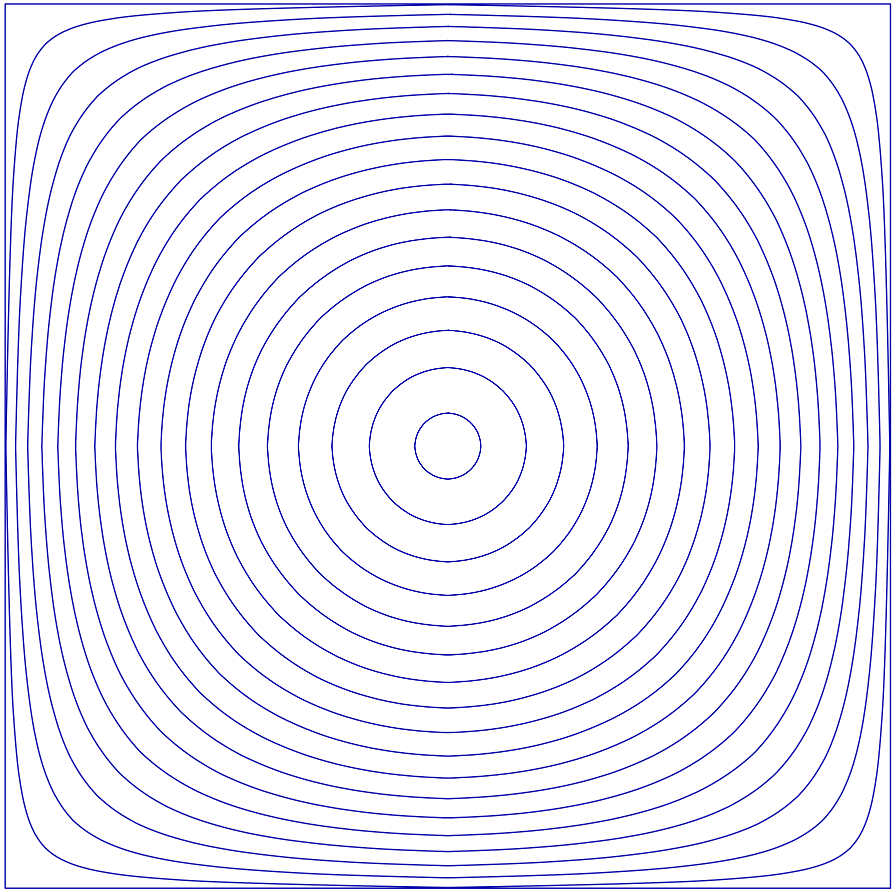

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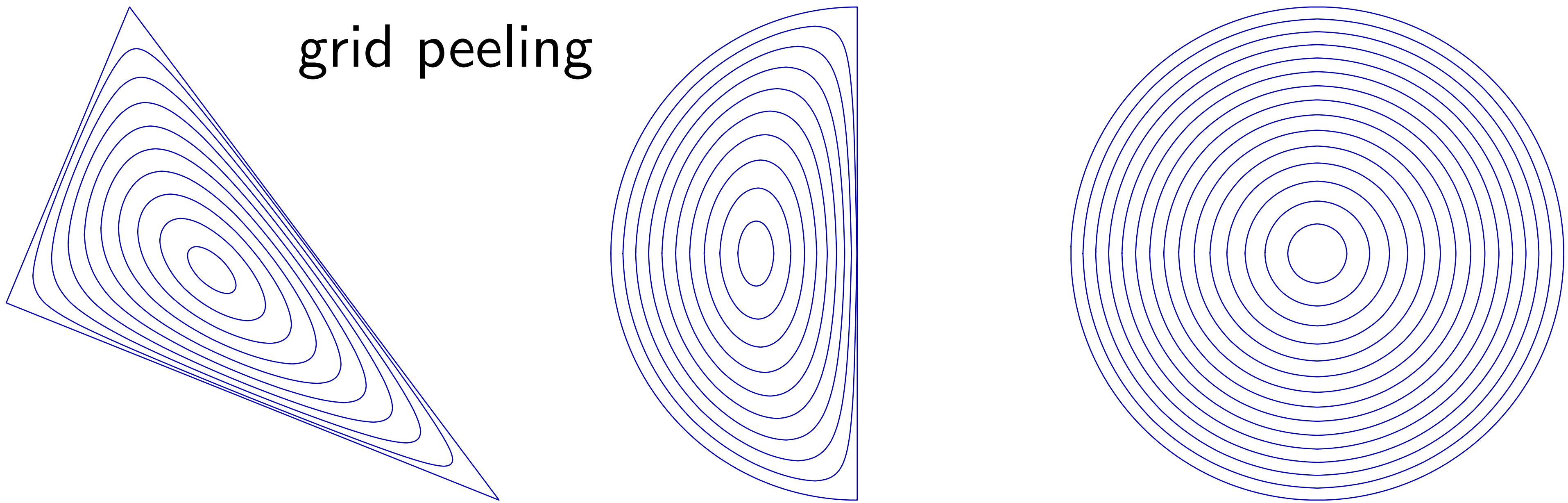

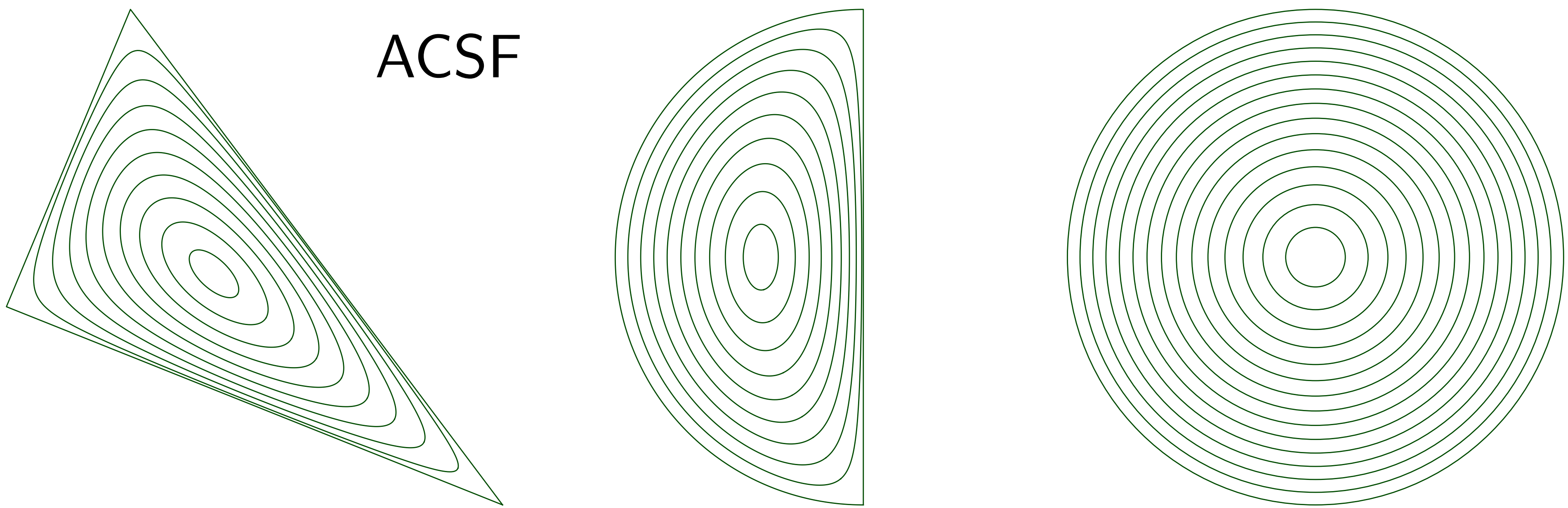

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ACSF at time $t \approx$ Grid peeling on $\frac{1}{n}$-grid after $C_g t n^{4/3}$ steps.

Conjecture: (Moritz Rüber and Günter Rote)

$$C_g = \sqrt[3]{\frac{\pi^2}{2\zeta(3)}} \approx 1.60120980542577$$

Peeling and the ACSF

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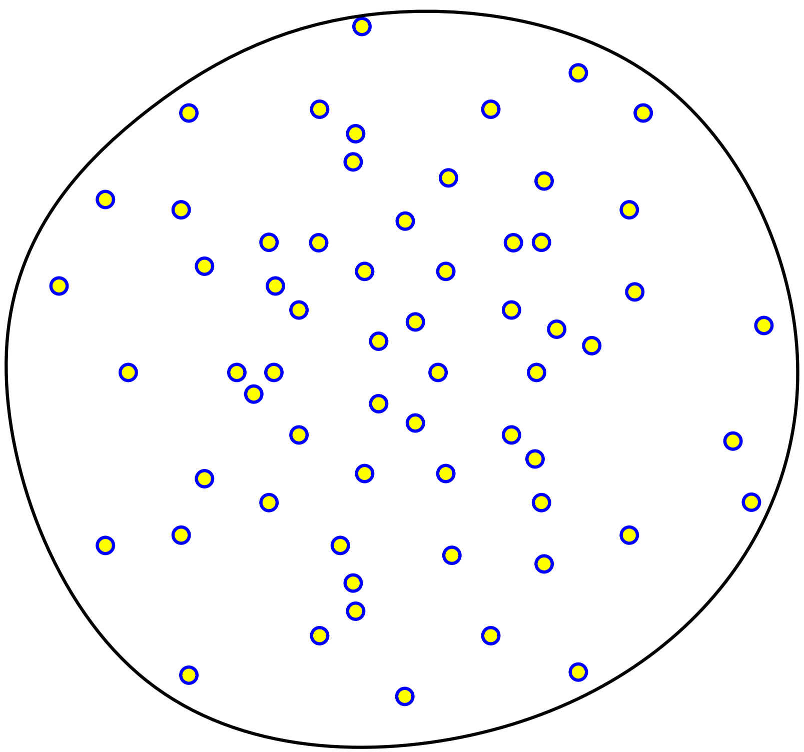
→ Jeff Calder and Charles K Smart. The limit shape of convex hull peeling. Duke Math. J. (2020)

random points



Peeling and the ACSF

Freie Universität Berlin

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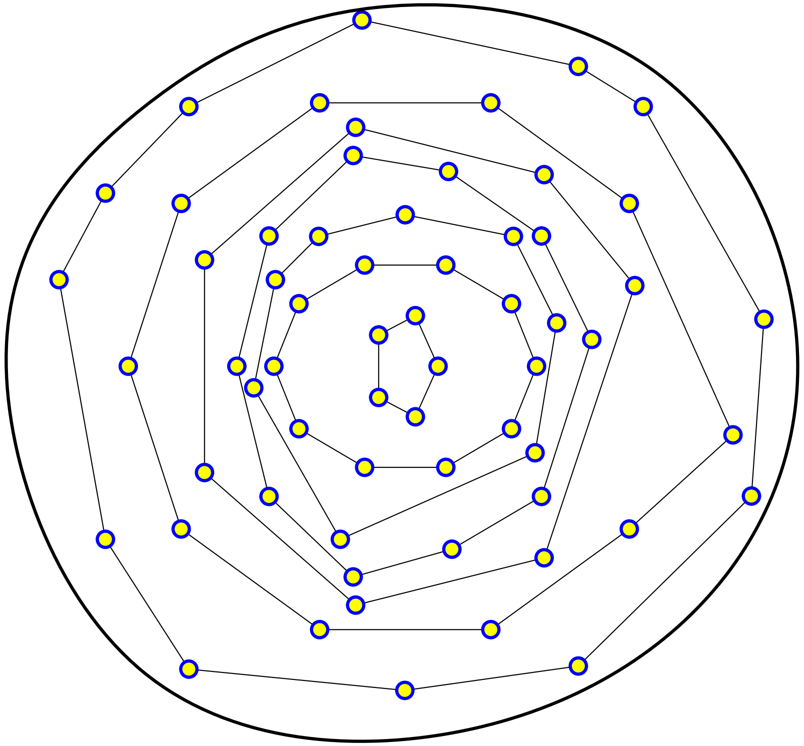
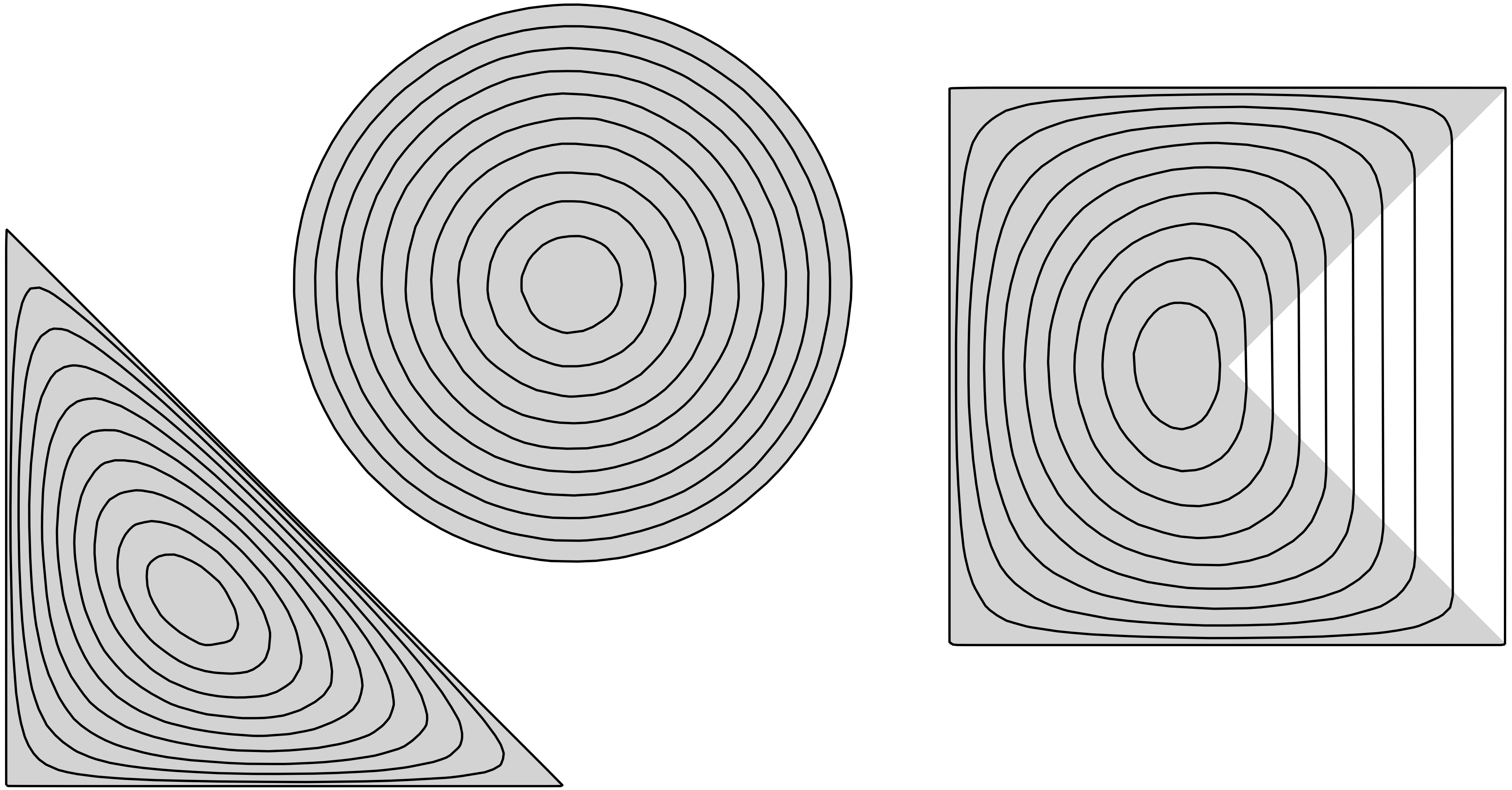


Peeling and the ACSF

Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020

10000 random points in the shaded region



Peeling and the ACSF

Freie Universität Berlin

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Theorem:

ACSF at time $t \approx$ Peeling on density-n^2 set after $C_r t n^{4/3}$ steps.

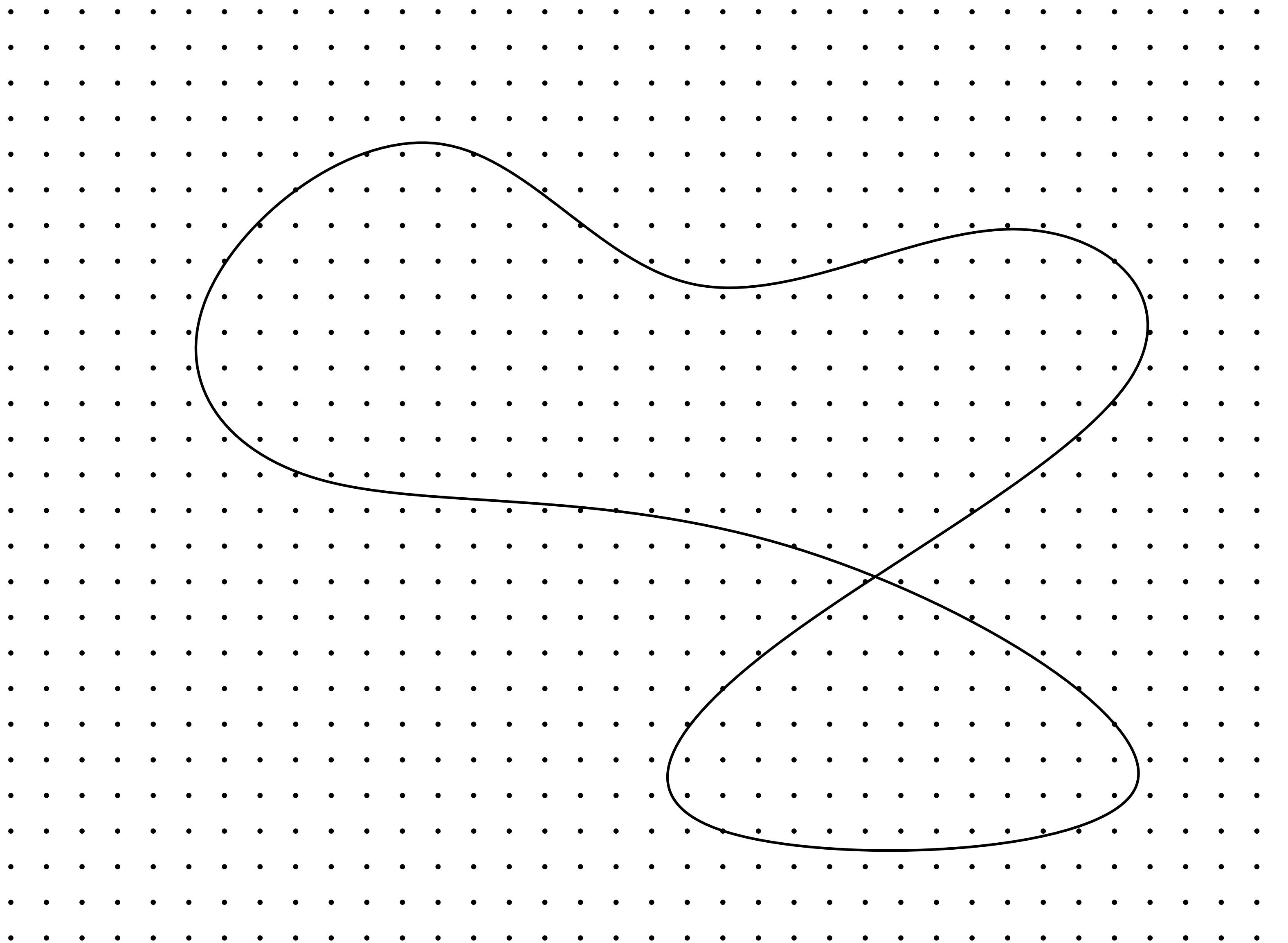
$$C_g \approx 1.6, \;\; C_r \approx 1.3$$

- Invariant under affine transformations?

Homotopic peeling

Freie Universität Berlin

[Sergey Avvakumov and Gabriel Nivasch 2019]



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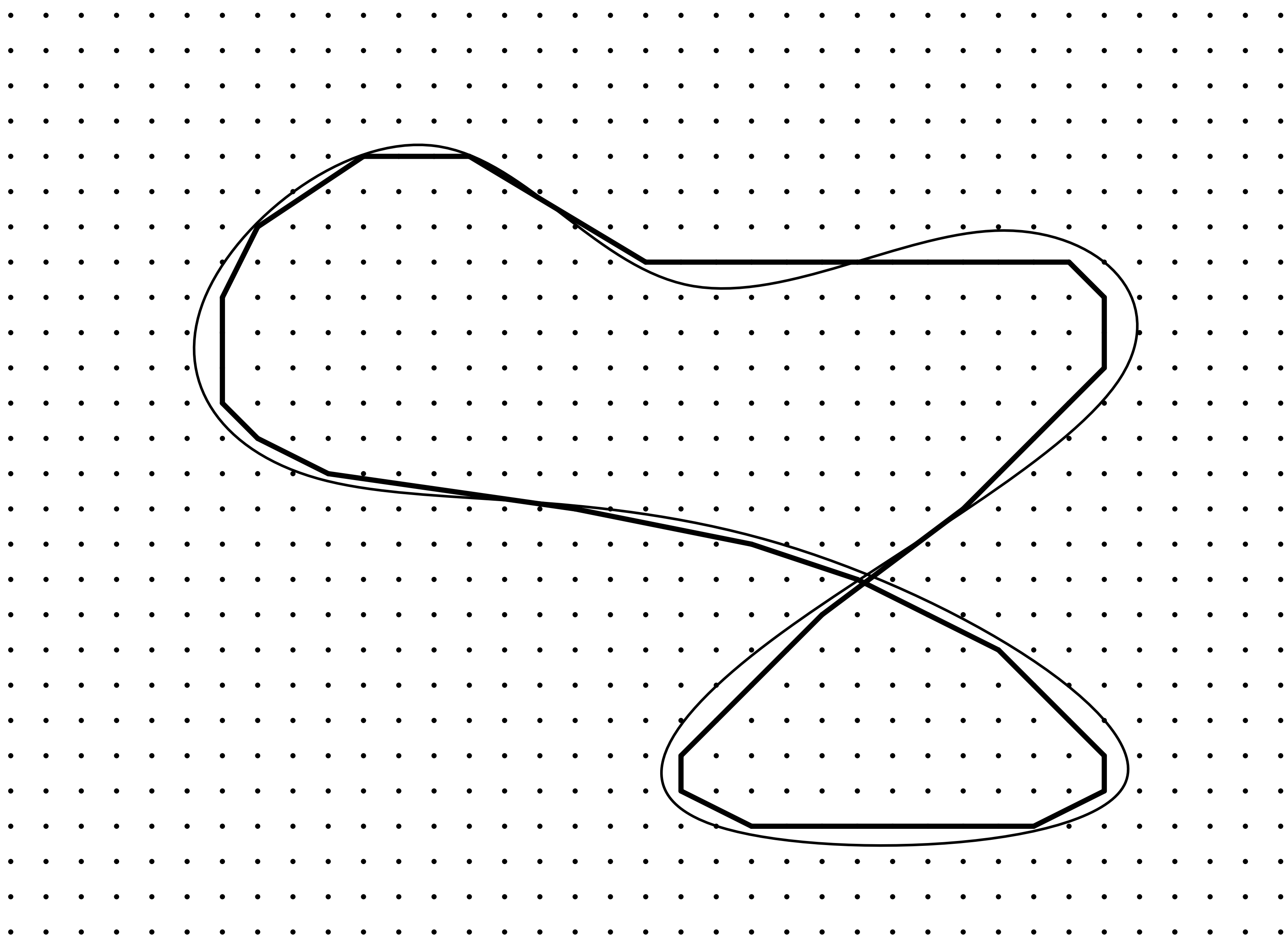

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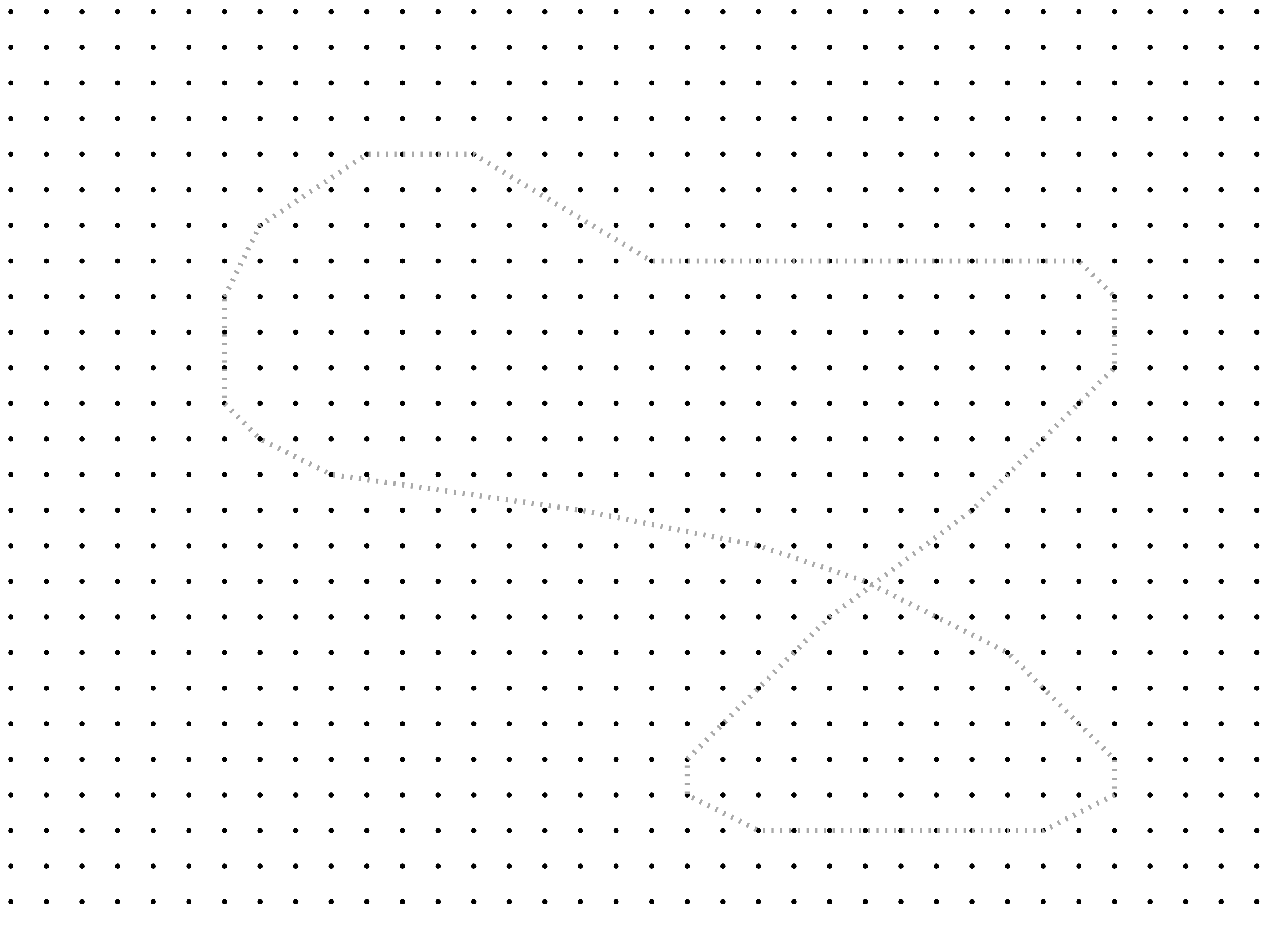

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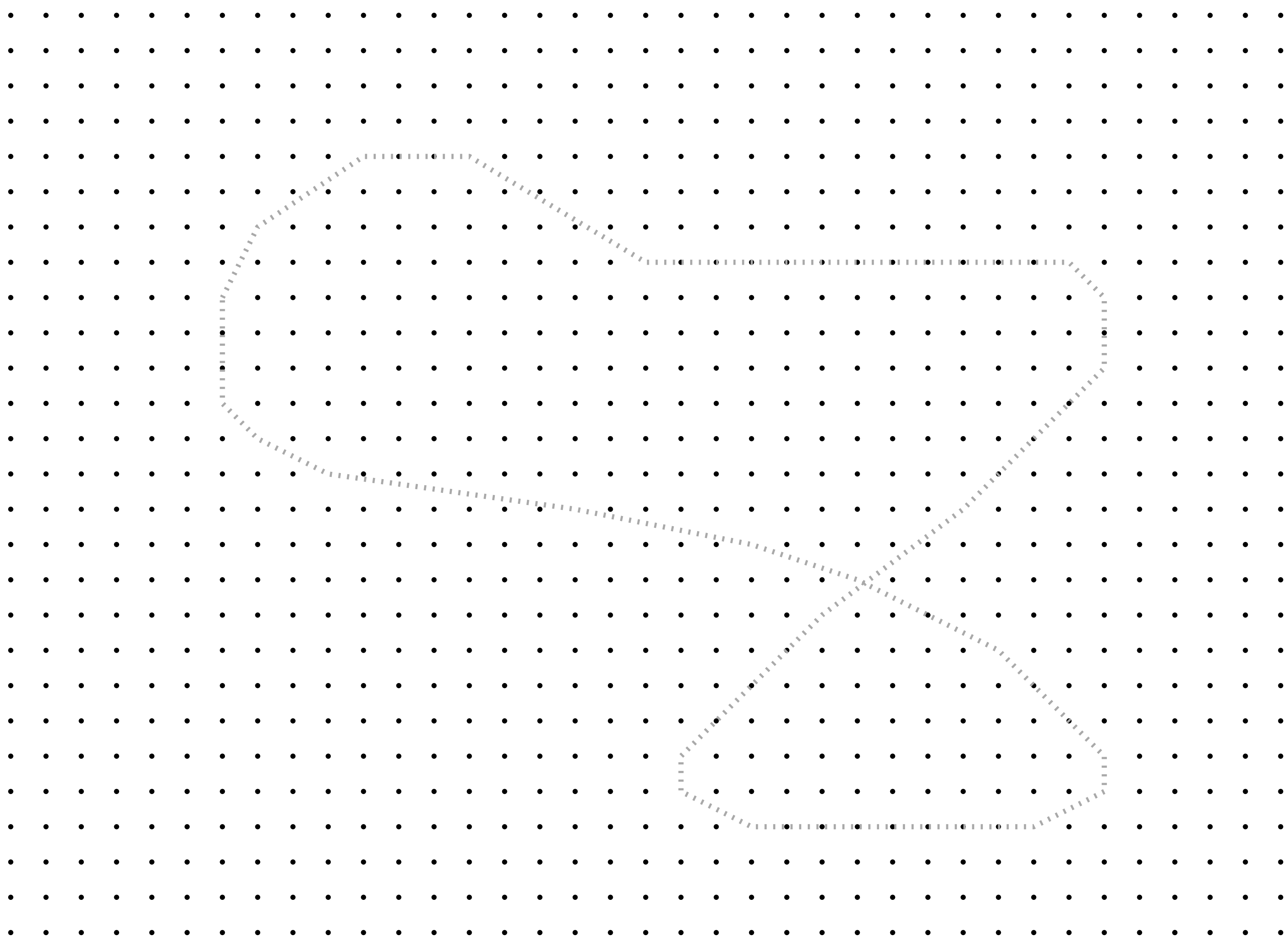

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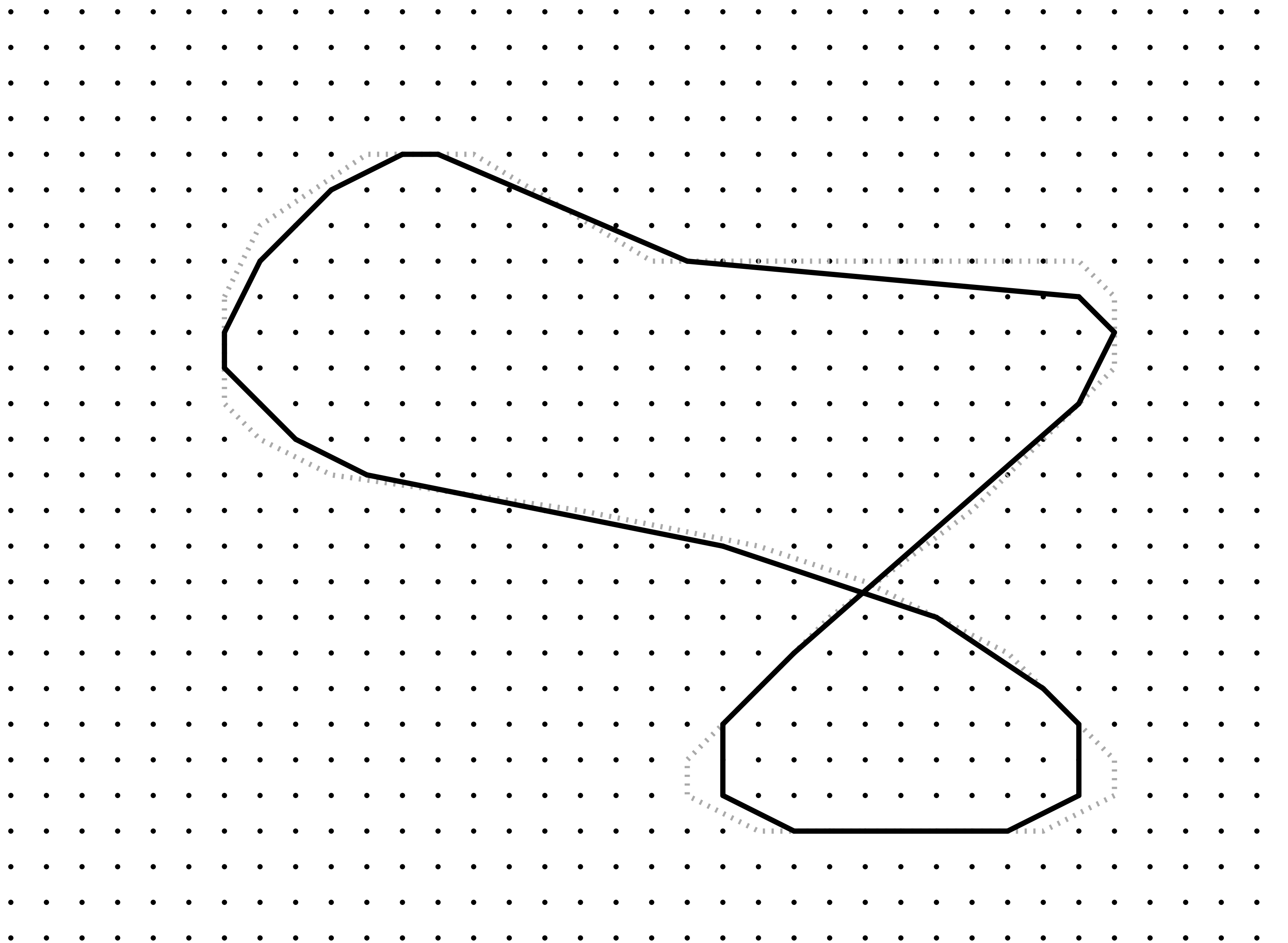

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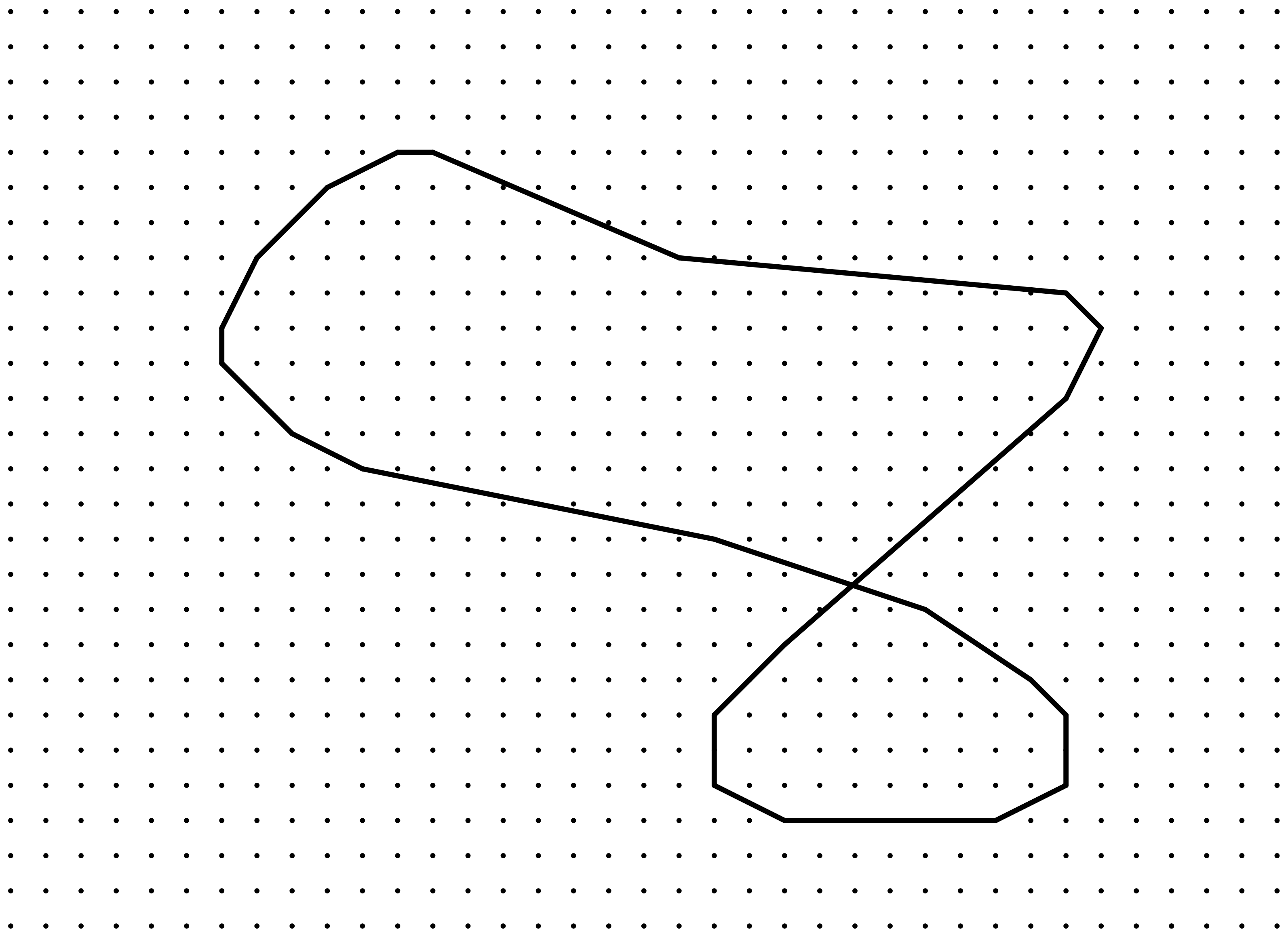

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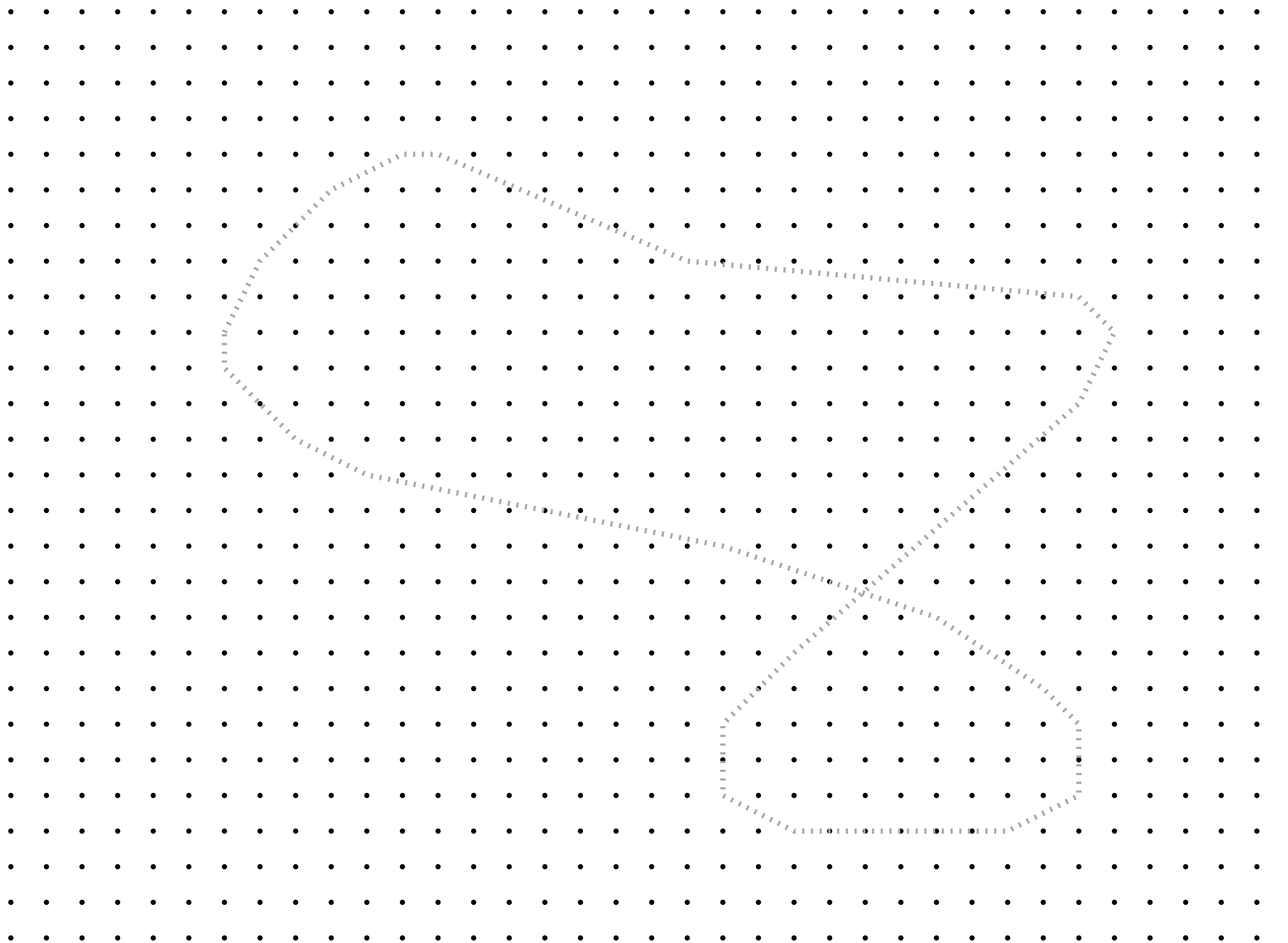

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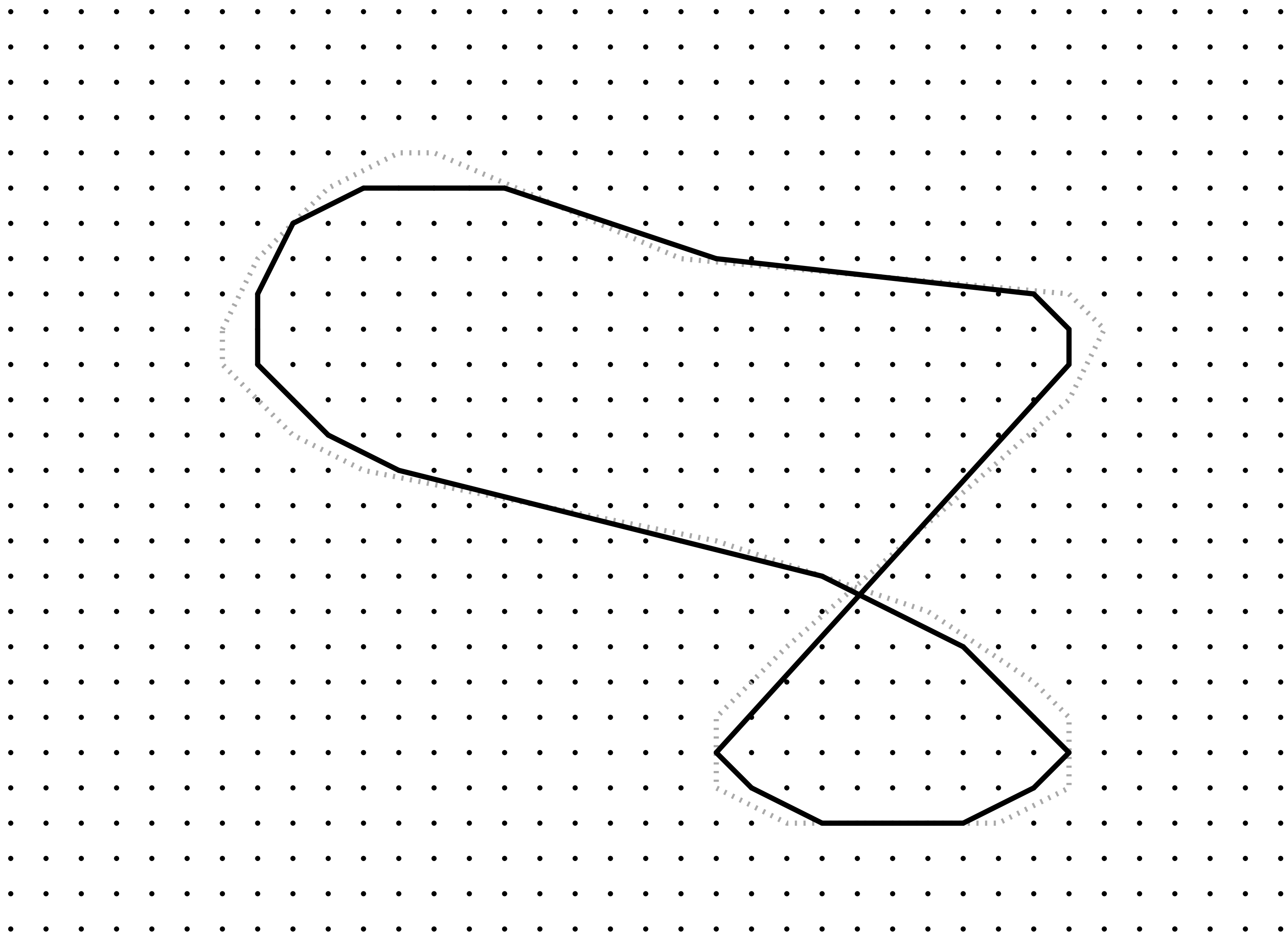

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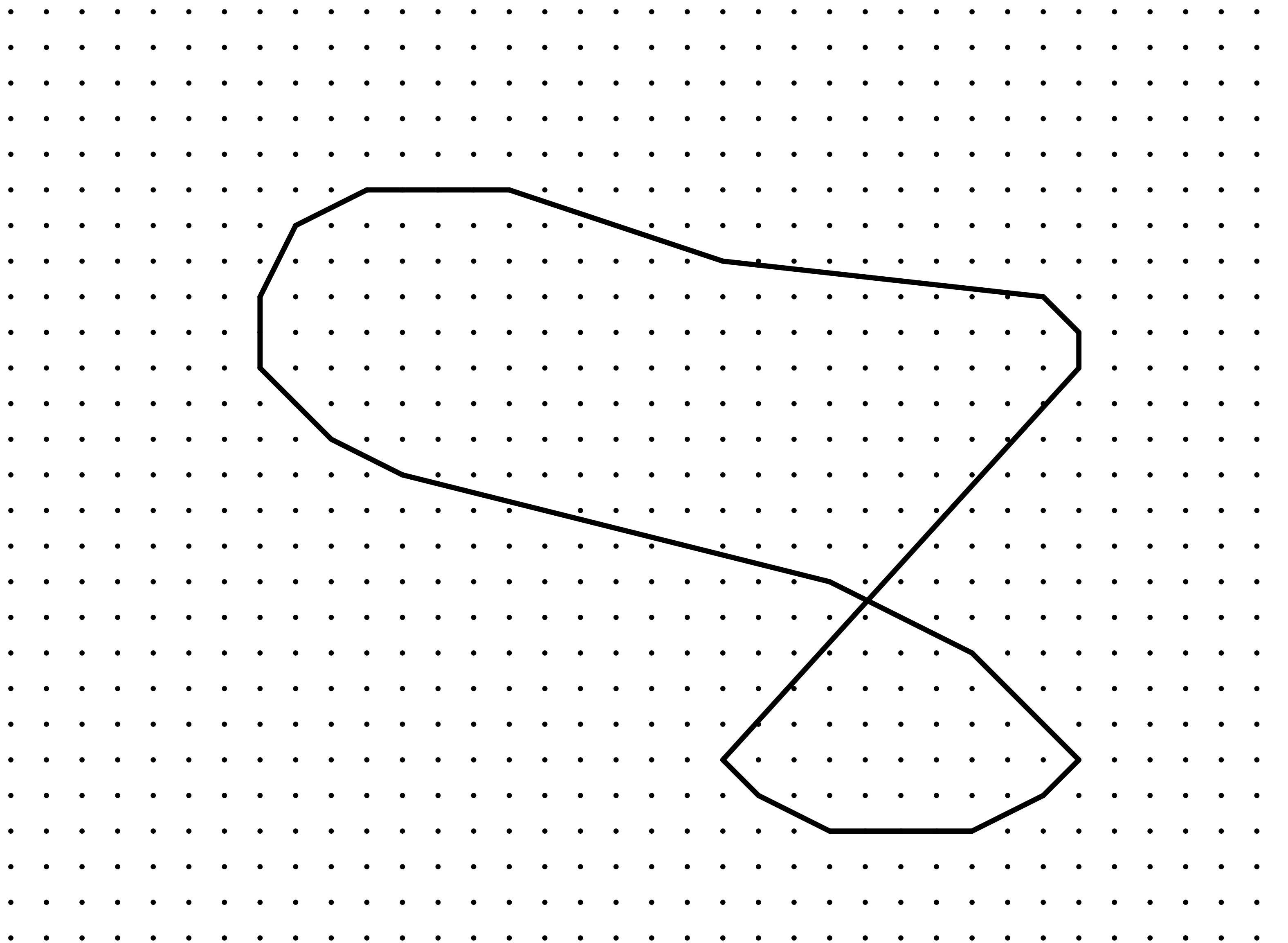

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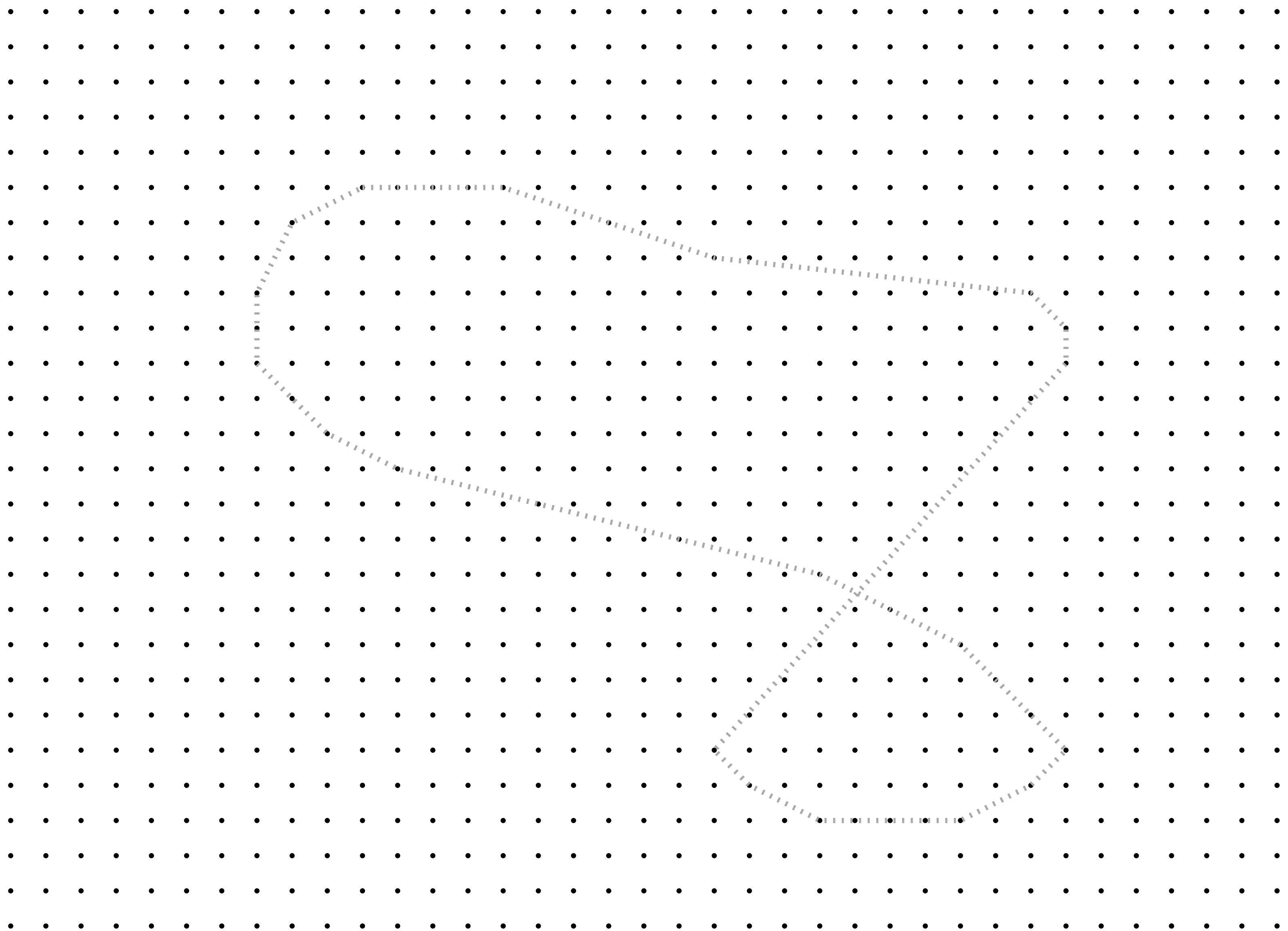

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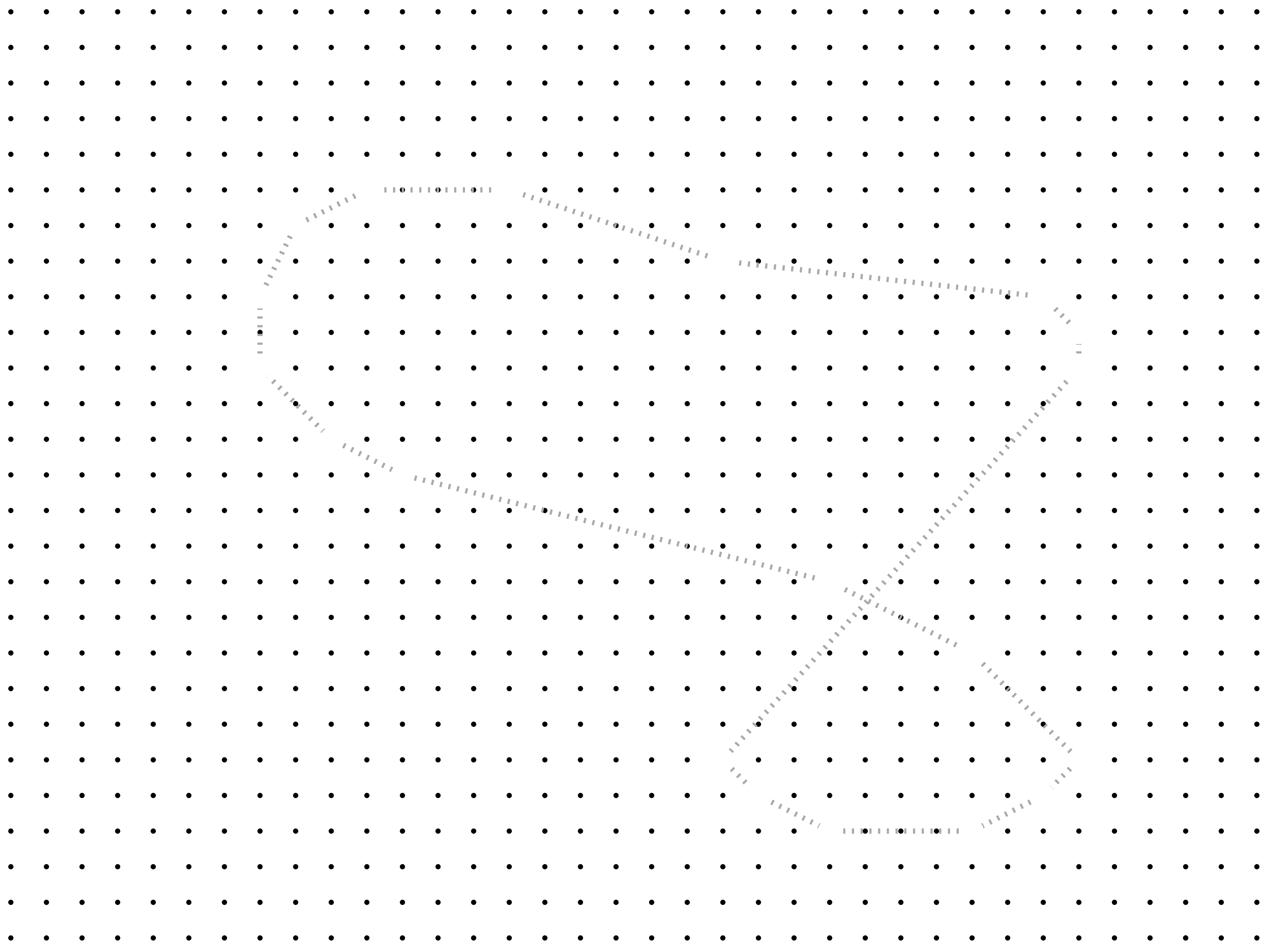

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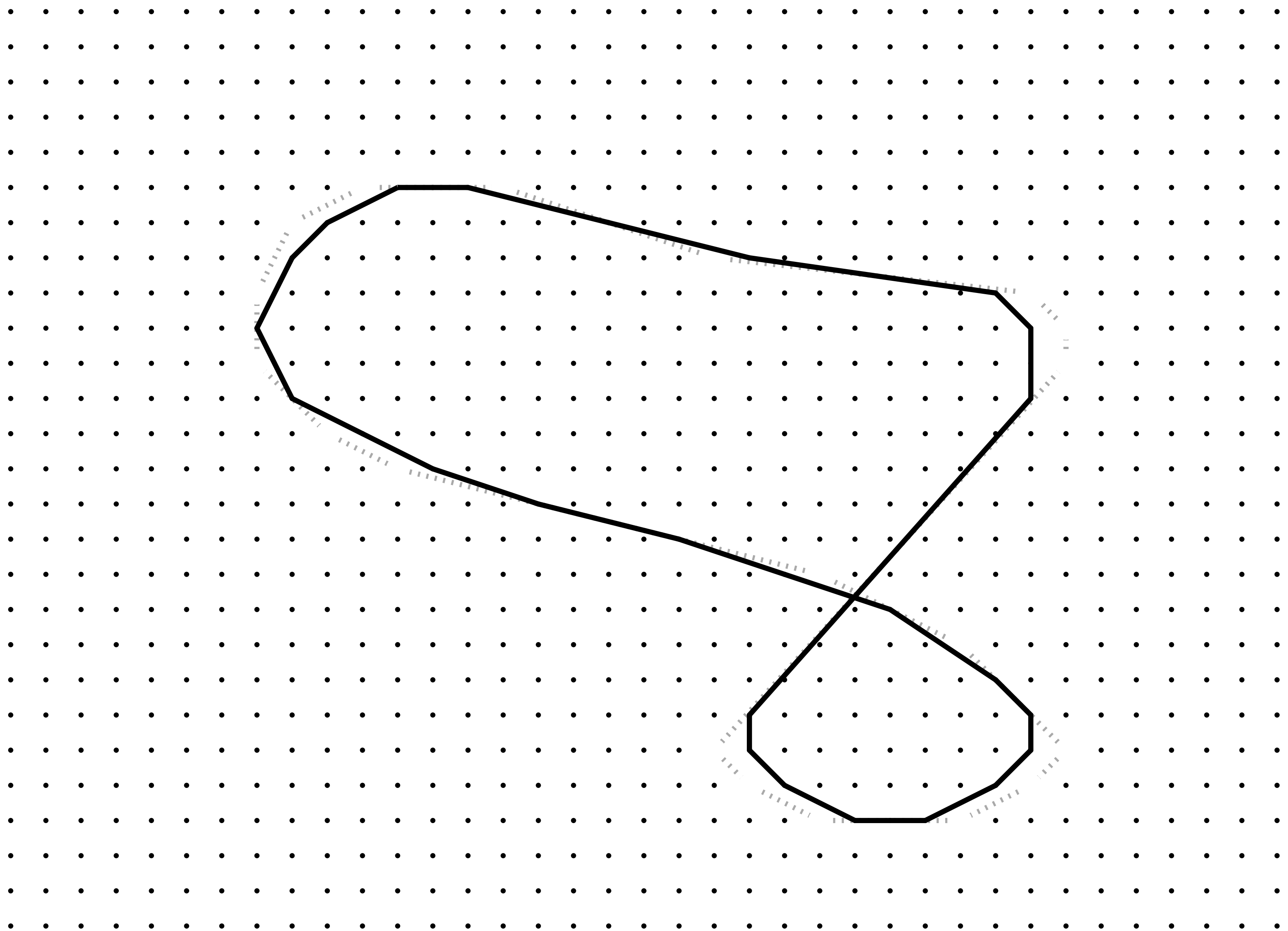

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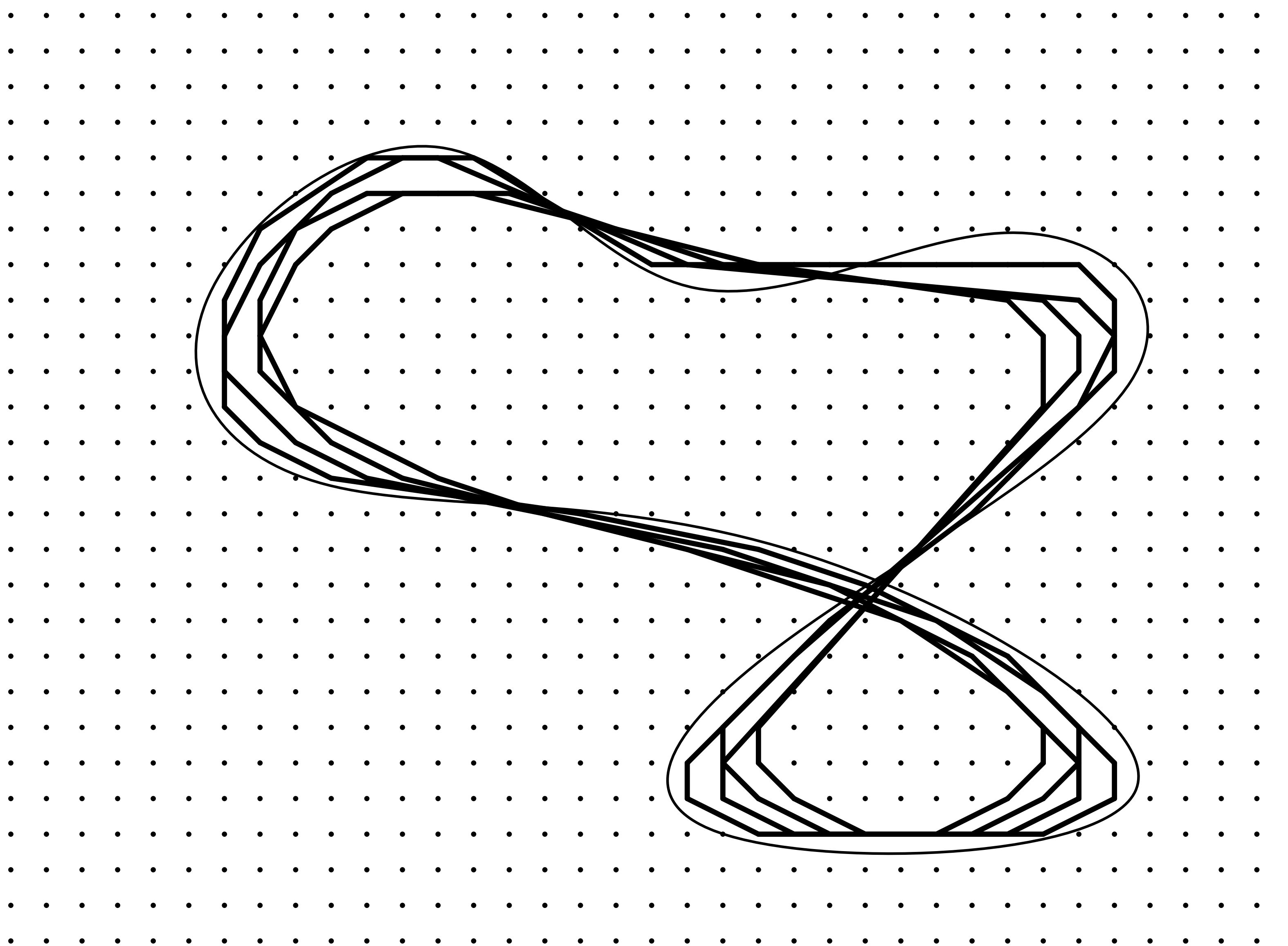
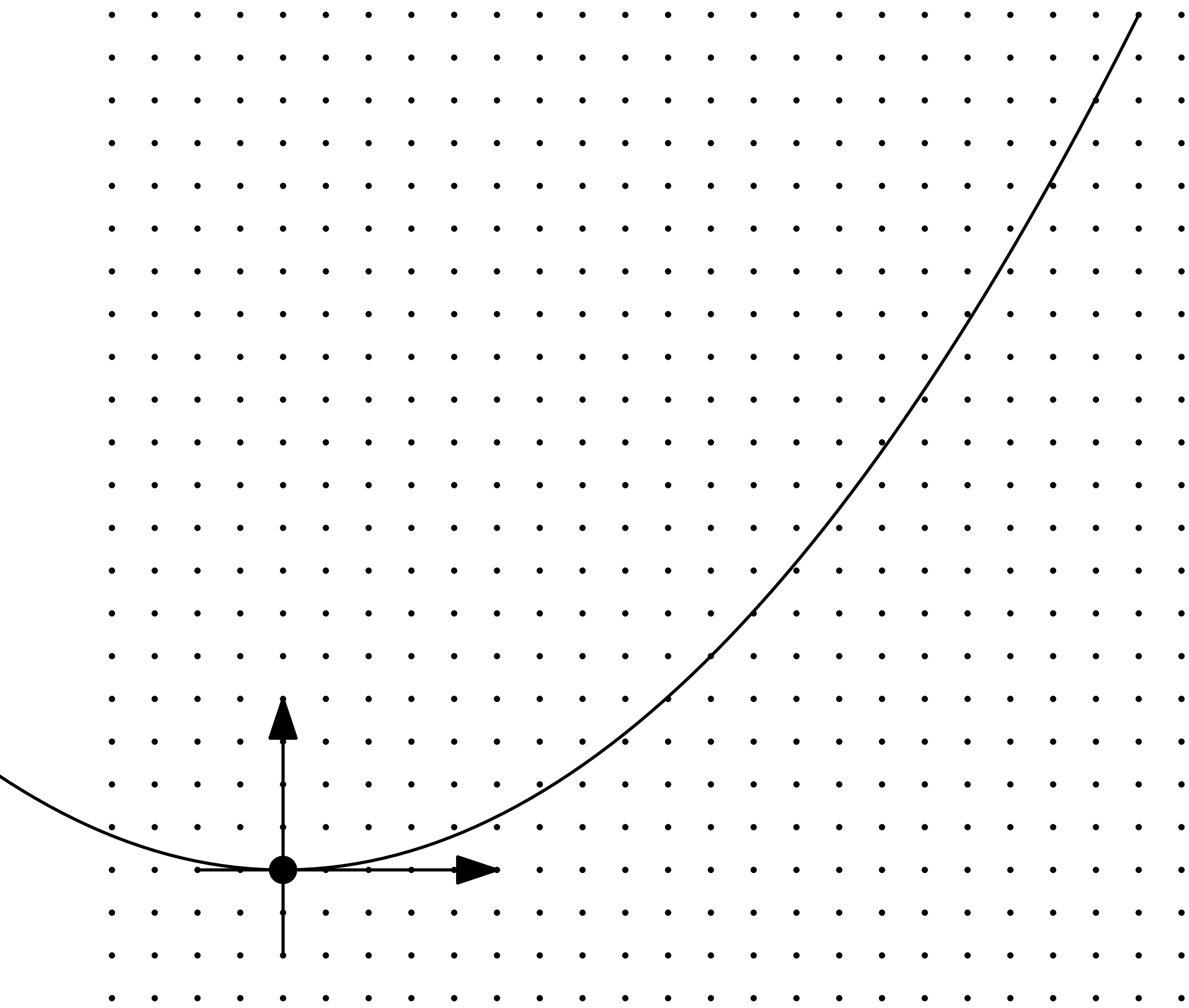


The parabola!



$$y = \frac{1}{20}x^2$$

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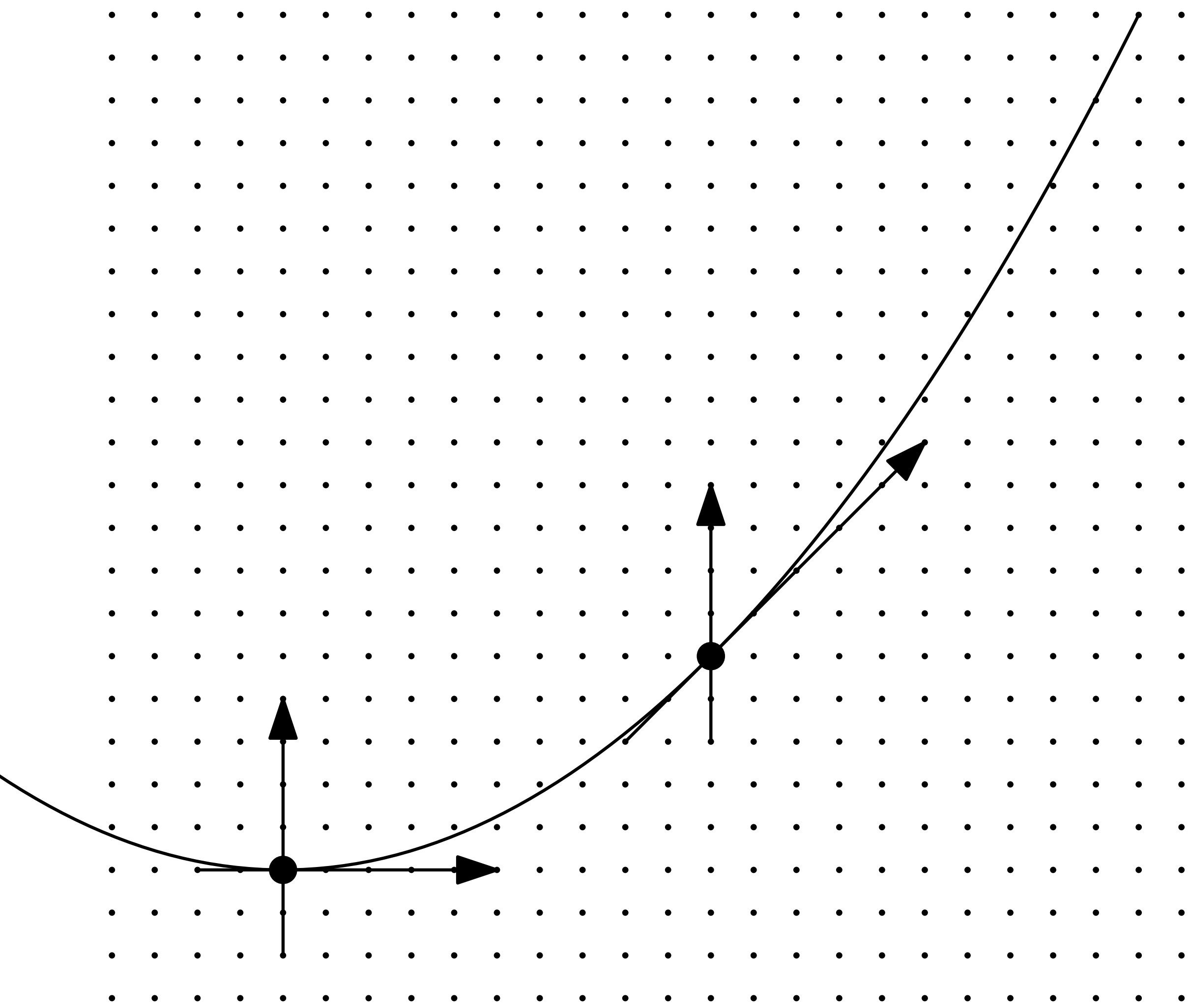

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affine lattice-preserving
shearing transformations

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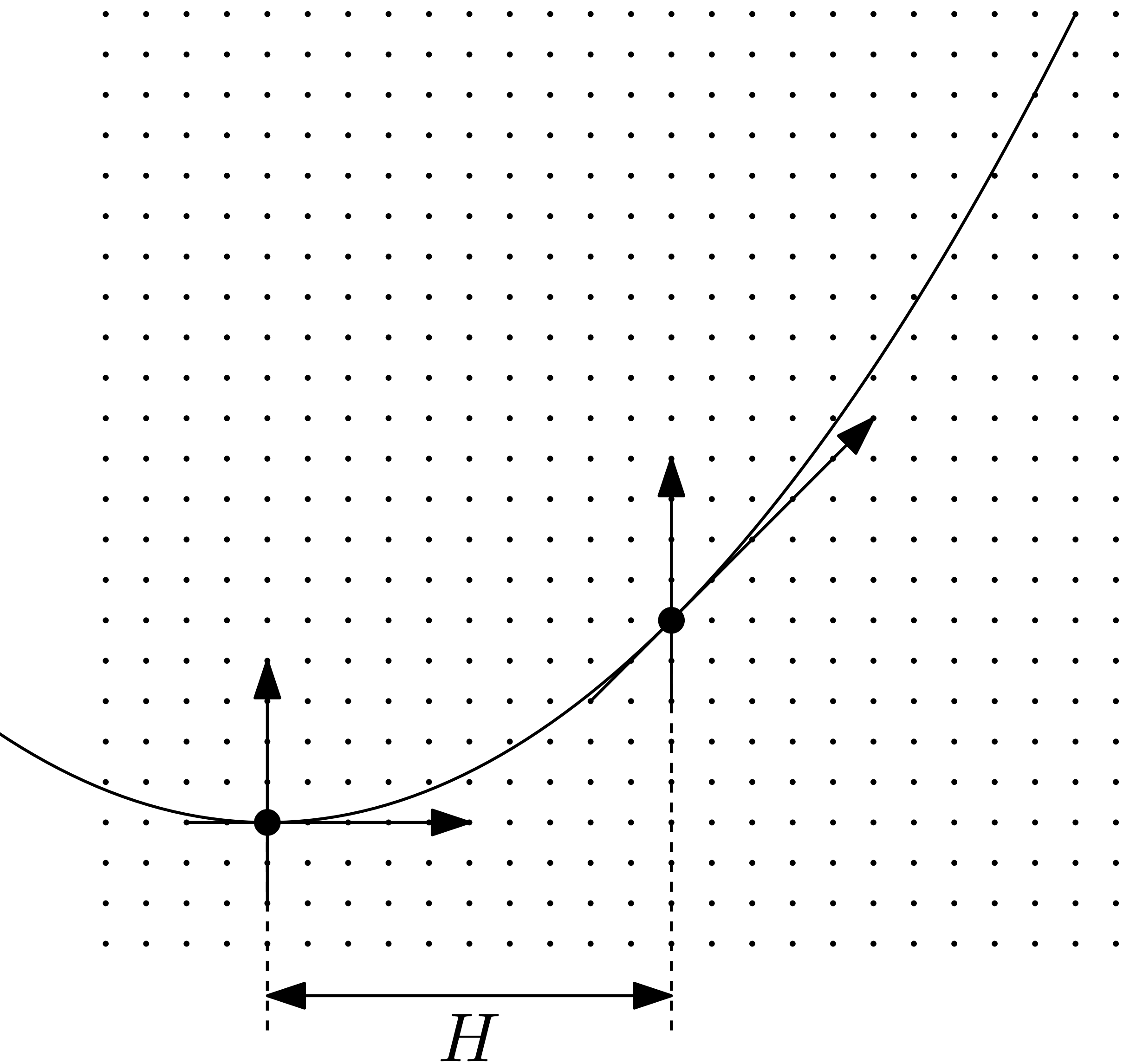


$$y = \frac{1}{20}x^2$$

affine lattice-preserving
shearing transformations

$$y = \frac{a_N}{a_D}x^2 + \frac{b_N}{b_D}x + c$$

Lemma:
Horizontal period $H = \operatorname{lcm}(a_D, b_D)$ or $H = \operatorname{lcm}(a_D, b_D)/2$

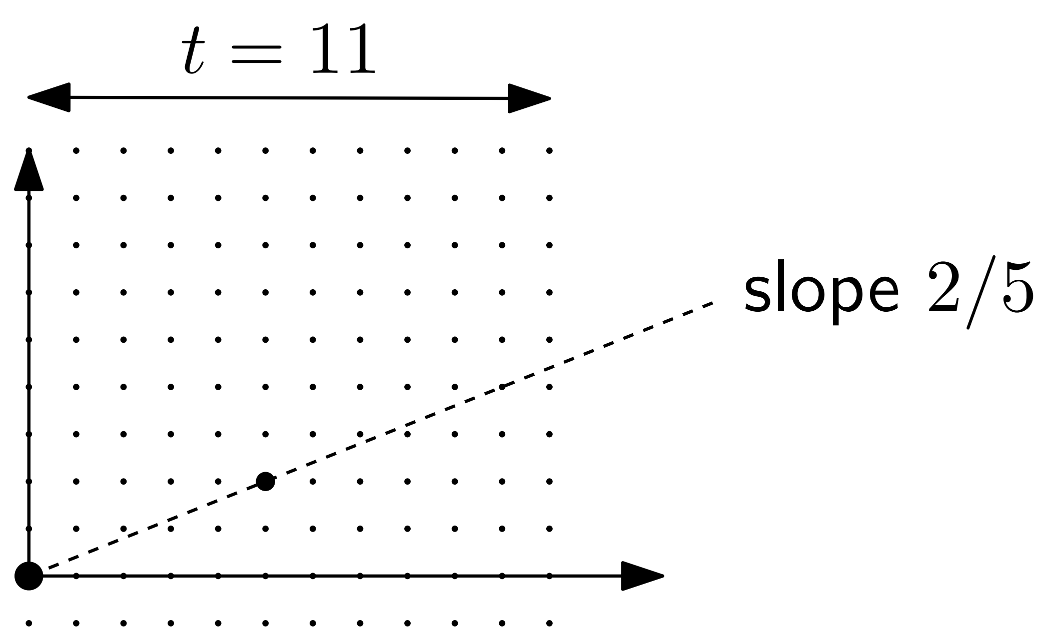
"The grid parabola"

- Parameter $t \geq 1$
- Take all slopes a/b with $0 < b \leq t$
- For each slope a/b, take the longest integer vector

$$\binom{x}{y} = f\binom{b}{a} \quad (f \in \mathbb{Z})$$

with $0 < x \leq t$

Example



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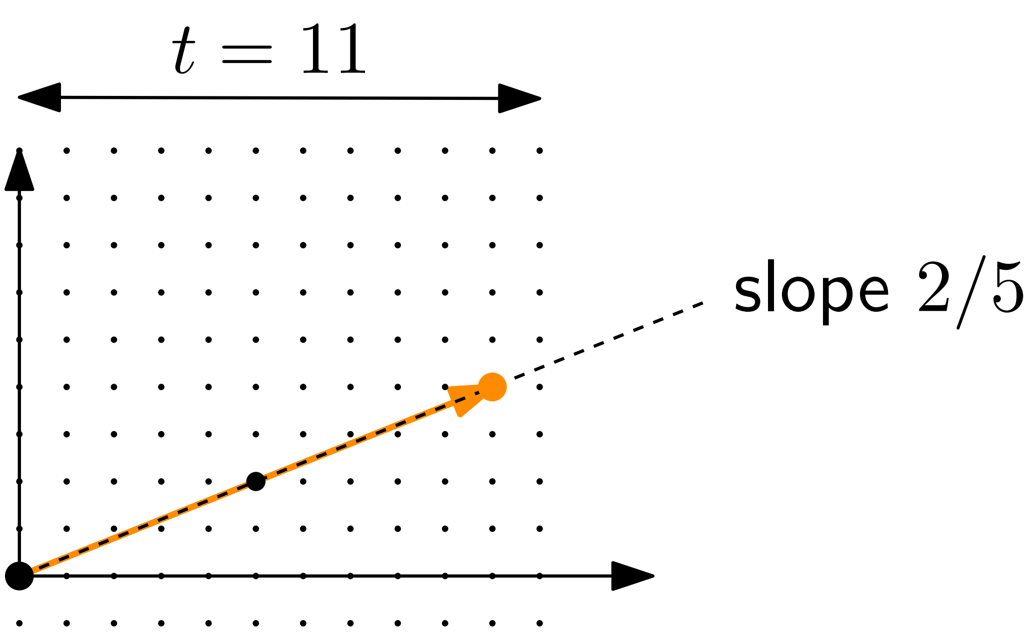

“The grid parabola”

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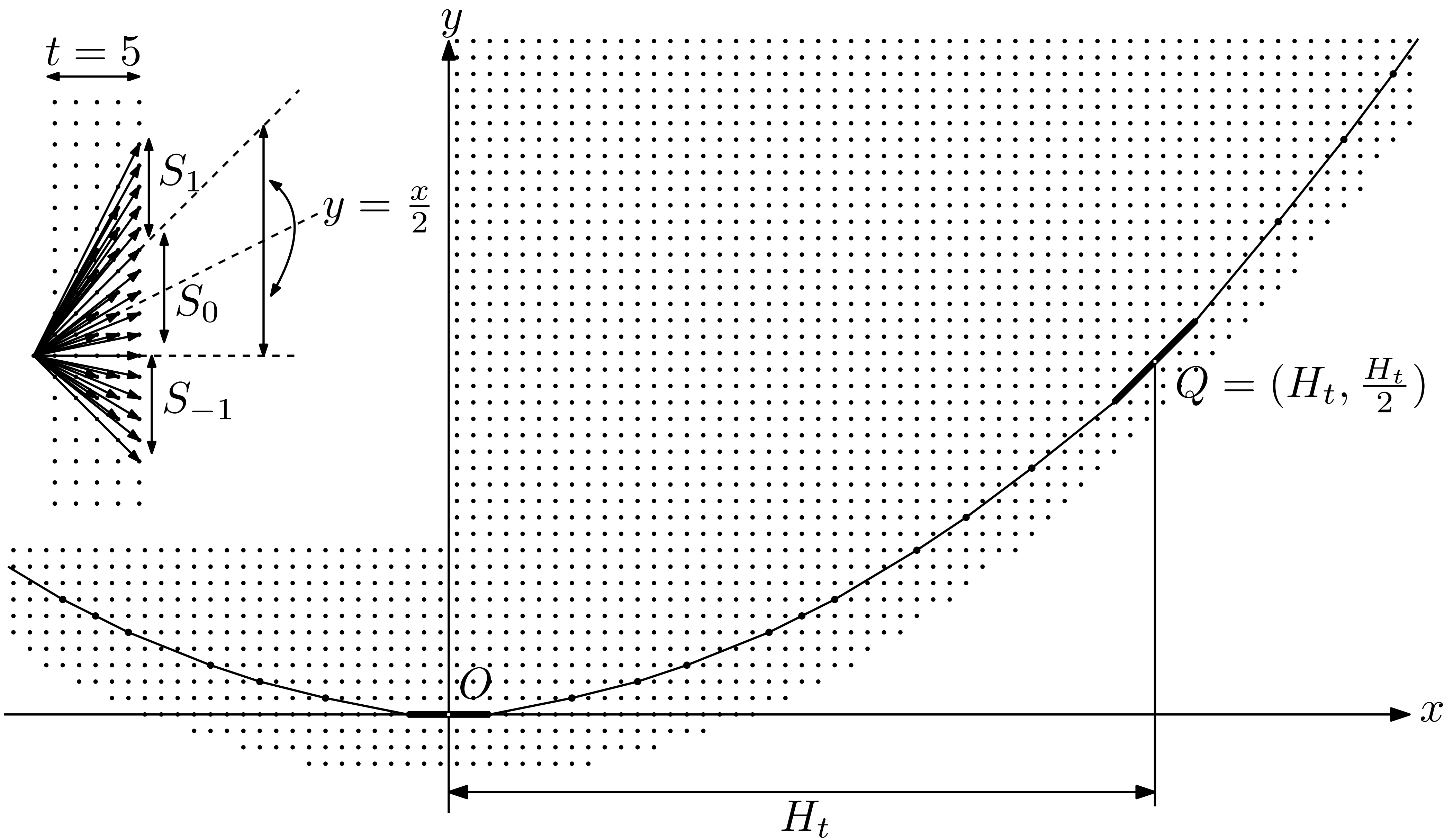

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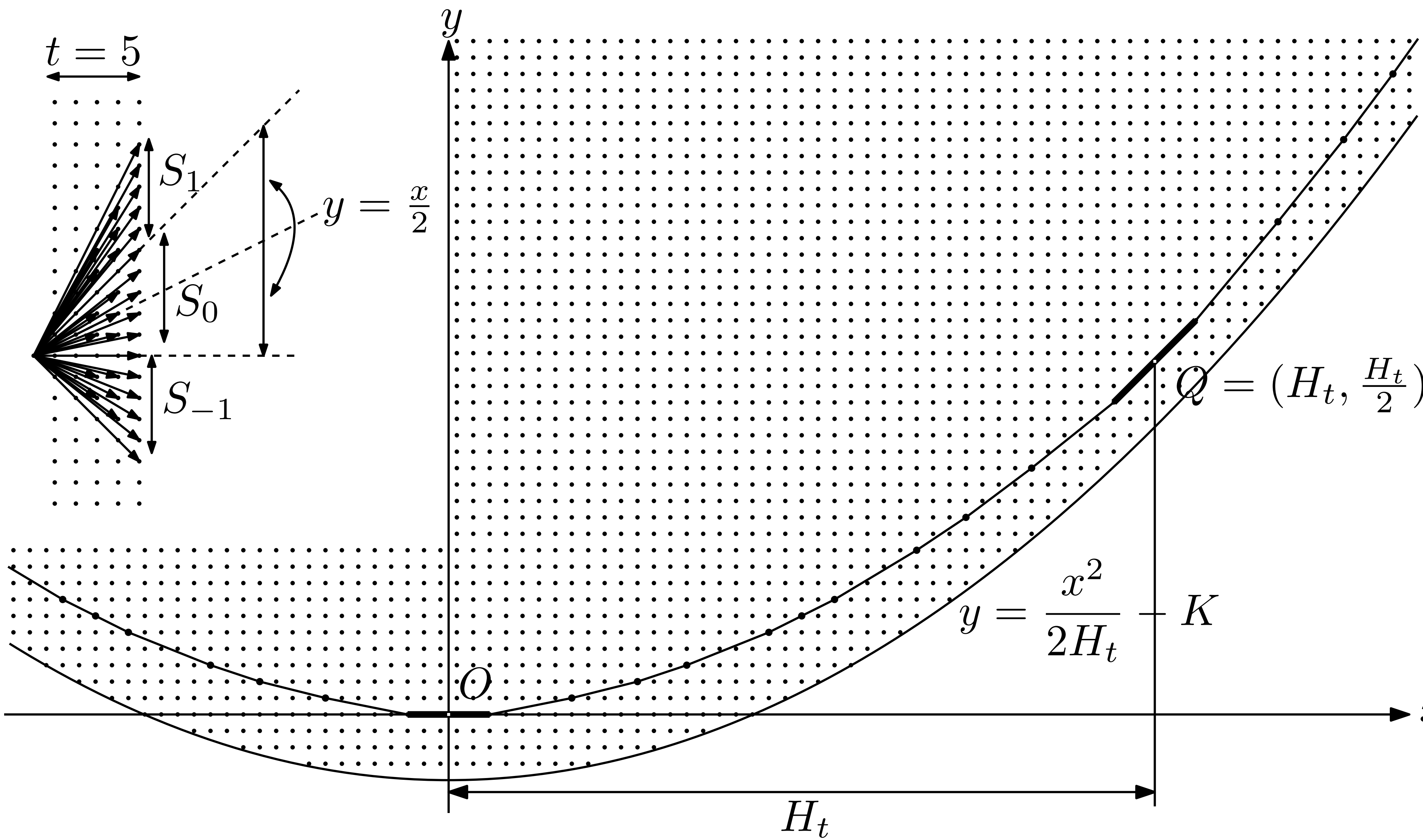

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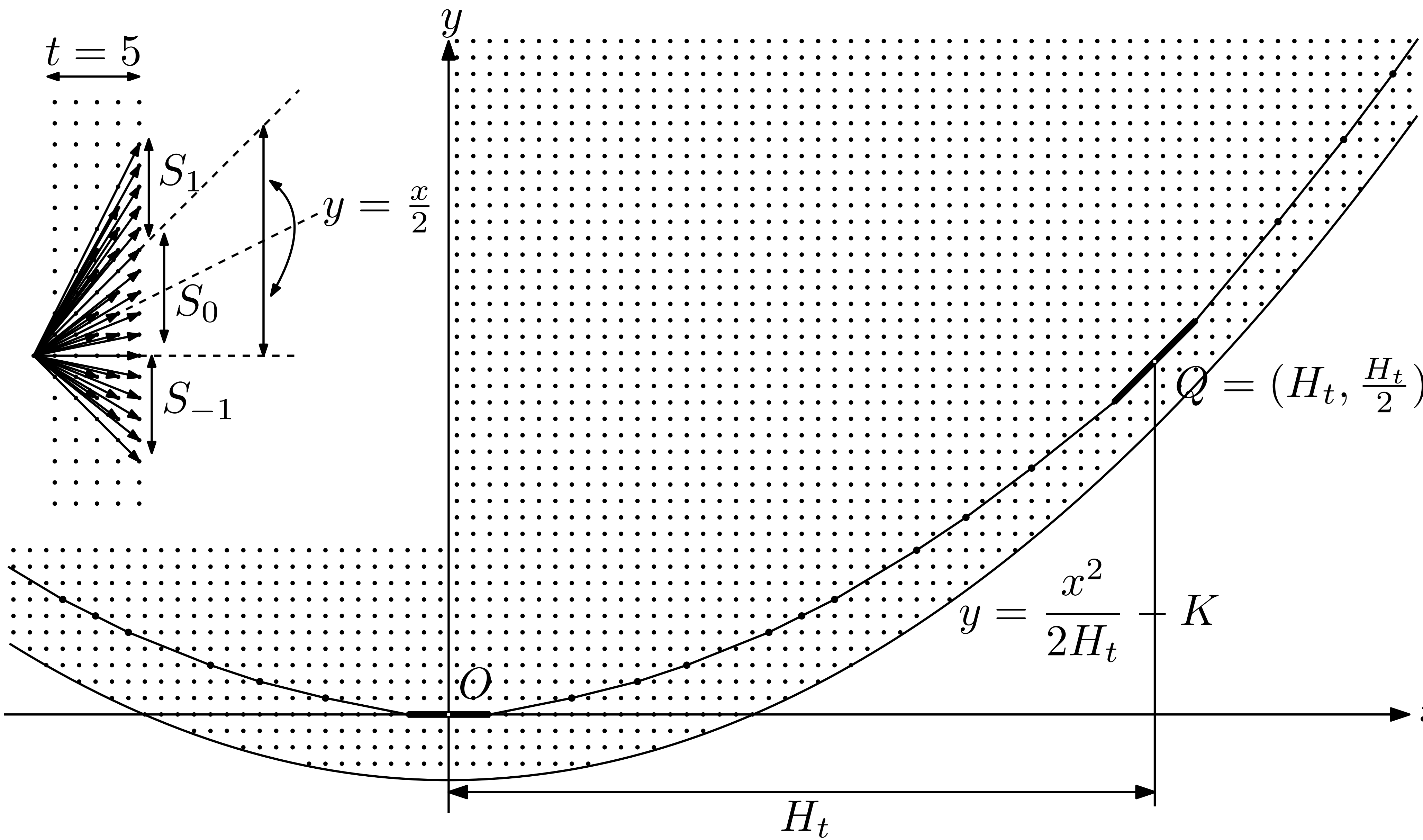
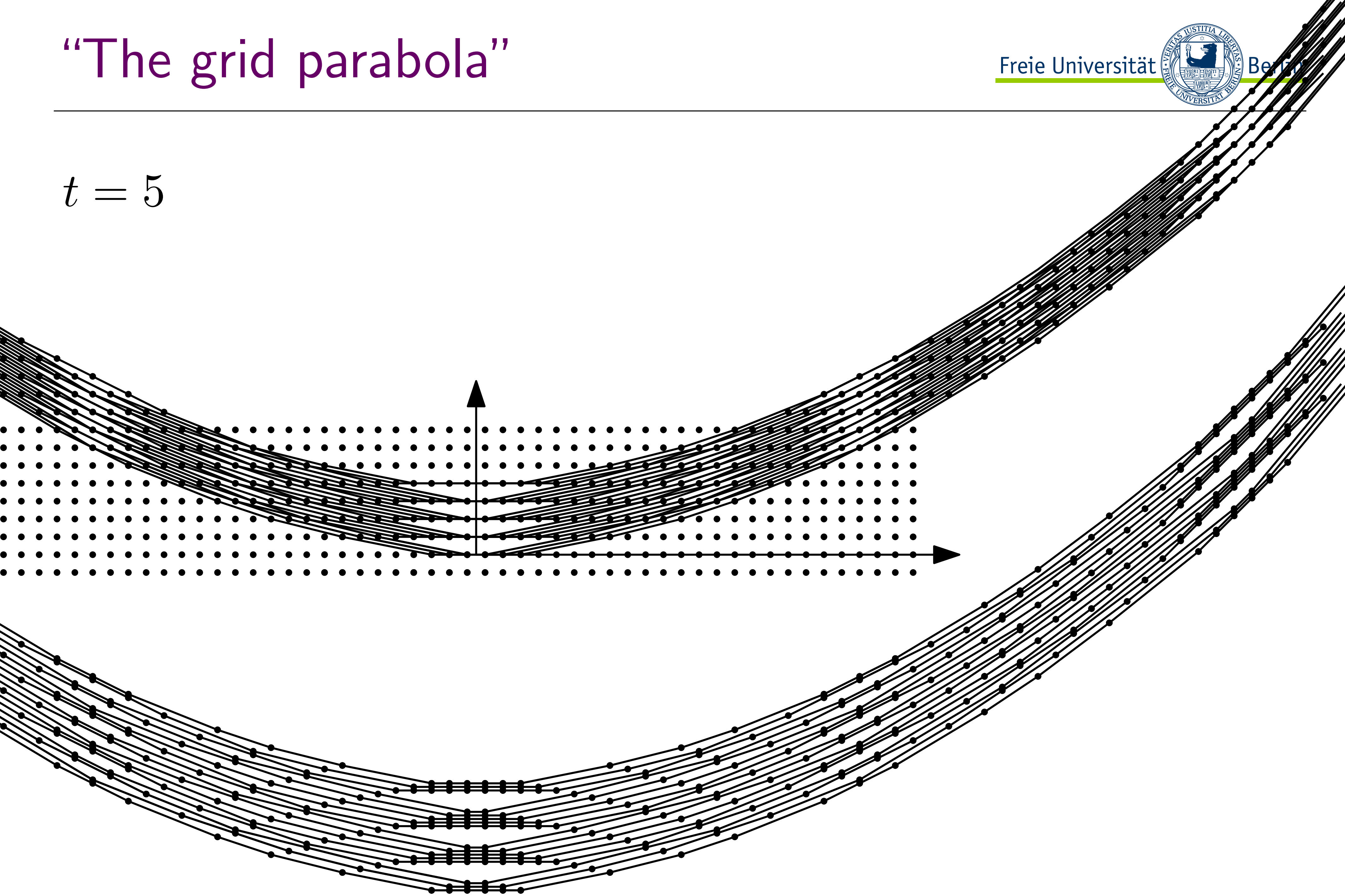


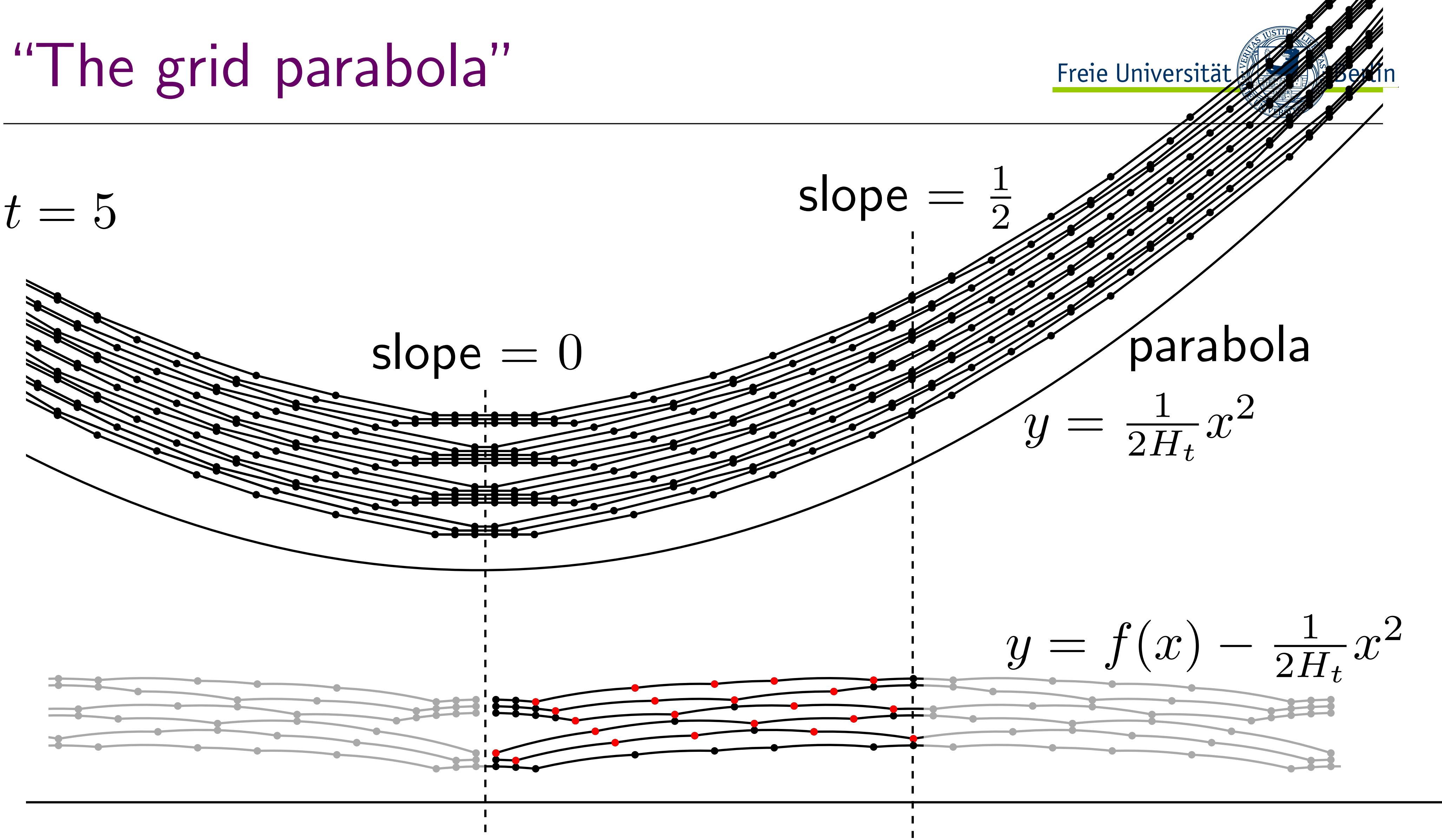
$$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$$

"The grid parabola"

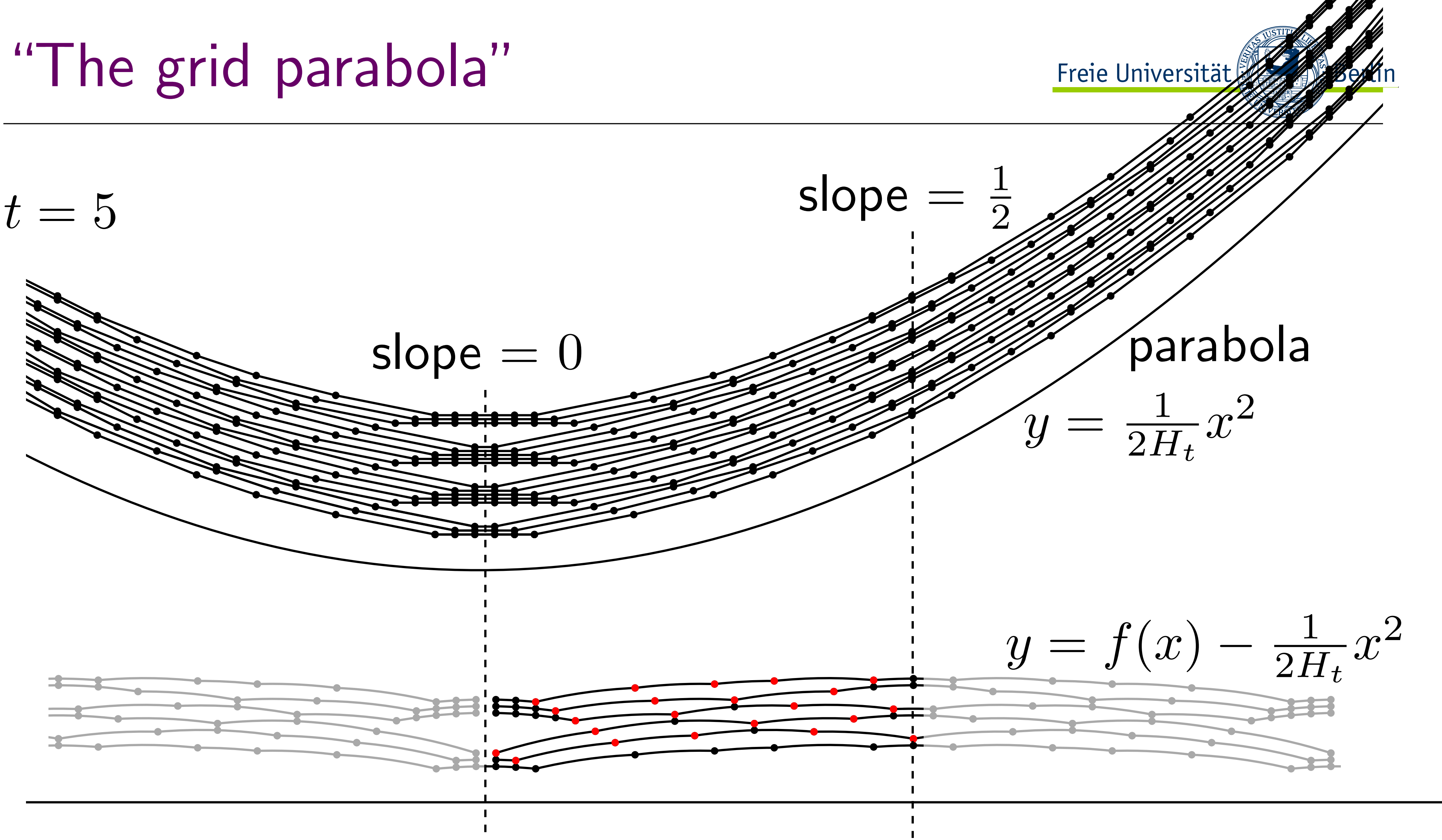
$t = 5$



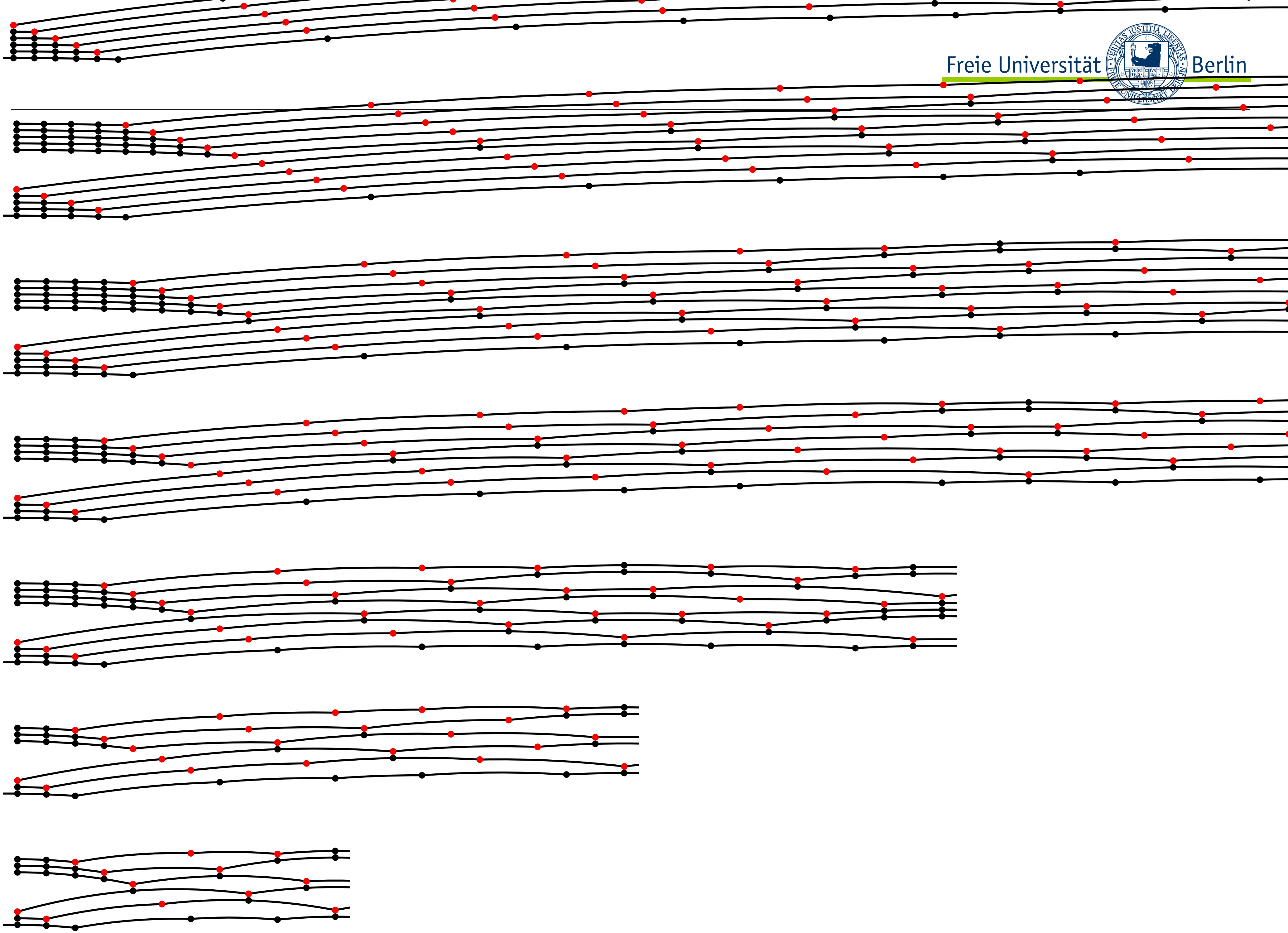
"The grid parabola"



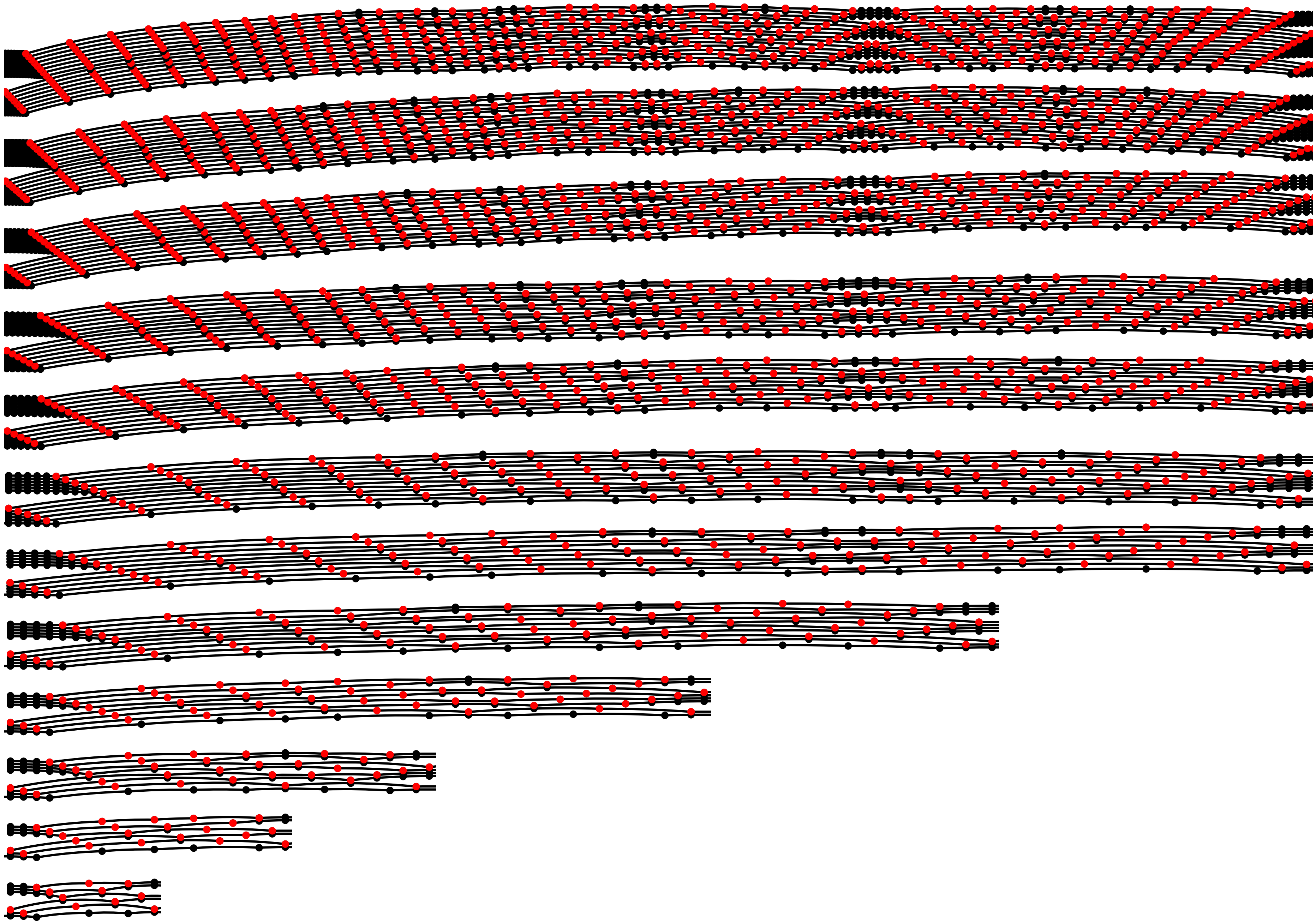
“The grid parabola”



Conjecture: The polygon repeats after t steps, one level higher.
(After $t + 1$ steps if t is even.)



$t = 4, 5, 6, \ldots$



Asymptotic period

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$$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$$

[OEIS A174405]

$$H_t := \sum_{\substack{0<y\le x\le t\\ \gcd(x,y)=1}} \left\lfloor \frac{t}{x} \right\rfloor x = \sum_{1\le i\le t} \sum_{d|i} d\varphi(d)$$

$$H_t = \frac{2\zeta(3)}{\pi^2} t^3 + O(t^2 \log t)$$

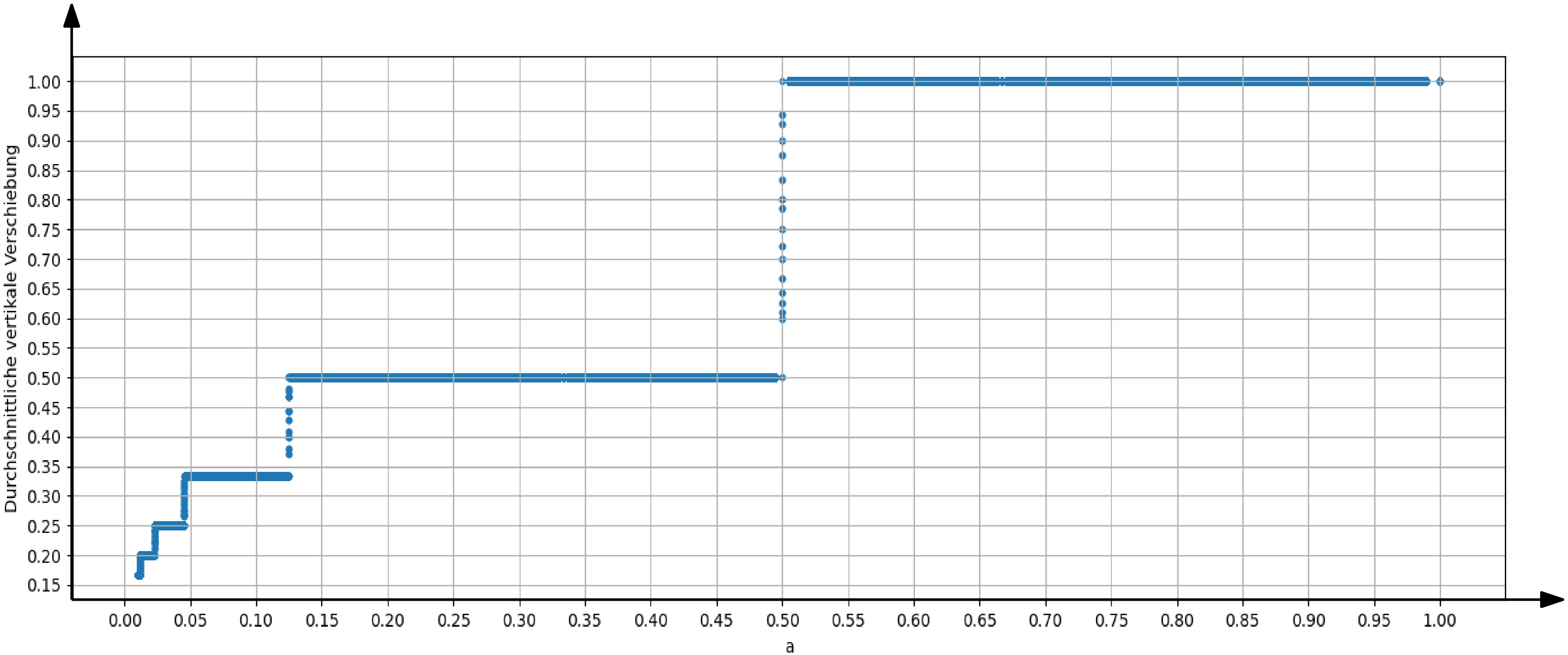
with $\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.2020569$

[Sándor and Kramer 1999]

Time period for various parabolas

$$y = ax^2 + bx$$

average vertical speed depending on a (various values of b)



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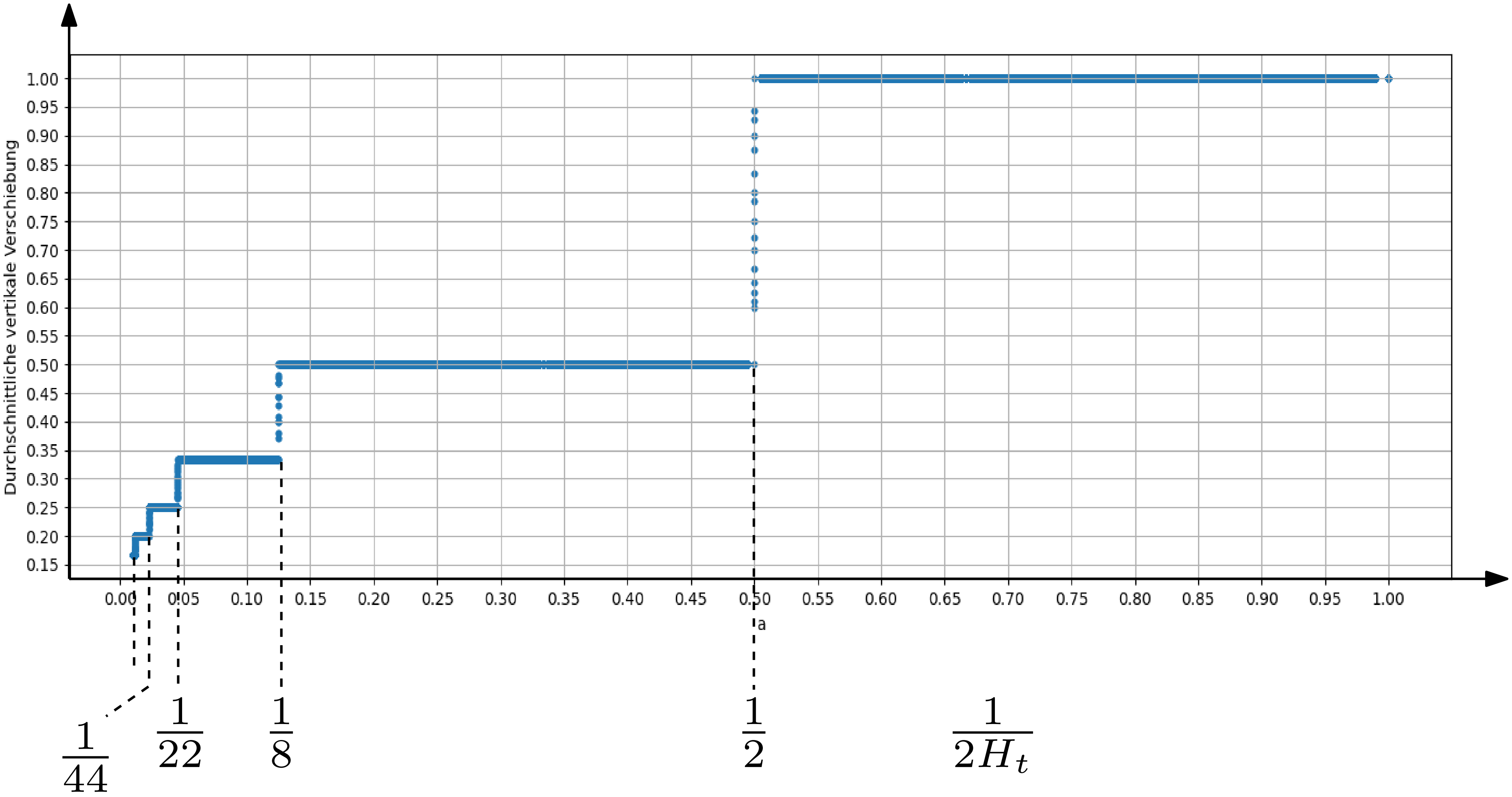
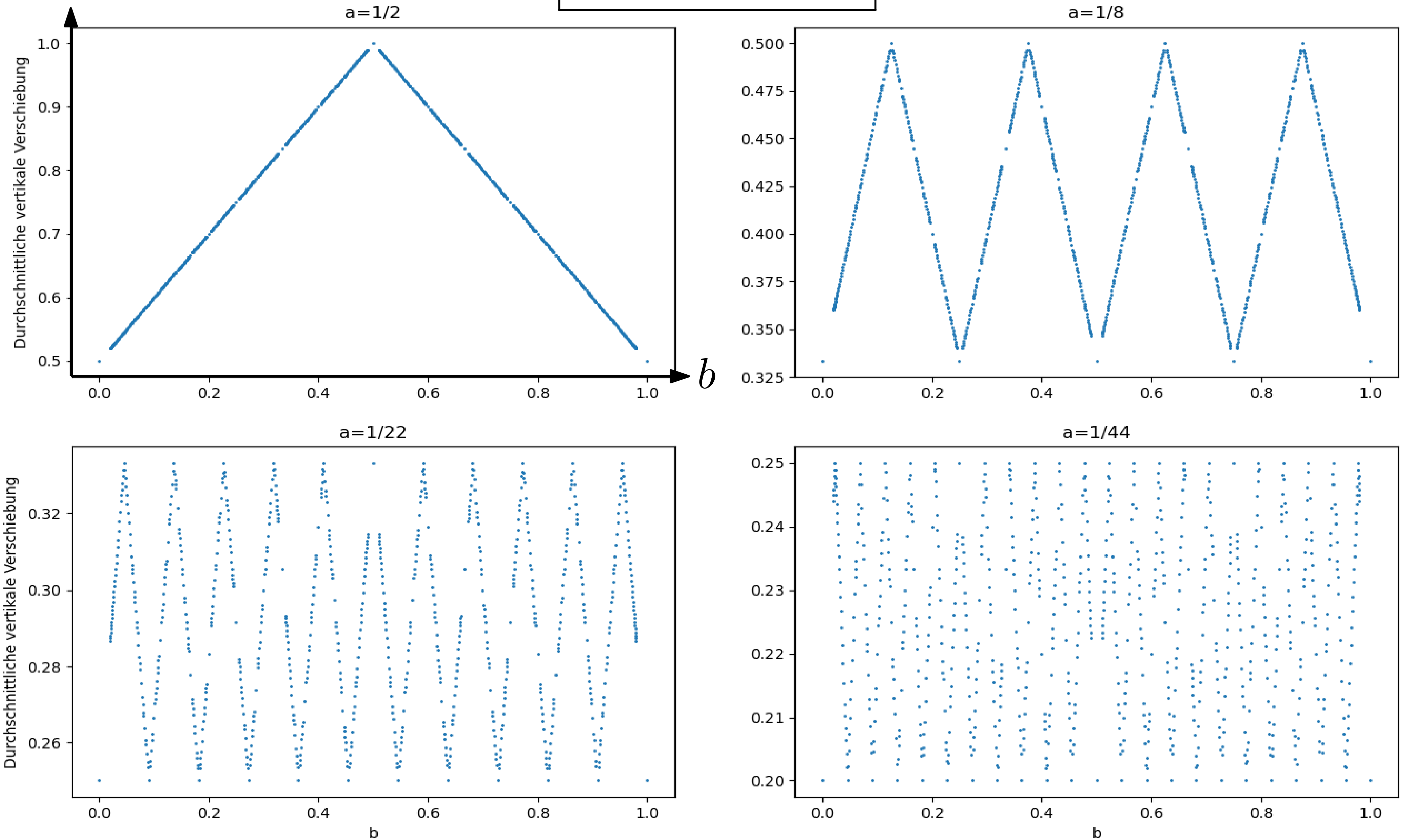


Time period for various parabolas

vertical speed

$$y = ax^2 + bx$$



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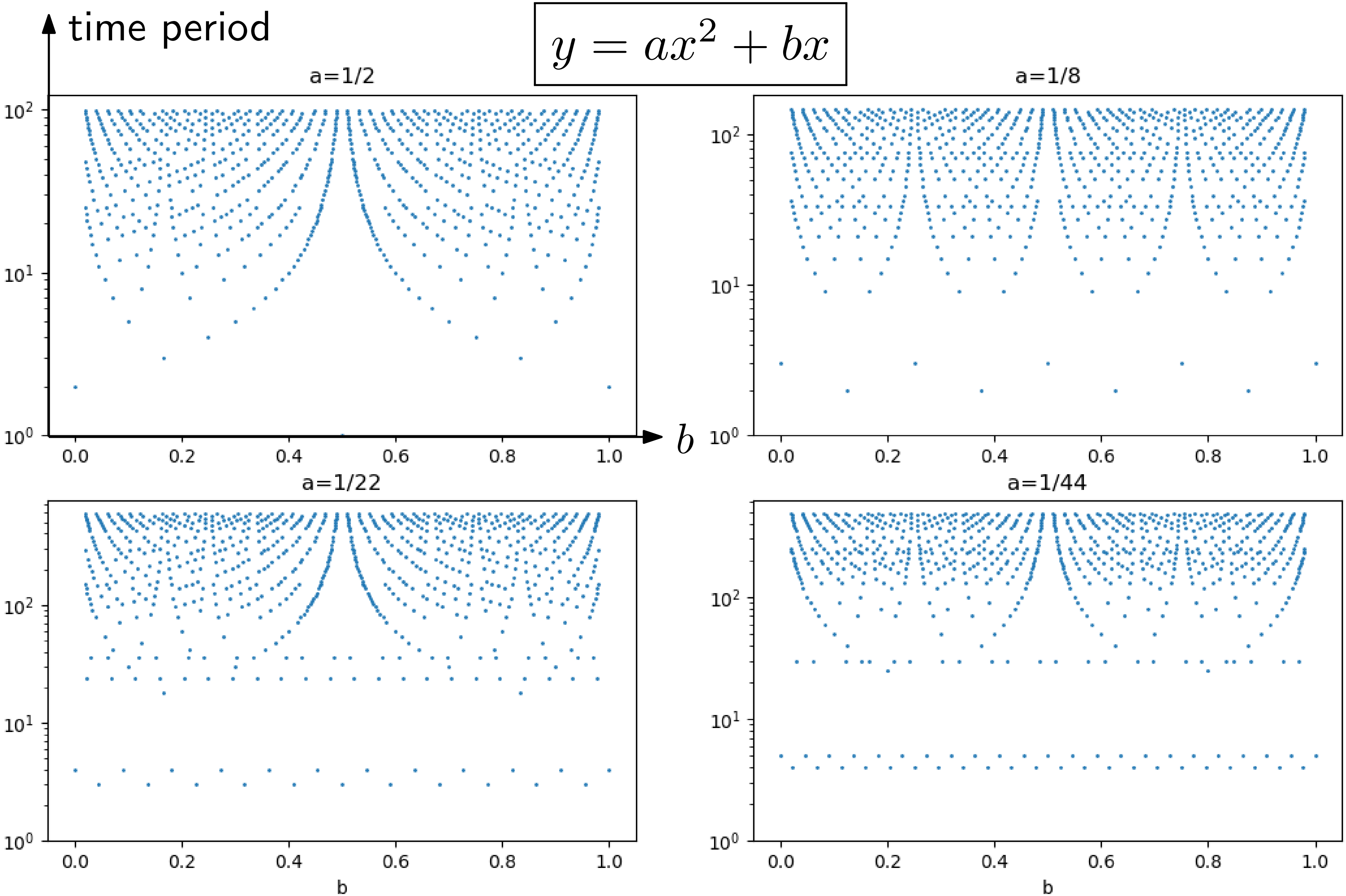

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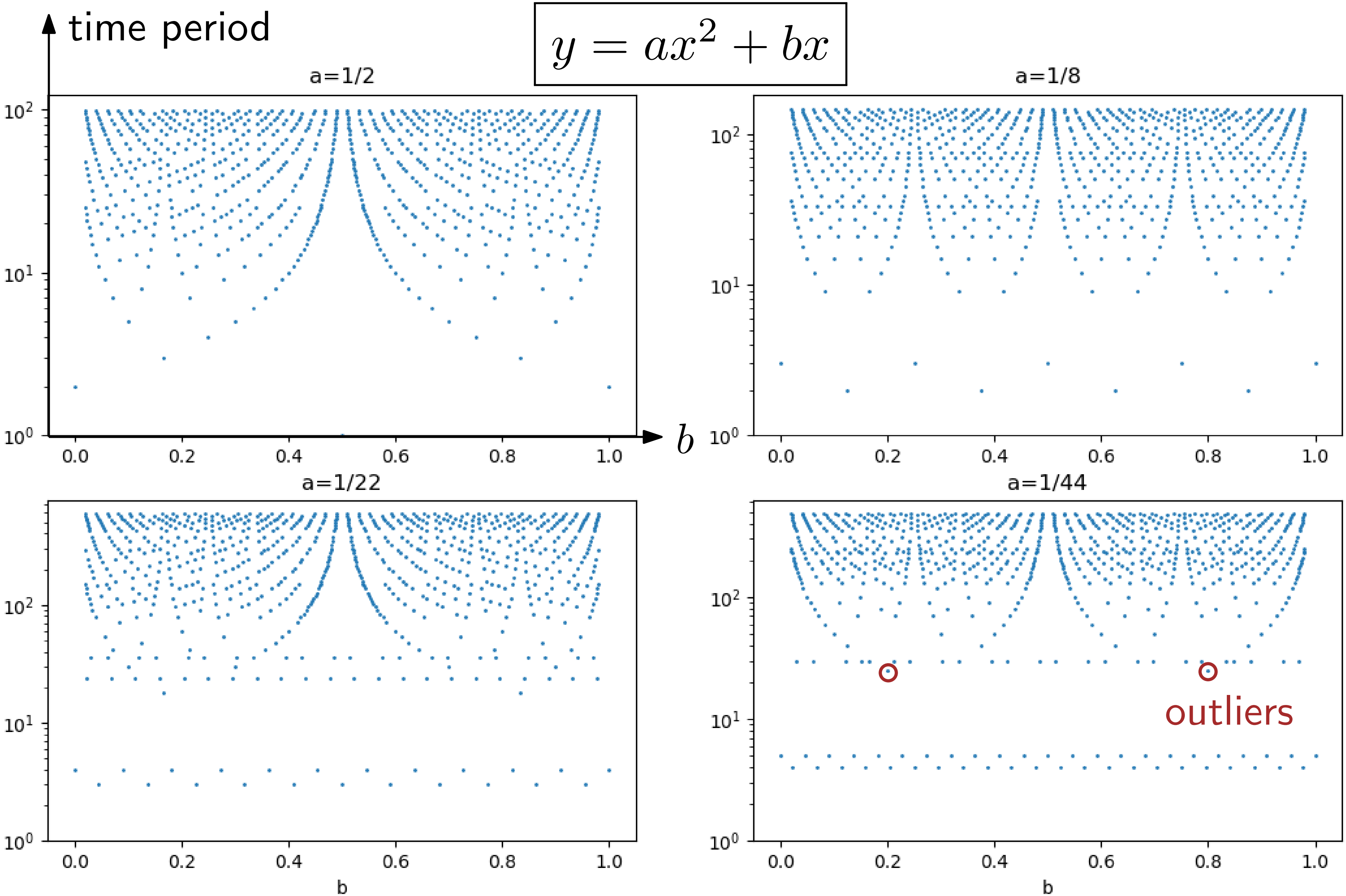
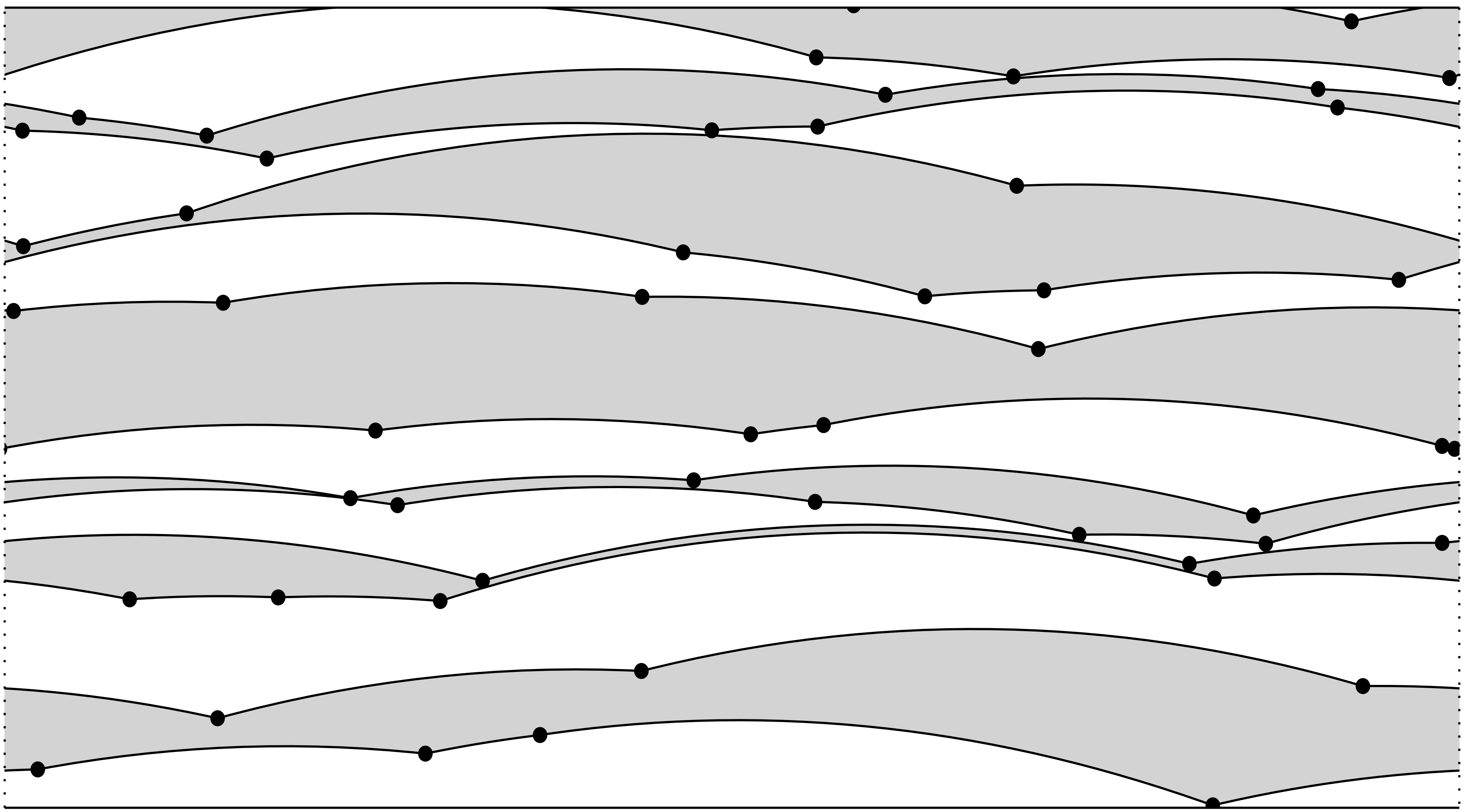


Random-set peeling

Freie Universität Berlin

Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020



semiconvex peeling, on a cylinder