How many compositions of two polyominoes?

Günter Rote
Freie Universität Berlin

joint work with Andrei Asinowski, Gill Barequet,
Gil Ben-Shachar, Martha Carolina Osegueda

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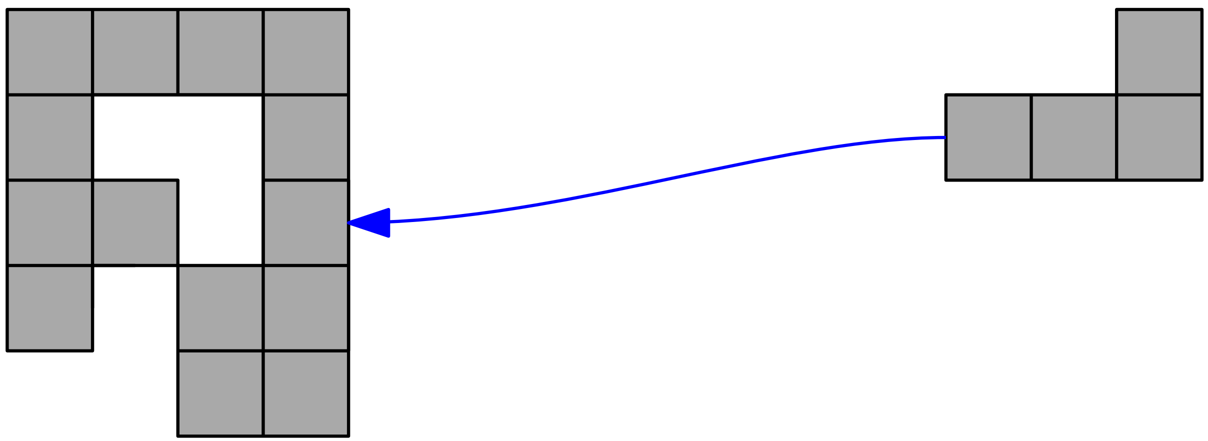

P_1, size $n_1 = 14$

P_2, size $n_2 = 4$

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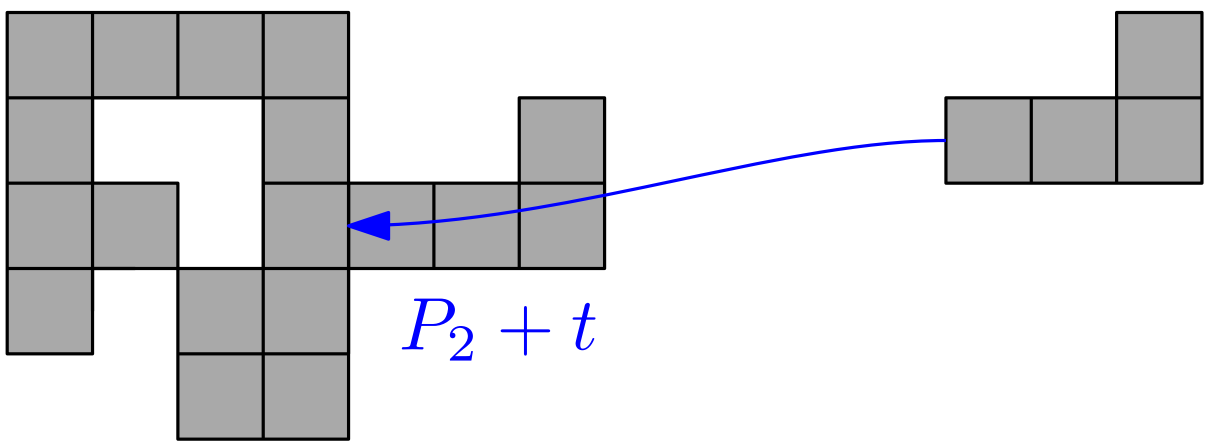

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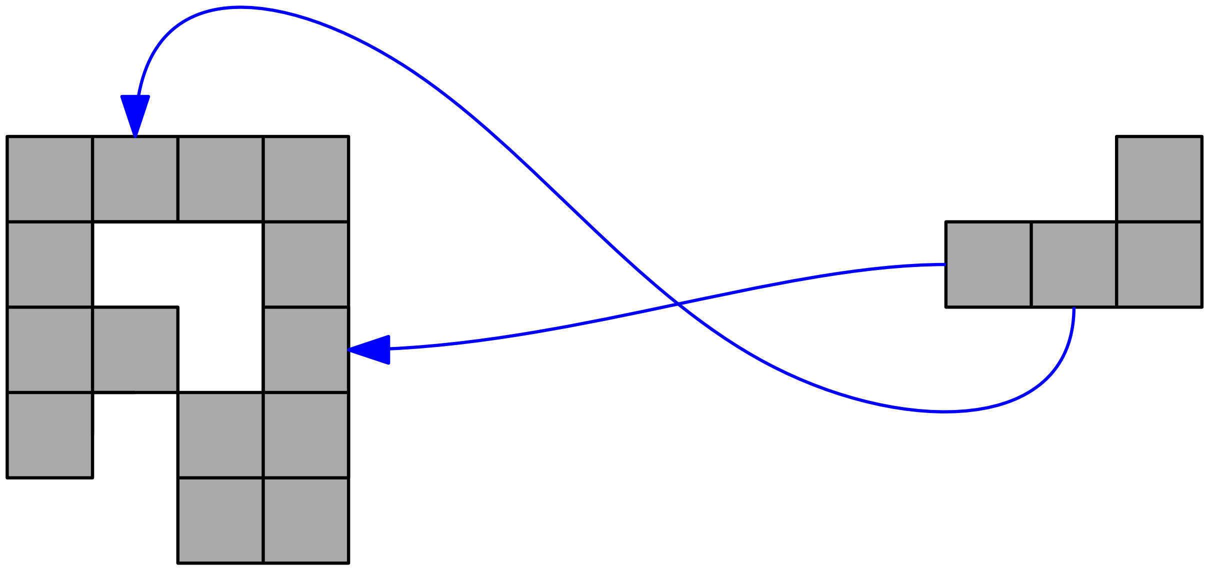

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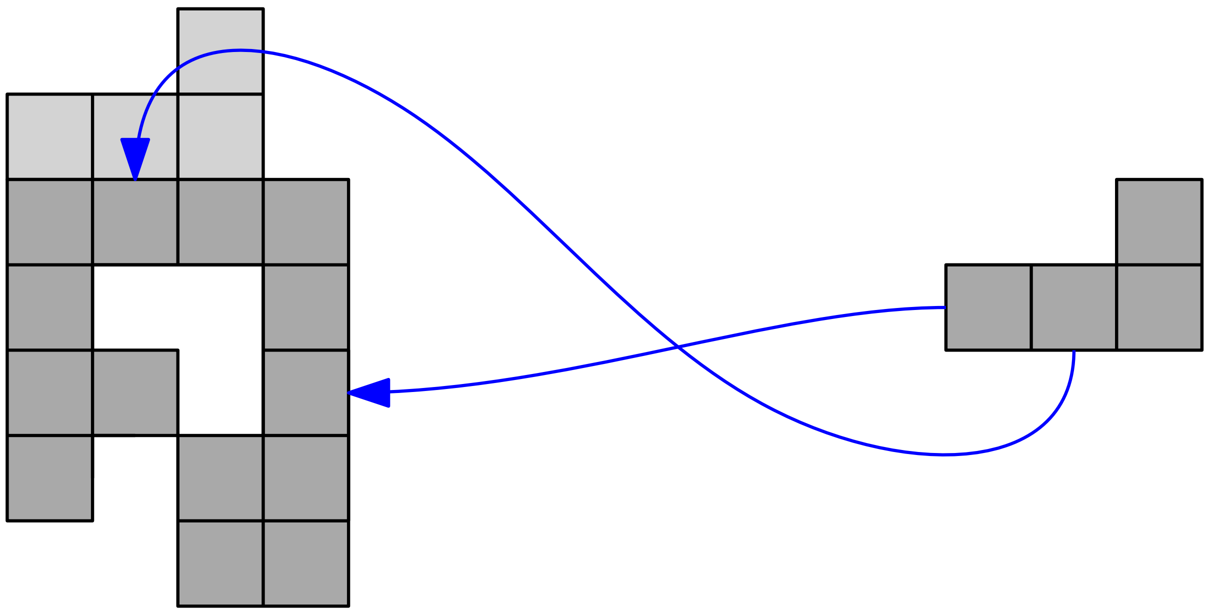

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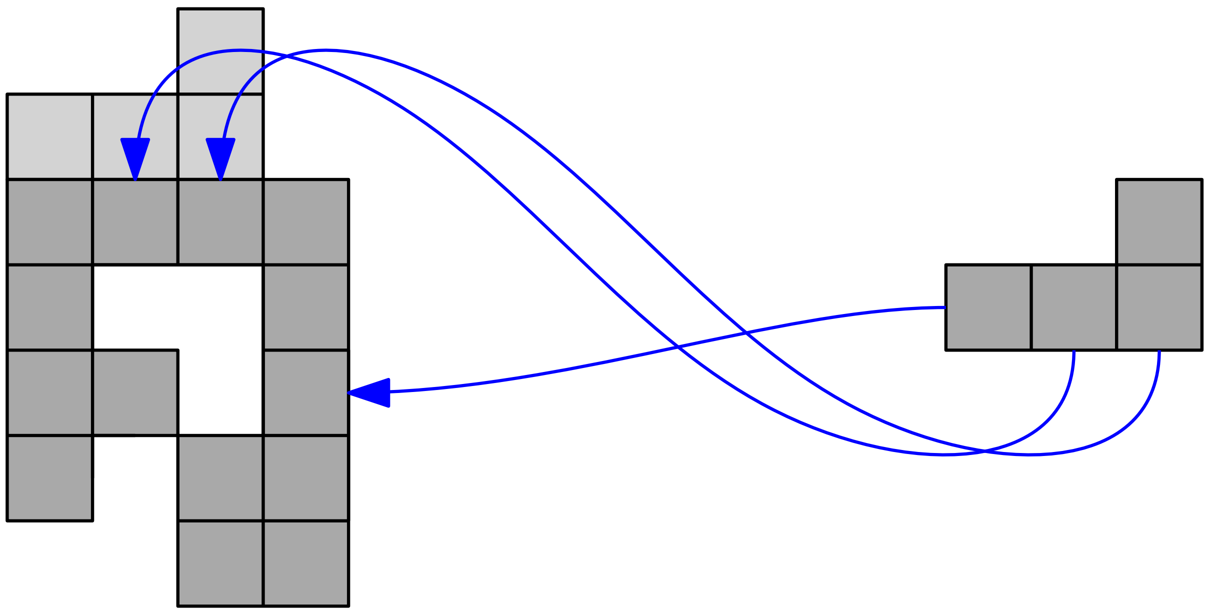

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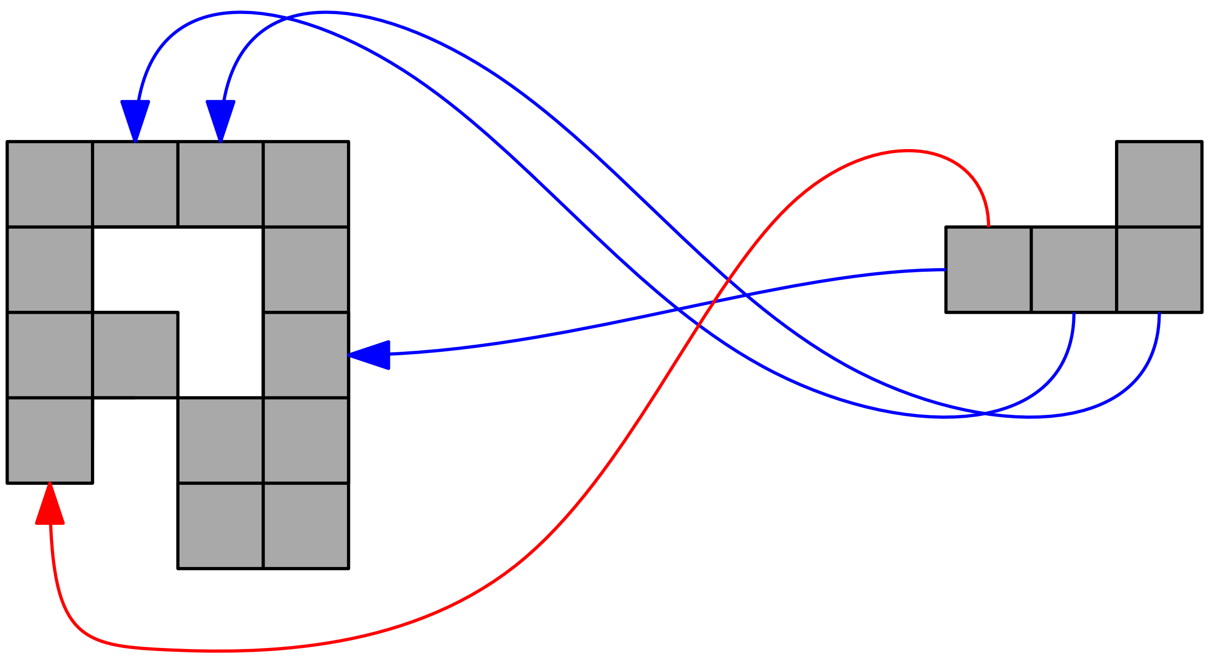

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Background

(Wrong) LEMMA. Two polyominoes of total size $n_1 + n_2 = n$ have at most $2n$ compositions.

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A_n = the number of polyominoes of size n

$$A_n \leq \sum_{n_1=n/4}^{3n/4} A_{n_1} A_{n-n_1} 2n$$

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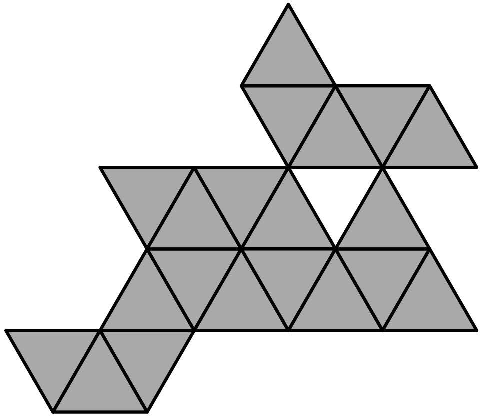
[G. Barequet, G. Rote, Mira Shalah 2019]:
Improved bounds on the number of polyiamonds



Almost tight bounds

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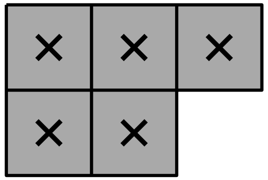
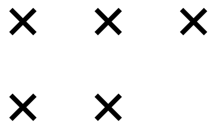
THEOREM. Two polyominoes of size n can have as many as

$$\frac{n^2}{2^{8\cdot\sqrt{\log_2 n}}}$$

compositions.

Compositions & Minkowski difference

Represent polyomino P by the set A of square centers

P_1  A_1 

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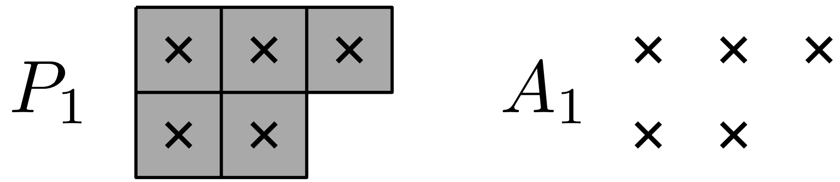

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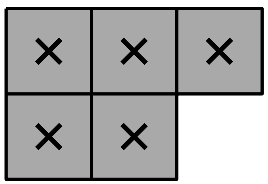

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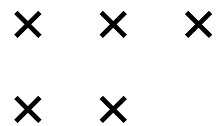

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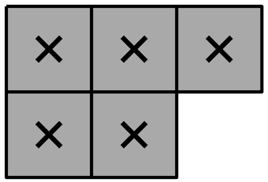
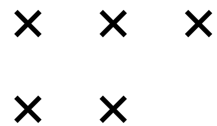
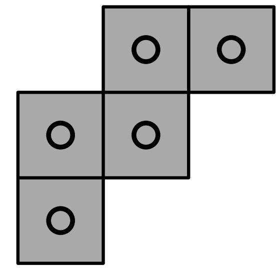
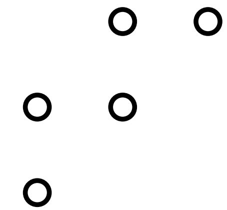
P_1  A_1  P_2  A_2 

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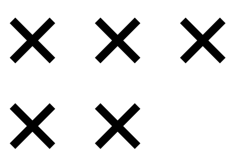

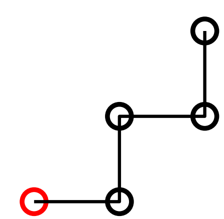

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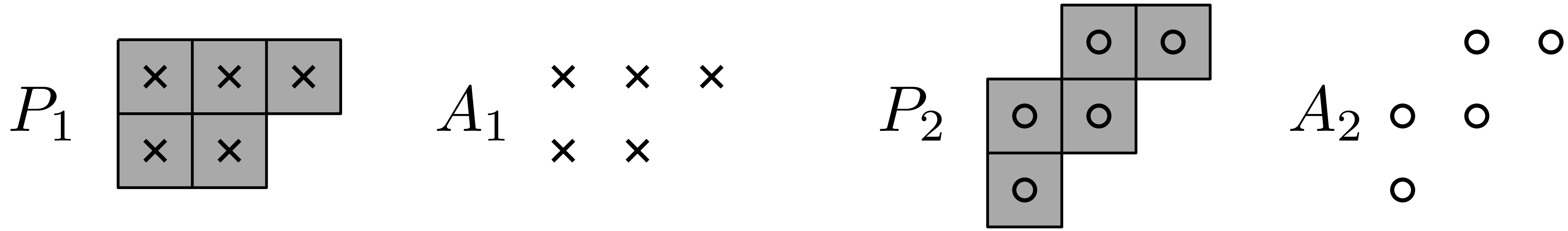

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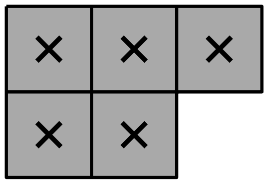
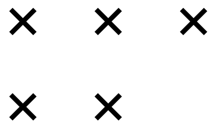
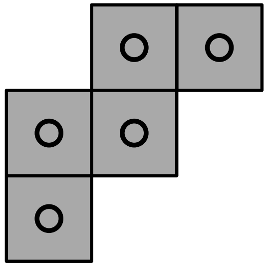
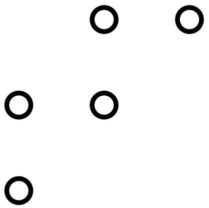
P_1  A_1  P_2  A_2 

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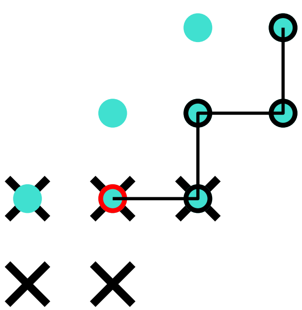

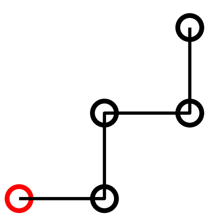

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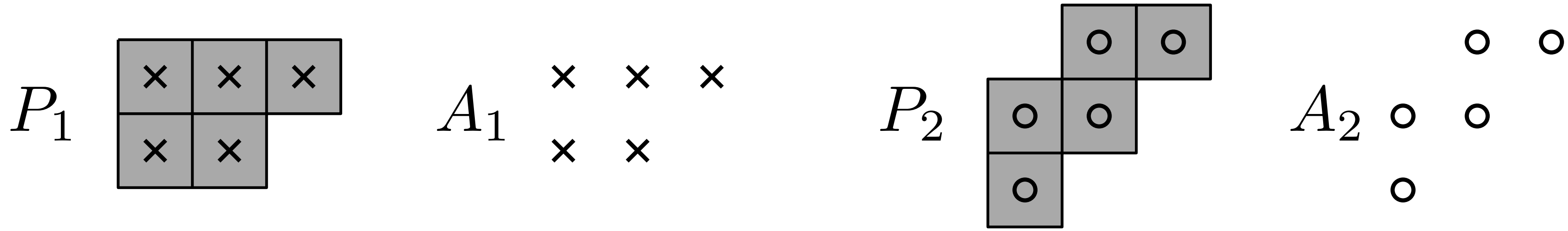

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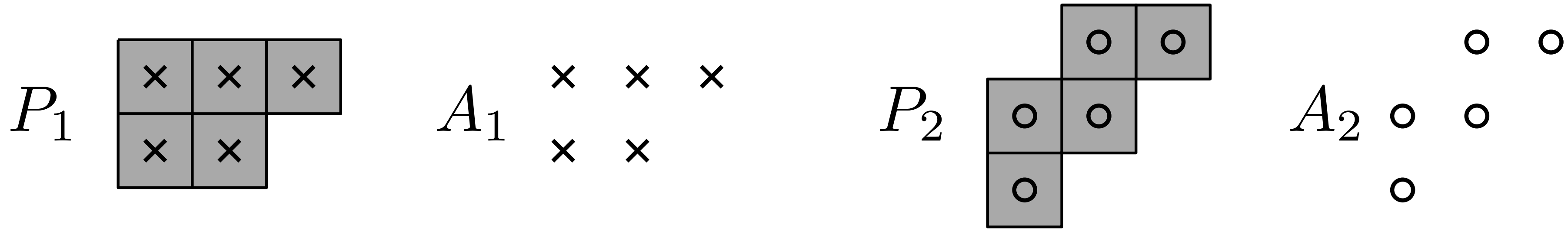

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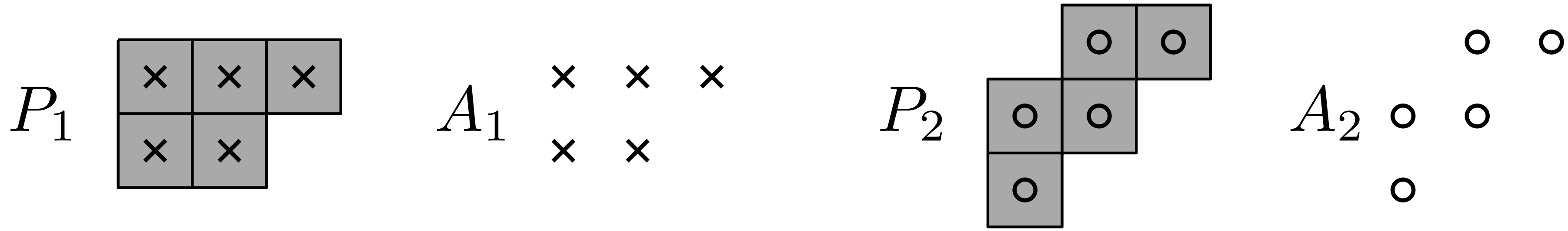

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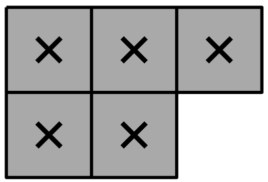
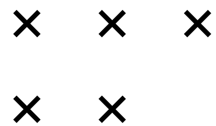
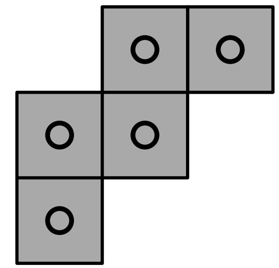
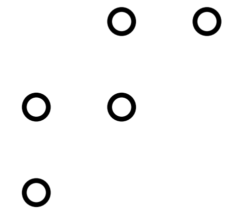
P_1  A_1  P_2  A_2 

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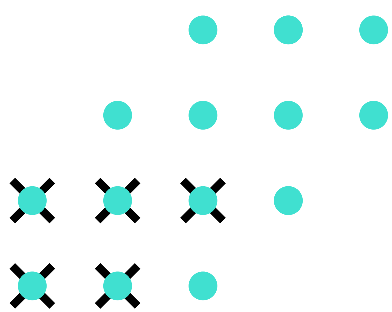

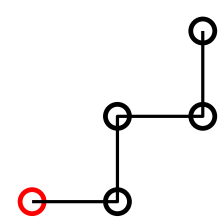 $-A_2$

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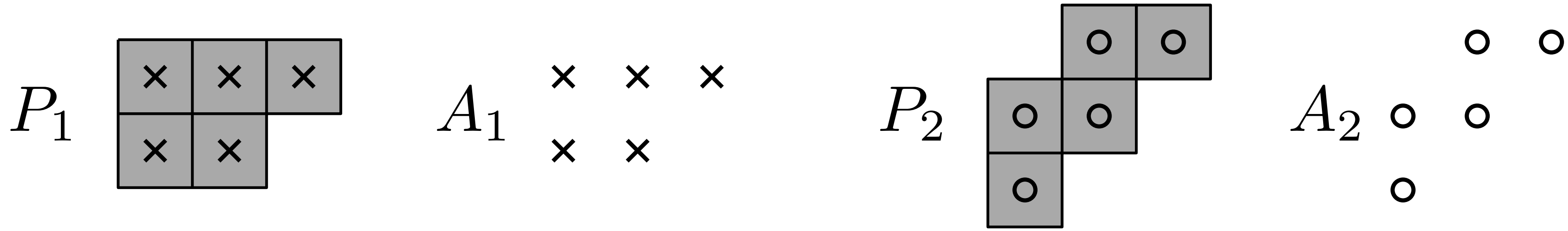

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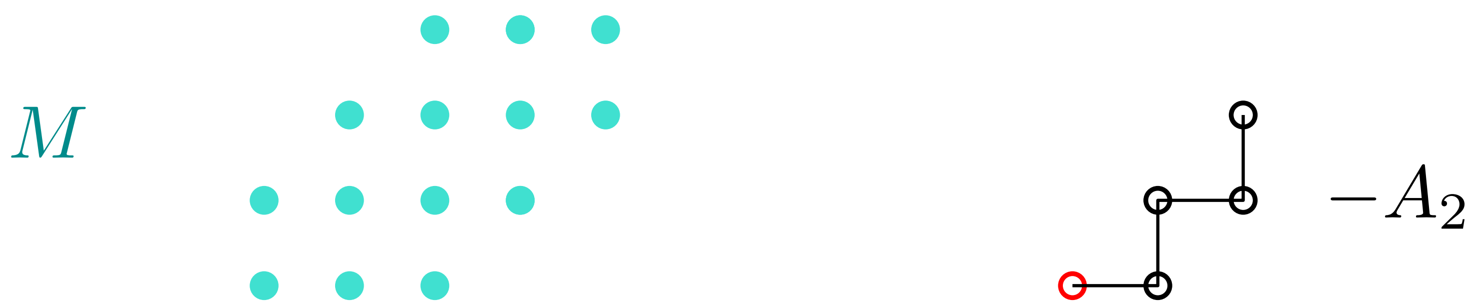

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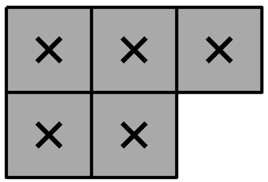
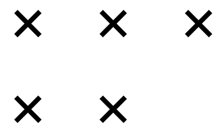
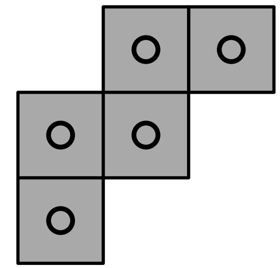
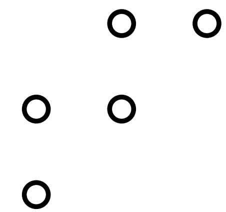
P_1  A_1  P_2  A_2 

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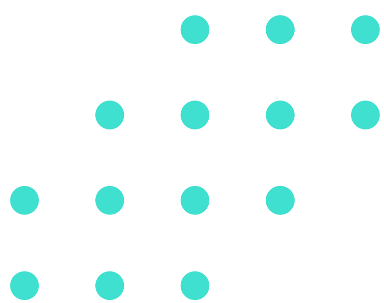
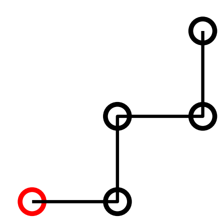
M 

 $-A_2$

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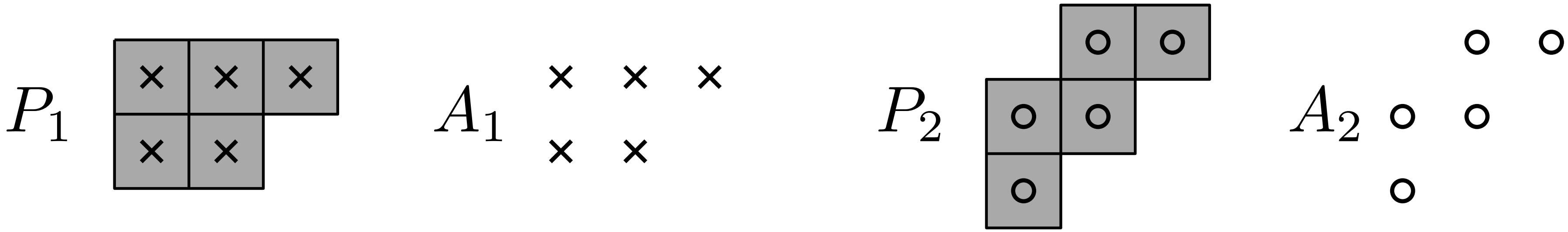

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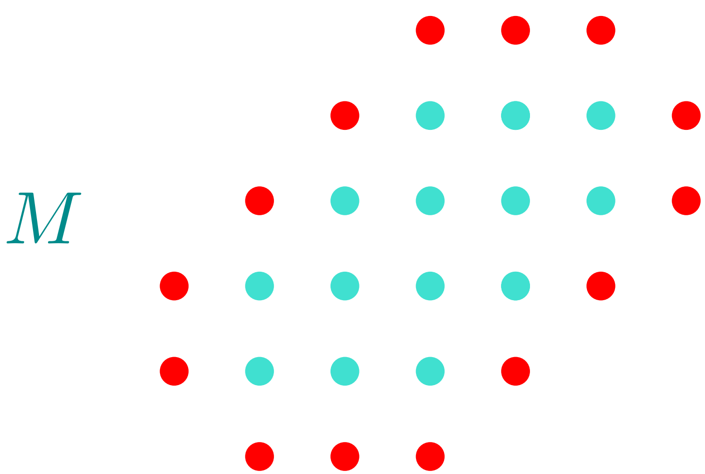

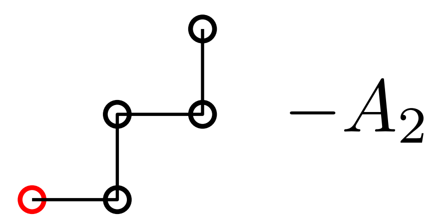

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Motion Planning

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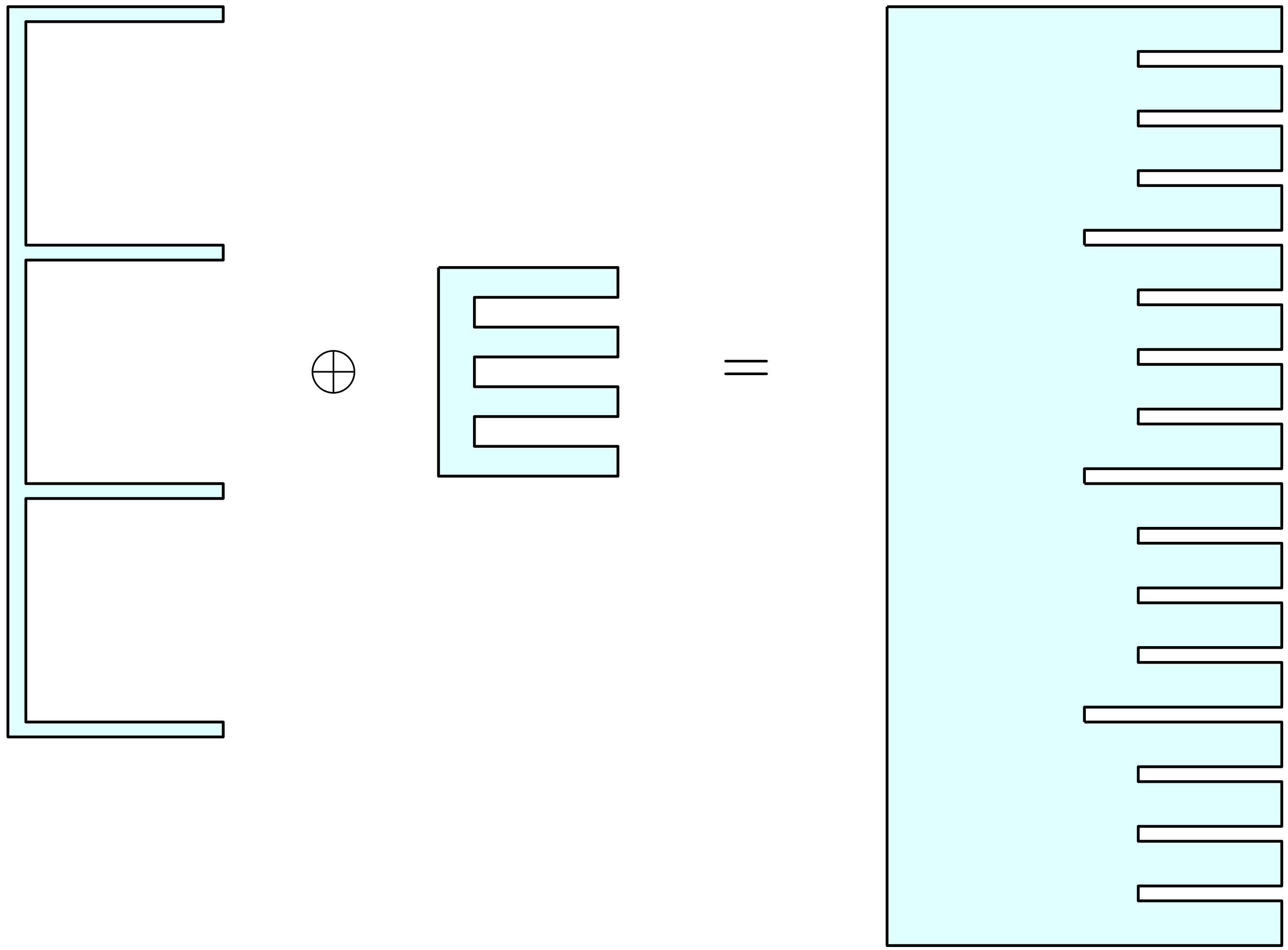
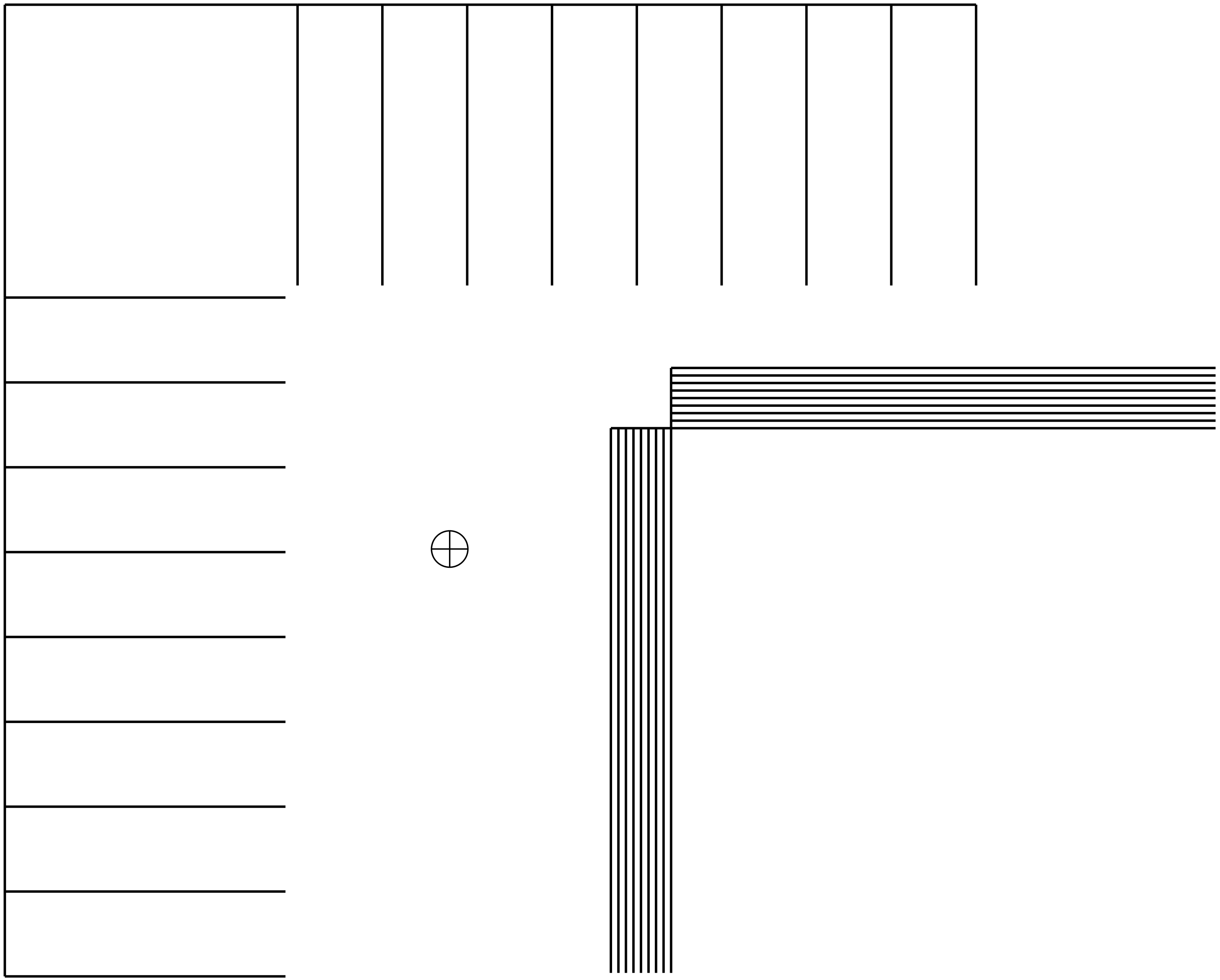


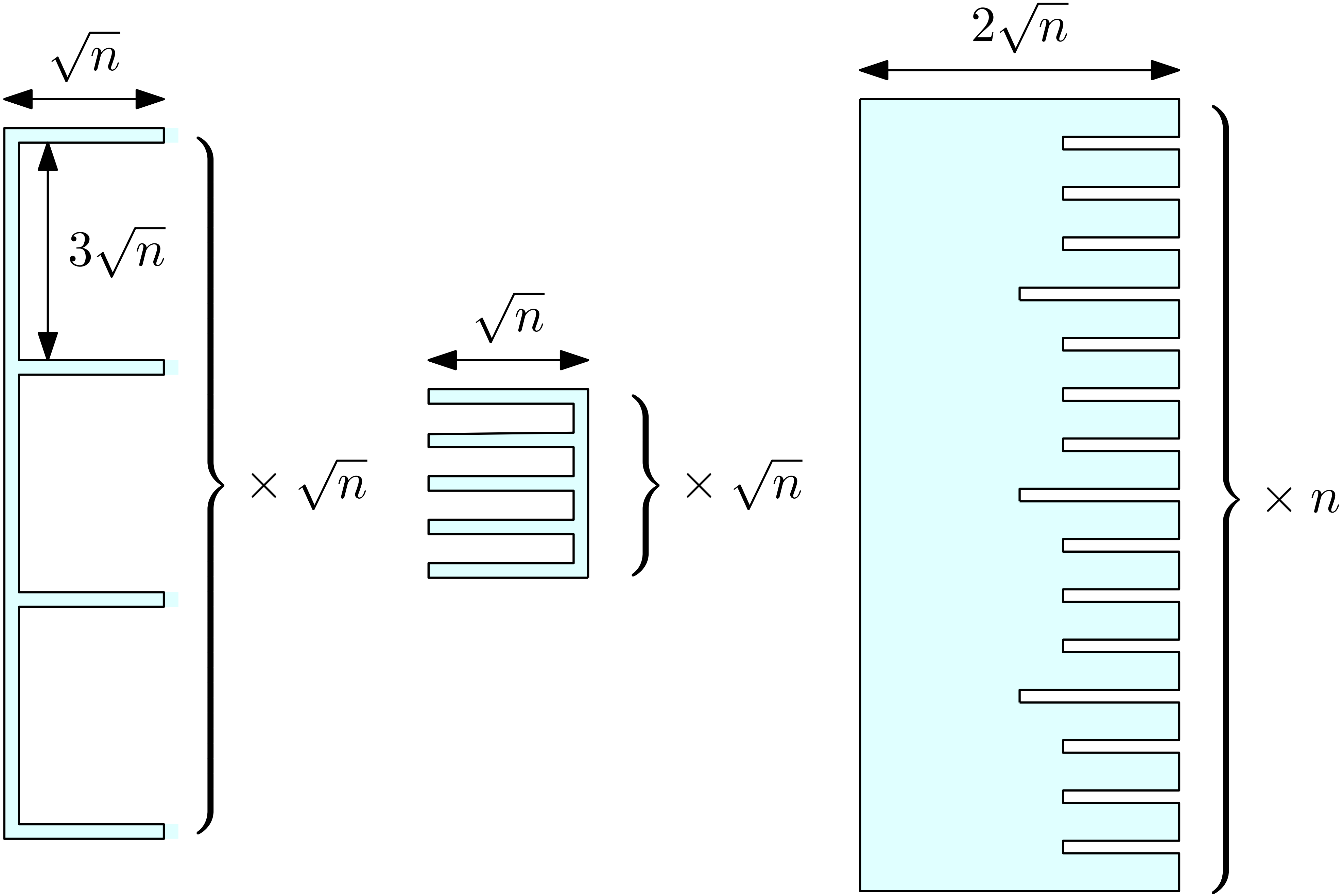
$\Theta(n^2)$

Motion Planning



$\Theta(n^4)$

Many Compositions



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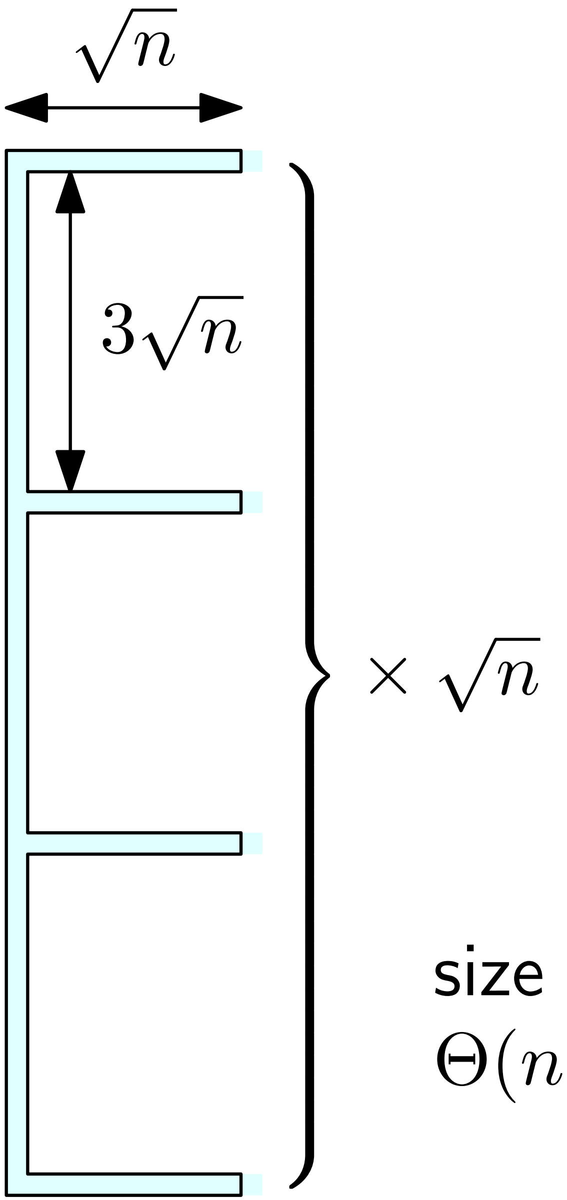
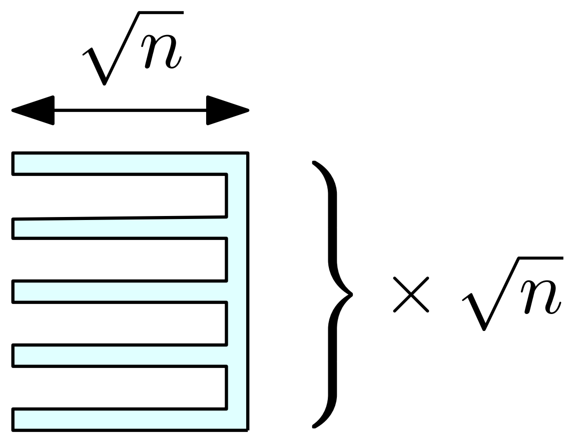
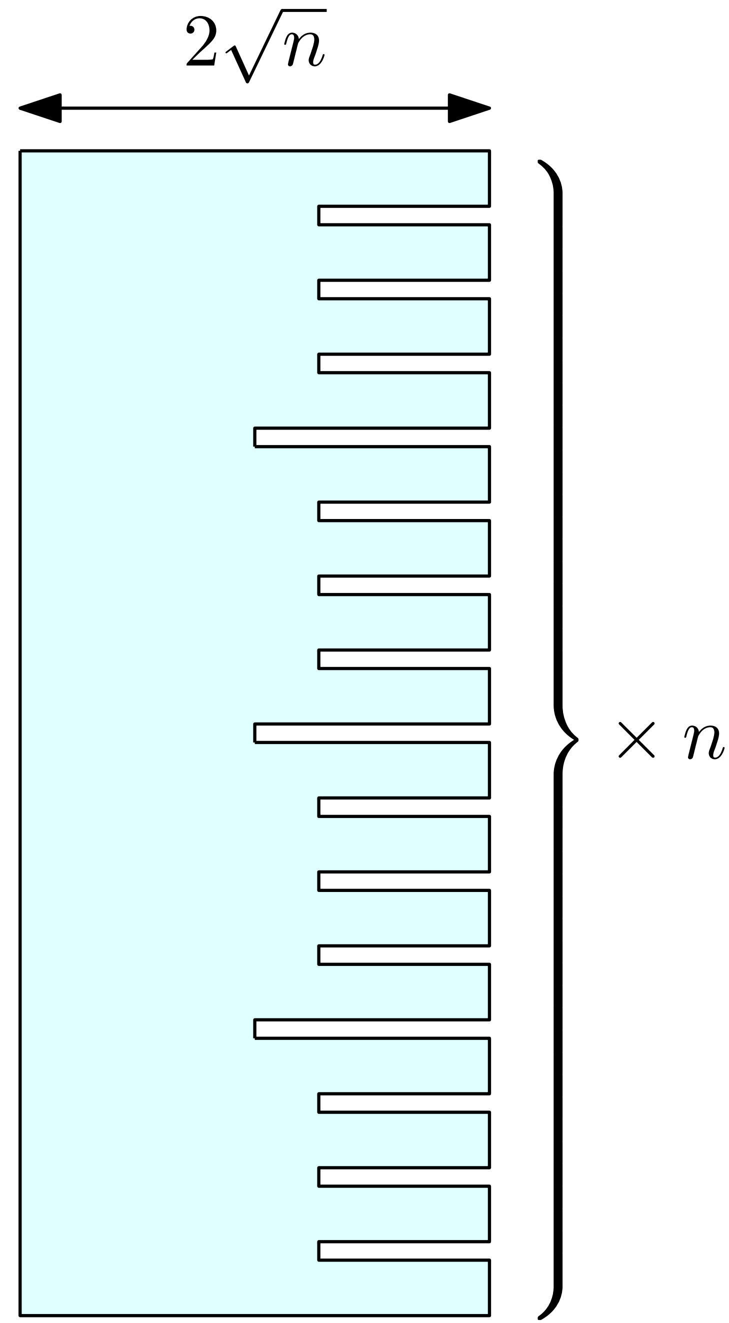


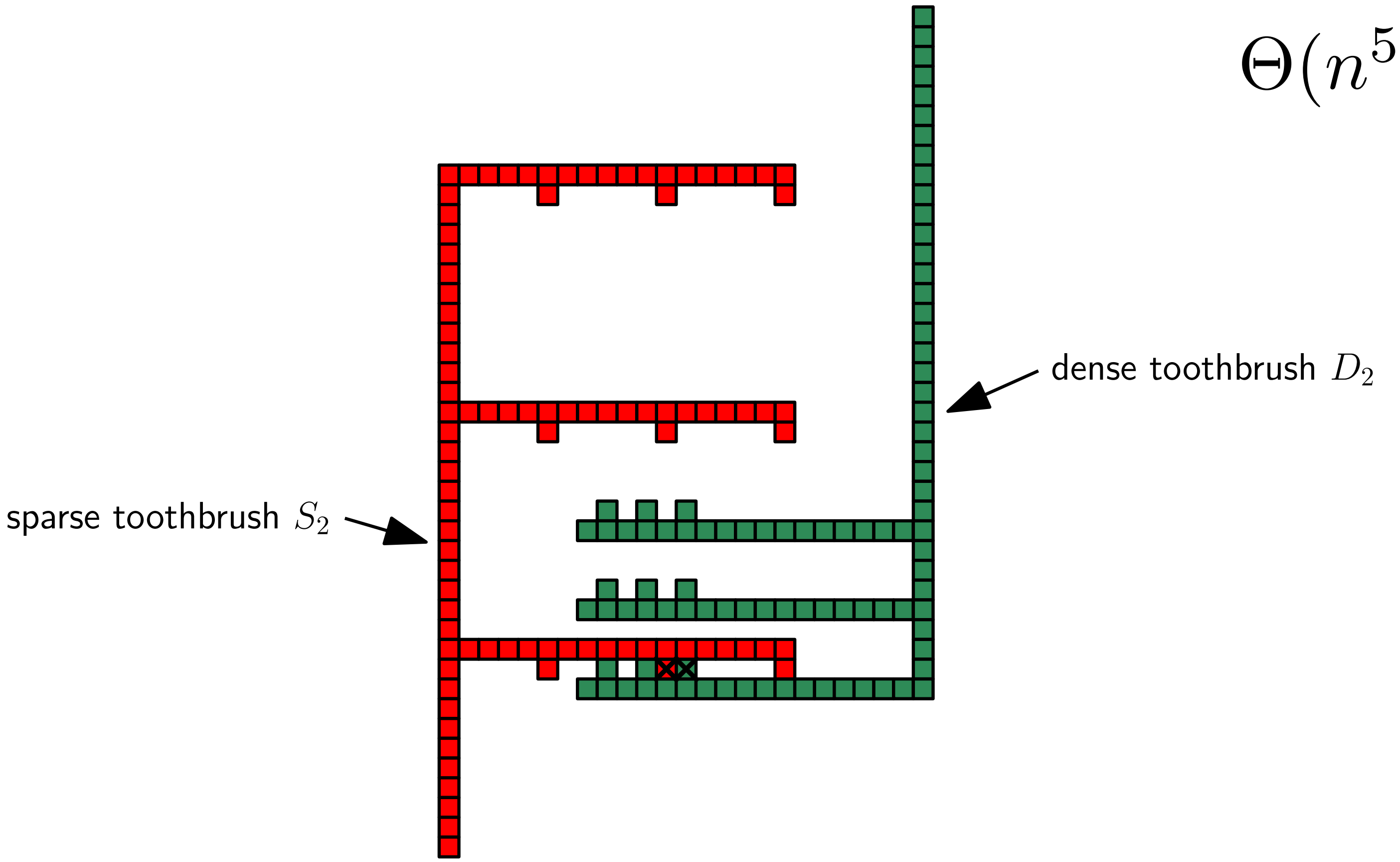




size $= \Theta(n)$

$\Theta(n^{3/2})$ compositions

Even more compositions



$$\Theta(n^{5/3})$$

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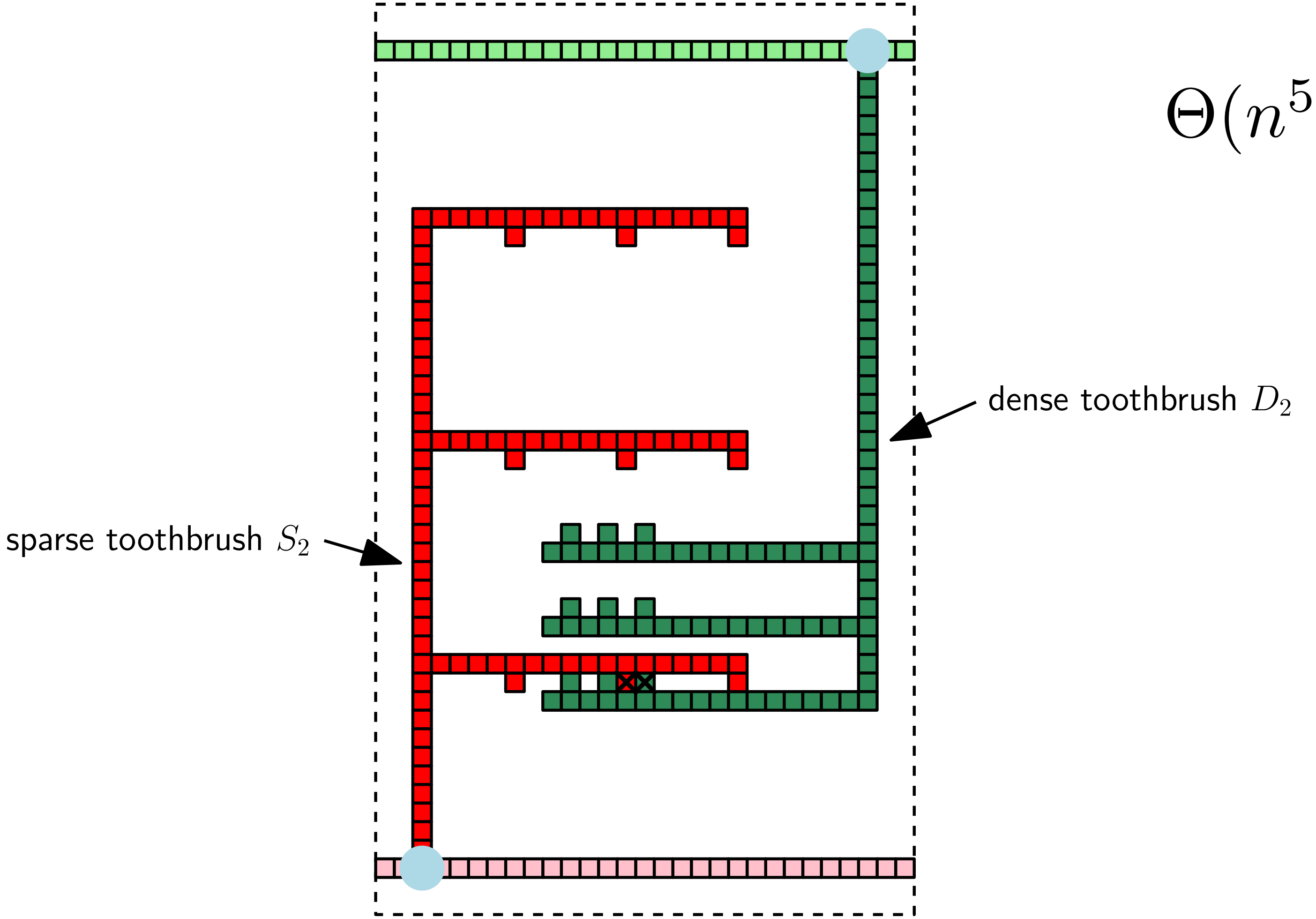
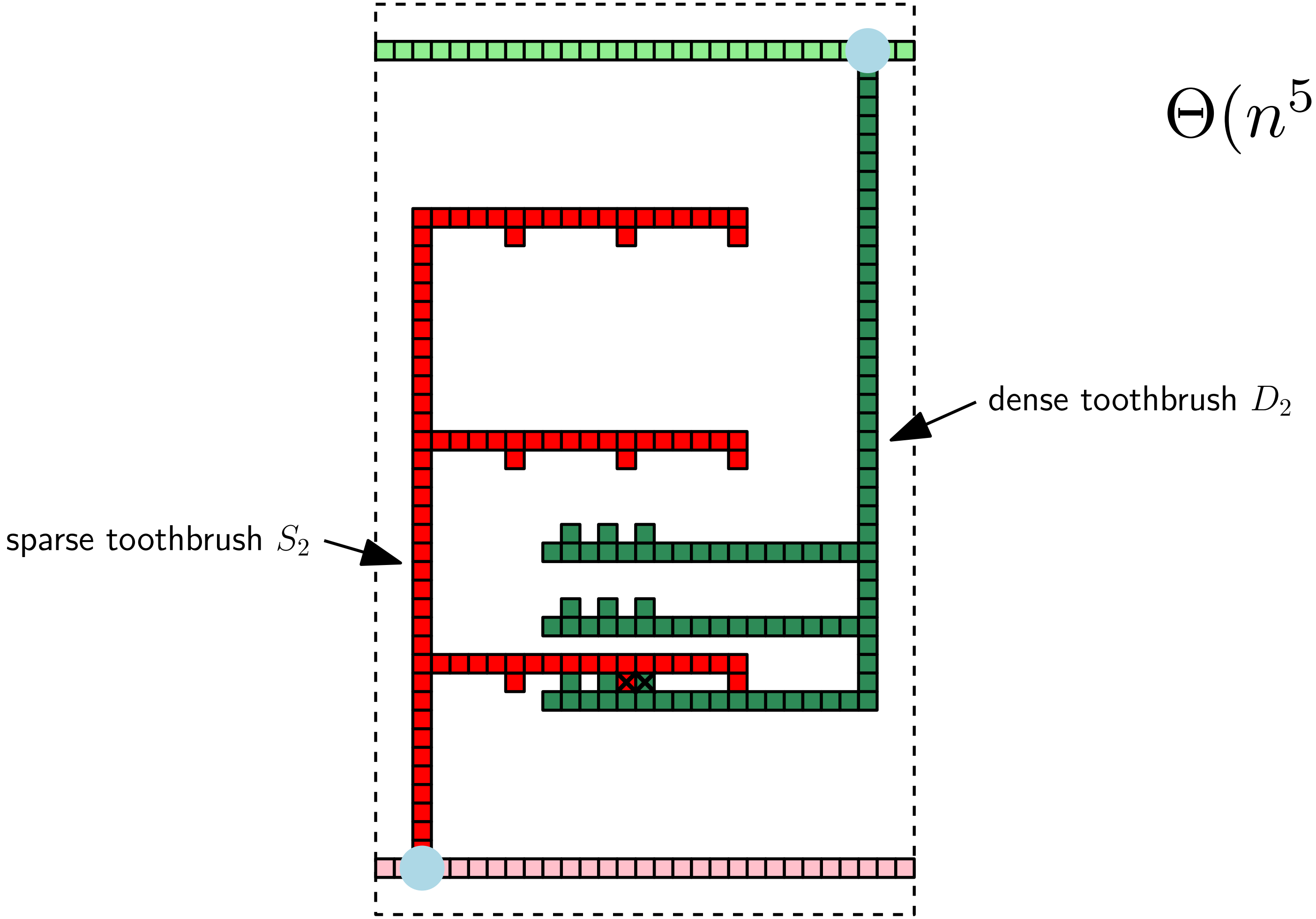


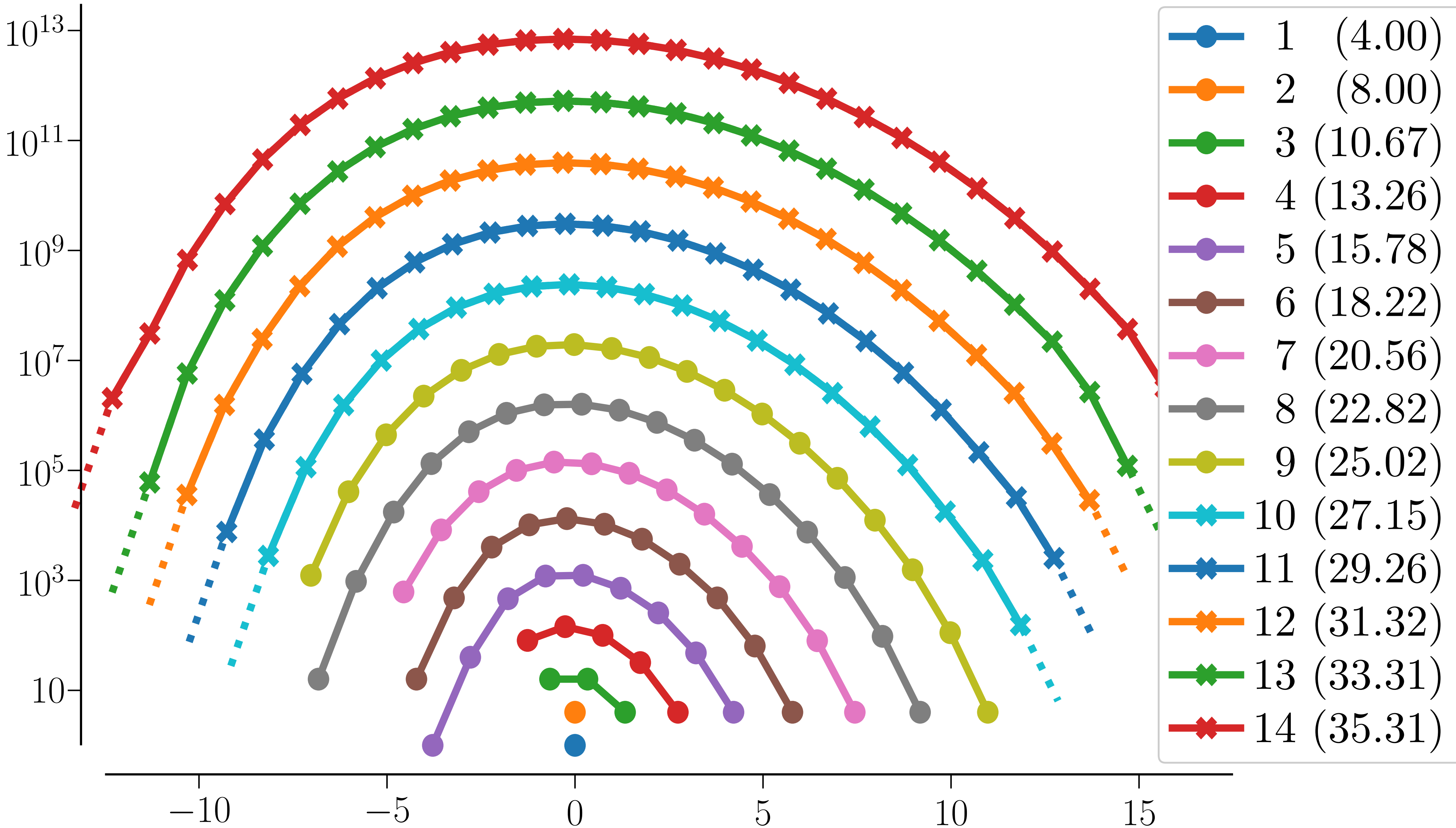
$$\Theta(n^{5/3})$$

Even more compositions



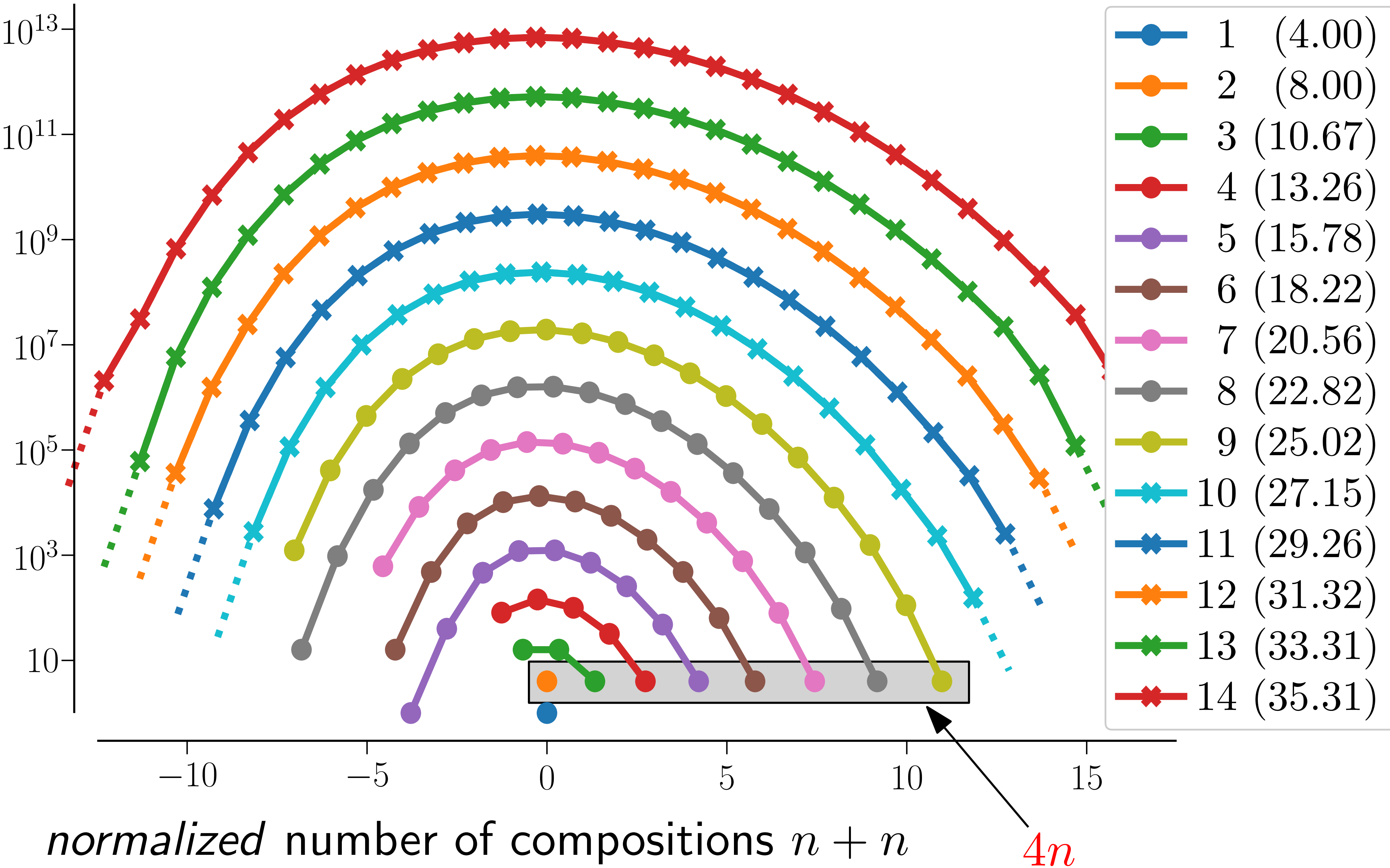
$$\Theta(n^{5/3})$$

Numerical experiments



normalized number of compositions $n + n$

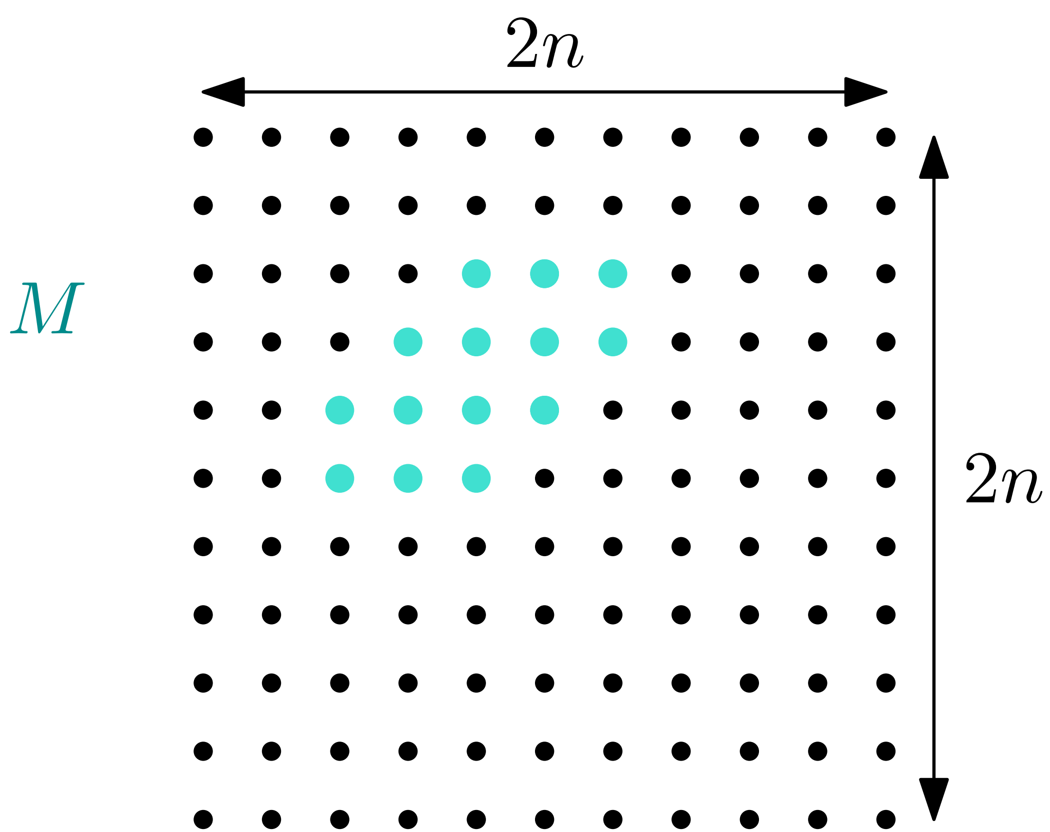
Numerical experiments



normalized number of compositions $n + n$

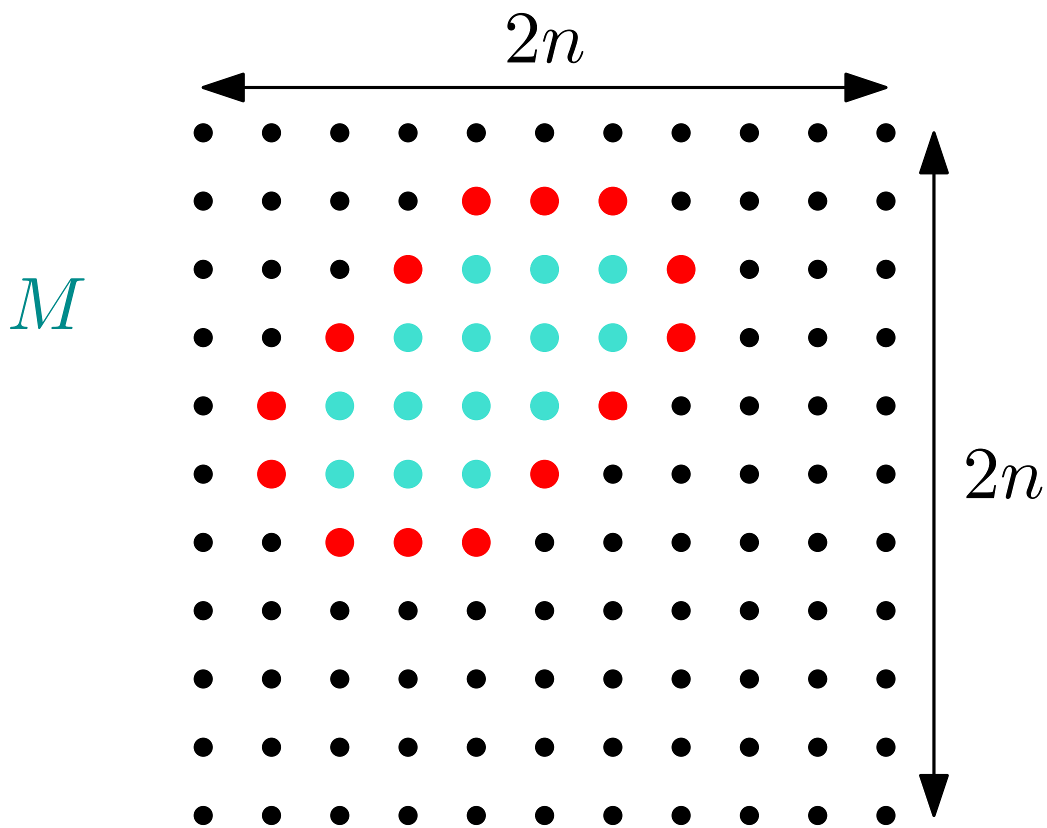
Compute the (number of) compositions

Find $M := A_1 \oplus (-A_2)$ and find all its neighbors



Compute the (number of) compositions

Find $M := A_1 \oplus (-A_2)$ and find all its neighbors



Compute the (number of) compositions

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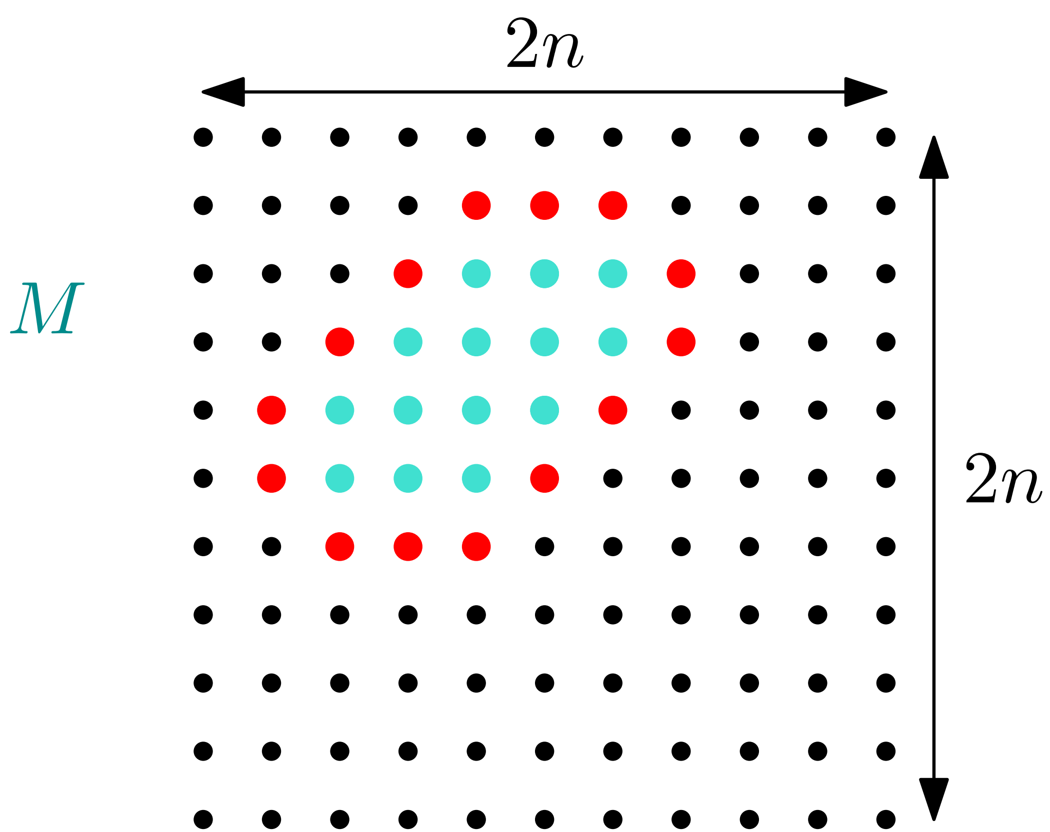

$O(n^2)$ space
for a $2n \times 2n$ array

$O(n^2)$ time

In d dimensions: $O(n^2 d)$ space and $O(n^2 d^2)$ time. (Radix sort)